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Very I-favorable spaces

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ABSTRACT

We prove that a Hausdorff space X is very I-favorable if and only if X is the almost limit space of a σ-complete inverse system consisting of (not necessarily Hausdorff) second countable spaces and surjective d-open bonding maps. It is also shown that the class of Tychonoff very I-favorable spaces with respect to the co-zero sets coincides with the d-openly generated spaces.

1. Introduction

The classes of I-favorable and very I-favorable spaces were introduced by P. Daniels, K. Kunen and H. Zhou [2]. Let us recall the corresponding definitions. Two players are playing the so-called *open–open game* in a space $(X, \mathcal{T}_X)$, a round consists of player I choosing a nonempty open set $U \subset X$ and player II a nonempty open set $V \subset U$; I wins if the union of II's open sets is dense in X, otherwise II wins. A space X is called I-*favorable* if player I has a winning strategy. This means that there exists a function $\sigma : \bigcup\{\mathcal{T}_X^n\colon n \geqslant 0\} \to \mathcal{T}_X$ such that for each game

$$\sigma(\varnothing), B_0, \sigma(B_0), B_1, \sigma(B_0, B_1), B_2, \ldots, B_n, \sigma(B_0, \ldots, B_n), B_{n+1}, \ldots$$

the union $\bigcup_{n\geqslant 0} B_n$ is dense in X, where $\emptyset \neq \sigma(\varnothing) \in \mathcal{T}_X$ and $B_{k+1} \subset \sigma(B_0, B_1, \ldots, B_k) \neq \emptyset$ and $\emptyset \neq B_k \in \mathcal{T}_X$ for $k \geqslant 0$.

A family $\mathcal{C} \subset [\mathcal{T}_X]^{\leqslant\omega}$ is said to be a *club* if: (i) $\mathcal{C}$ is closed under increasing ω-chains, i.e., if $C_1 \subset C_2 \subset \cdots$ is an increasing ω-chain from $\mathcal{C}$, then $\bigcup_{n\geqslant 1} C_n \in \mathcal{C}$; (ii) for any $B \in [\mathcal{T}_X]^{\leqslant\omega}$ there exists $C \in \mathcal{C}$ with $B \subset C$.

Let us recall [7, p. 218], that $C \subset_c \mathcal{T}_X$ means that for any nonempty $V \in \mathcal{T}_X$ there exists $W \in C$ such that if $U \in C$ and $U \subset W$, then $U \cap V \neq \varnothing$. A space X is I-*favorable* if and only if the family

$$\{\mathcal{P} \in [\mathcal{T}_X]^{\leqslant\omega}\colon \mathcal{P} \subset_c \mathcal{T}_X\}$$

contains a club, see [2, Theorem 1.6].

A space X is called *very* I-*favorable* if the family

$$\{\mathcal{P} \in [\mathcal{T}_X]^{\leqslant\omega}\colon \mathcal{P} \subset_! \mathcal{T}_X\}$$

contains a club. Here, $\mathcal{P} \subset_! \mathcal{T}_X$ means that for any $\mathcal{S} \subset \mathcal{P}$ and $x \notin \operatorname{cl}_X \bigcup\mathcal{S}$, there exists $W \in \mathcal{P}$ such that $x \in W$ and $W \cap \bigcup\mathcal{S} = \emptyset$. It is easily seen that $\mathcal{P} \subset_! \mathcal{T}_X$ implies $\mathcal{P} \subset_c \mathcal{T}_X$.

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It was shown by the first two authors in [5] that a compact Hausdorff space is I-favorable if and only if it can be represented as the limit of a σ-complete (in the sense of Shchepin [10]) inverse system consisting of I-favorable compact metrizable spaces and skeletal bonding maps, see also [4] and [6]. For similar characterization of I-favorable spaces with respect to co-zero sets, see [14]. Recall that a continuous map $f : X \to Y$ is called *skeletal* if the set $\mathrm{Int}_Y \mathrm{cl}_Y f(U)$ is nonempty, for any $U \in \mathcal{T}_X$, see [8].

In this paper we show that there exists an analogy between the relations I-favorable spaces—skeletal maps and very I-favorable spaces—d-open maps (see Section 2 for the definition of d-open maps). The following two theorems are our main results:

Theorem 3.3. *A Hausdorff space X is very* I*-favorable if and only if $X = \text{a-}\varprojlim S$, where $S = \{X_A, q^A_B, \mathcal{C}\}$ is a σ-complete inverse system such that all X_A are (not-necessarily Hausdorff) spaces with countable weight and the bonding maps q^A_B are d-open and onto.*

Theorem 4.1. *A completely regular space X is very* I*-favorable with respect to the co-zero sets if and only if X is* d*-openly generated.*

We say that a space X is *an almost limit* of the inverse system $S = \{X_\sigma, \pi^\sigma_\varrho, \Gamma\}$, if X can be embedded in $\varprojlim S$ such that $\pi_\sigma(X) = X_\sigma$ for each $\sigma \in \Gamma$. We denote this by $X = \text{a-}\varprojlim S$, and it implies that X is a dense subset of $\varprojlim S$. A completely regular space X is d-*openly generated* if there exists a σ-complete inverse system $S = \{X_\sigma, \pi^\sigma_\varrho, \Gamma\}$ consisting of separable metric spaces X_σ and d-open surjective bonding maps π^σ_ϱ such that $X = \text{a-}\varprojlim S$.

Theorem 4.1 implies the following characterization of κ-metrizable compacta (see Corollary 4.3), which provides an answer of a question from [14]: A compact Hausdorff space is very I-favorable with respect to the co-zero sets if and only if X is κ-metrizable.

2. Very I-favorable spaces and d-open maps

T. Byczkowski and R. Pol [1] introduced nearly open sets and nearly open maps as follows. A subset of a topological space is *nearly open* if it is in the interior of its closure. A map is *nearly open* if the image of every open subset is nearly open. Continuous nearly open maps were called d-*open* by M. Tkachenko [12]. Obviously, every d-open map is skeletal.

Proposition 2.1. *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and $f : X \to Y$ a continuous function. Then the following conditions are equivalent*:

(1) *f is d-open*;
(2) $\mathrm{cl}_X f^{-1}(V) = f^{-1}(\mathrm{cl}_Y V)$ *for any open $V \subset Y$*;
(3) $f(U) \subset \mathrm{Int}_Y \mathrm{cl}_Y f(U)$ *for every open subset $U \subset X$*;
(4) $\{f^{-1}(V)\colon V \in \mathcal{T}_Y\} \subset_! \mathcal{T}_X$.

Proof. The implication (1) ⇒ (2) was established in [12, Lemma 5]. Obviously (3) ⇒ (1). Let us prove the implication (2) ⇒ (3). Suppose $U \subset X$ is open. Then we have $X \setminus f^{-1}(\mathrm{Int}_Y \mathrm{cl}_Y f(U)) \subset X \setminus U$. Indeed, $Y \setminus \mathrm{Int}_Y \mathrm{cl}_Y f(U) = \mathrm{cl}_Y(Y \setminus \mathrm{cl}_Y f(U))$ and by (2) we get

$$f^{-1}\big(\mathrm{cl}_Y\big(Y \setminus \mathrm{cl}_Y f(U)\big)\big) = \mathrm{cl}_X\big(f^{-1}\big(Y \setminus \mathrm{cl}_Y f(U)\big)\big).$$

But $\mathrm{cl}_X(f^{-1}(Y \setminus \mathrm{cl}_Y f(U))) = \mathrm{cl}_X(X \setminus f^{-1}(\mathrm{cl}_Y f(U)))$ and

$$X \setminus f^{-1}\big(\mathrm{cl}_Y f(U)\big) \subset X \setminus \mathrm{cl}_X f^{-1}\big(f(U)\big) \subset X \setminus \mathrm{cl}_X U \subset X \setminus U.$$

Hence $f(U) \cap Y \setminus \mathrm{Int}_Y \mathrm{cl}_Y f(U) = \emptyset$ and $f(U) \subset \mathrm{Int}_Y \mathrm{cl}_Y f(U)$.

To show (4) ⇒ (2), assume that $\{f^{-1}(V)\colon V \in \mathcal{T}_Y\} \subset_! \mathcal{T}_X$. Since f is continuous we get $\mathrm{cl}_X f^{-1}(V) \subset f^{-1}(\mathrm{cl}_Y V)$ for any open set $V \subset Y$. We shall show that $f^{-1}(\mathrm{cl}_Y V) \subset \mathrm{cl}_X f^{-1}(V)$ for any open $V \subset Y$. Suppose there exists an open set $V \subset Y$ such that

$$f^{-1}(\mathrm{cl}_Y V) \setminus \mathrm{cl}_X f^{-1}(V) \neq \emptyset.$$

Let $x \in f^{-1}(\mathrm{cl}_Y V) \setminus \mathrm{cl}_X f^{-1}(V)$ and $\mathcal{S} = \{f^{-1}(V)\}$. Since $x \notin \mathrm{cl}_X \bigcup \mathcal{S} = \mathrm{cl}_X f^{-1}(V)$, there is an open set $U \in \mathcal{B}_Y$ such that $x \in f^{-1}(U)$ and $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. Therefore, $f(x) \in U \cap \mathrm{cl}_Y V$ which contradicts $V \cap U = \emptyset$.

Finally, we can show that (2) yields $\{f^{-1}(V)\colon V \in \mathcal{T}_Y\} \subset_! \mathcal{T}_X$. Indeed, let $\mathcal{S} \subset \{f^{-1}(V)\colon V \in \mathcal{T}_Y\}$ and $x \notin \mathrm{cl}_X \bigcup \mathcal{S}$. Then there is $U \in \mathcal{T}_Y$ such that $\bigcup \mathcal{S} = f^{-1}(U)$. Hence, $\mathrm{cl}_X \bigcup \mathcal{S} = f^{-1}(\mathrm{cl}_Y U)$. Put

$$W = f^{-1}(Y \setminus \mathrm{cl}_Y U).$$

We have $x \in W$ and $W \cap \mathrm{cl}_X \bigcup \mathcal{S} = \emptyset$. □

Remark 2.2. If, under the hypotheses of Proposition 2.1, there exists a base $\mathcal{B}_Y \subset \mathcal{T}_Y$ with $\{f^{-1}(V)\colon V \in \mathcal{B}_Y\} \subset_! \mathcal{T}_X$, then f is d-open.

Indeed, we can follow the proof of the implication (4) ⇒ (2) from Proposition 2.1. The only difference is the choice of the family $\mathcal{S}$. If there exists $x \in f^{-1}(\mathrm{cl}_Y V) \setminus \mathrm{cl}_X f^{-1}(V)$ for some open $V \subset Y$, we choose $\mathcal{S} = \{f^{-1}(W)\colon W \in \mathcal{B}_Y \text{ and } W \subset V\}$.

Next lemma was established in [12, Lemma 9].

Lemma 2.3. *Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps with f being surjective. Then g is d-open provided so is $g \circ f$.* □

Let X be a topological space equipped with a topology $\mathcal{T}_X$ and $\mathcal{Q} \subset \mathcal{T}_X$. Suppose that there exists a function $\sigma : \bigcup\{\mathcal{Q}^n\colon n \geqslant 0\} \to \mathcal{Q}$ such that if $B_0, B_1, \ldots$ is a sequence of nonempty elements of $\mathcal{Q}$ with $B_0 \subset \sigma(\varnothing)$ and $B_{n+1} \subset \sigma((B_0, B_1, \ldots, B_n))$ for all $n \in \omega$, then $\{B_n\colon n \in \omega\} \cup \{\sigma((B_0, B_1, \ldots, B_n))\colon n \in \omega\} \subset_! \mathcal{Q}$. The function σ is called a *strong winning strategy in* $\mathcal{Q}$. If $\mathcal{Q} = \mathcal{T}_X$, σ is called a strong winning strategy. It is clear that if σ is strong winning strategy, then it is a winning strategy for player I in the open–open game.

Lemma 2.4. *Let $\sigma : \bigcup\{\mathcal{Q}^n\colon n \geqslant 0\} \to \mathcal{Q}$ be a strong winning strategy in $\mathcal{Q}$, where $\mathcal{Q}$ is a family of open subsets of X. Then $\mathcal{P} \subset_! \mathcal{Q}$ for every family $\mathcal{P} \subset \mathcal{Q}$ such that $\mathcal{P}$ is closed under σ and finite intersections.*

Proof. Let $\mathcal{P} \subset \mathcal{Q}$ be closed under σ and finite intersections. Fix a family $S \subset \mathcal{P}$ and $x \notin \mathrm{cl} \bigcup S$. If $\sigma(\varnothing) \cap \bigcup S \neq \emptyset$, then take an element $U \in S$ such that $\sigma(\varnothing) \cap U \neq \emptyset$ and put $V_0 = \sigma(\varnothing) \cap U \in \mathcal{P}$. If $\sigma(\varnothing) \cap \bigcup S = \emptyset$, then put $V_0 = \sigma(\varnothing) \in \mathcal{P}$. Assume that sets $V_0, \ldots, V_n \in \mathcal{P}$ are just defined. If $\sigma(V_0, \ldots, V_n) \cap \bigcup S \neq \emptyset$, then take an element $U \in S$ such that $\sigma(V_0, \ldots, V_n) \cap U \neq \emptyset$ and put $V_{n+1} = \sigma(V_0, \ldots, V_n) \cap U \in \mathcal{P}$. If $\sigma(V_0, \ldots, V_n) \cap \bigcup S = \emptyset$, then put $V_{n+1} = \sigma(V_0, \ldots, V_n) \in \mathcal{P}$. Take a subfamily

$$\mathcal{U} = \left\{V_k\colon V_k \cap \bigcup S \neq \emptyset \text{ and } k \in \omega\right\} \subset \mathcal{Q}.$$

Since σ is strong strategy, then $\bigcup\{V_n\colon n \in \omega\}$ is dense in X. Hence $\mathrm{cl} \bigcup \mathcal{U} = \mathrm{cl} \bigcup S$. Since $\{V_n\colon n \in \omega\} \cup \{\sigma((V_0, V_1, \ldots, V_n))\colon n \in \omega\} \subset_! \mathcal{Q}$ there exists $V \in \{V_n\colon n \in \omega\} \cup \{\sigma((V_0, V_1, \ldots, V_n))\colon n \in \omega\} \subset \mathcal{P}$ such that $x \in V$ and $V \cap \bigcup S = \emptyset$. □

Proposition 2.5. *Let X be a topological space and $\mathcal{Q} \subset \mathcal{T}_X$ be a family closed under finite intersection. Then there is a strong winning strategy $\sigma : \bigcup\{\mathcal{Q}^n\colon n \geqslant 0\} \to \mathcal{Q}$ in $\mathcal{Q}$ if and only if the family $\{\mathcal{P} \in [\mathcal{Q}]^{\leqslant\omega}\colon \mathcal{P} \subset_! \mathcal{Q}\}$ contains a club $\mathcal{C}$ such that every $A \in \mathcal{C}$ is closed under finite intersections.*

Proof. If there is a club $\mathcal{C} \subset \{\mathcal{P} \in [\mathcal{Q}]^{\leqslant\omega}\colon \mathcal{P} \subset_! \mathcal{Q}\}$, then following the arguments from [2, Theorem 1.6] one can construct a strong winning strategy in $\mathcal{Q}$.

Suppose there exists a strong winning strategy $\sigma : \bigcup\{\mathcal{Q}^n\colon n \geqslant 0\} \to \mathcal{Q}$. Let $\mathcal{C}$ be the family of all countable subfamilies $A \subset \mathcal{Q}$ such that A is closed under σ and finite intersections. The family $\mathcal{C} \subset [\mathcal{Q}]^{\leqslant\omega}$ is a club. Obviously, $\mathcal{C}$ is closed under increasing ω-chains. If $B \in [\mathcal{Q}]^{\leqslant\omega}$, there exists a countable family $A_B \subset \mathcal{Q}$ which contains B and is closed under σ and finite intersections. So, $A_B \in \mathcal{C}$. According to Lemma 2.4, $A \subset_! \mathcal{Q}$ for all $A \in \mathcal{C}$. □

Corollary 2.6. *A Hausdorff space $(X, \mathcal{T})$ is very* I*-favorable if and only if the family $\{\mathcal{P} \in [\mathcal{T}]^{\leqslant\omega}\colon \mathcal{P} \subset_! \mathcal{T}\}$ contains a club $\mathcal{C}$ with the following properties*:

(i) *every $A \in \mathcal{C}$ covers X and it is closed under finite intersections*;
(ii) *for any two different points $x, y \in X$ there exists $A \in \mathcal{C}$ containing two disjoint elements $U_x, U_y \in A$ with $x \in U_x$ and $y \in U_y$*;
(iii) $\bigcup \mathcal{C} = \mathcal{T}$. □

The next proposition shows that every space X having a base $\mathcal{B}_X$ such that the family $\{\mathcal{P} \in [\mathcal{B}_X]^{\leqslant\omega}\colon \mathcal{P} \subset_! \mathcal{B}_X\}$ contains a club is very I-favorable.

Proposition 2.7. *If there exists a base $\mathcal{B}$ of X such that the family $\{\mathcal{P} \in [\mathcal{B}]^{\leqslant\omega}\colon \mathcal{P} \subset_! \mathcal{B}\}$ contains a club, then the family $\{\mathcal{P} \in [\mathcal{T}_X]^{\leqslant\omega}\colon \mathcal{P} \subset_! \mathcal{T}_X\}$ contains a club too.*

Proof. If there exists a base $\mathcal{B}$ of X such that the family $\{\mathcal{P} \in [\mathcal{B}]^{\leqslant\omega}\colon \mathcal{P} \subset_! \mathcal{B}\}$ contains a club, then there exists a strong winning strategy in $\mathcal{B}$. Therefore, player I has winning strategy in the open–open game $G(\mathcal{B})$ (i.e., the open–open game when each player chooses a set from $\mathcal{B}$). This implies that X satisfies the countable chain condition, otherwise the strategy for player II to choose at each stage a nonempty subset of a member of a fixed uncountable maximal disjoint collection of elements of $\mathcal{B}$ is winning (see [2, Theorem 1.1(ii)] for a similar situation). Consequently, every nonempty open subset $G \subset X$ contains a countable disjoint open family whose union is dense in G (just take a maximal disjoint open family in G).

Now, for each element $U \in \mathcal{T}_X \setminus \mathcal{B}$ we assign a countable family $\mathcal{A}_U \subset \mathcal{B}$ of pairwise disjoint open subsets of U such that $\operatorname{cl}\bigcup \mathcal{A}_U = \operatorname{cl} U$. If $U \in \mathcal{B}$, then we assign $\mathcal{A}_U = \{U\}$. Let $\mathcal{C} \subset \{\mathcal{P} \in [\mathcal{B}]^{\leqslant\omega}\colon \mathcal{P} \subset_! \mathcal{B}\}$ be a club. Put

$$\mathcal{C}' = \bigl\{A \cup \mathcal{Q}\colon \mathcal{Q} \in \mathcal{C} \text{ and } A \in [\mathcal{T}_X]^{\leqslant\omega} \text{ with } \mathcal{A}_U \subset \mathcal{Q} \text{ for all } U \in A\bigr\}.$$

First, observe that if $A \cup \mathcal{Q}_A \subset D \cup \mathcal{Q}_D$ and $A \cup \mathcal{Q}_A, D \cup \mathcal{Q}_D \in \mathcal{C}'$, then $\mathcal{Q}_A \subset \mathcal{Q}_D$. Indeed, if $U \in \mathcal{Q}_A \subset \mathcal{B}$ then $U \in D \cup \mathcal{Q}_D$ and $U \in \mathcal{B}$. If $U \in D$, then we get $\{U\} = \mathcal{A}_U \subset \mathcal{Q}_D$ (i.e. $U \in \mathcal{Q}_D$). Therefore, if we have a chain $\{A_n \cup \mathcal{Q}_{A_n}\colon n \in \omega\} \subset \mathcal{C}'$, then

$$\bigcup\{A_n \cup \mathcal{Q}_{A_n}\colon n \in \omega\} = \bigcup_{n\in\omega} A_n \cup \bigcup_{n\in\omega} \mathcal{Q}_{A_n} \in \mathcal{C}'.$$

The absorbing property (i.e. for every $A \in [\mathcal{T}_X]^{\leqslant\omega}$ there is an element $\mathcal{P} \in \mathcal{C}'$ such that $A \subset \mathcal{P}$) for $\mathcal{C}'$ is obvious. So, $\mathcal{C}' \subset [\mathcal{T}_X]^{\leqslant\omega}$ is a club.

It remains to prove that $A \cup \mathcal{Q} \subset_! \mathcal{T}_X$ for every $A \cup \mathcal{Q} \in \mathcal{C}'$. Fix a subfamily $\mathcal{S} \subset A \cup \mathcal{Q}$ and $x \notin \operatorname{cl}\bigcup \mathcal{S}$. Define

$$\mathcal{S}' = \{U \in \mathcal{S}\colon U \in \mathcal{Q}\} \cup \bigcup\{\mathcal{A}_U\colon U \in A\}$$

and note that $\operatorname{cl}\bigcup \mathcal{S} = \operatorname{cl}\bigcup \mathcal{S}'$. The last equality follows from the inclusion $\bigcup \mathcal{S}' \subset \bigcup \mathcal{S}$ and the fact that $\bigcup \mathcal{A}_U$ is dense in U for every $U \in A$. So, if $x \notin \operatorname{cl}\bigcup \mathcal{S}$ then $x \notin \operatorname{cl}\bigcup \mathcal{S}'$. Since $\mathcal{S}' \subset \mathcal{Q} \in \mathcal{C}$ there is $G \in \mathcal{Q}$ such that $x \in G$ and $G \cap \operatorname{cl}\bigcup \mathcal{S}' = \emptyset$. □

If X is a completely regular space, then Σ_X denotes the collection of all co-zero sets in X.

Corollary 2.8. *Let X be a completely regular space and $\mathcal{B} \subset \Sigma_X$ a base for X. If $\{\mathcal{P} \in [\mathcal{B}]^{\leqslant\omega}\colon \mathcal{P} \subset_! \mathcal{B}\}$ contains a club, then the family $\{\mathcal{P} \in [\Sigma_X]^{\leqslant\omega}\colon \mathcal{P} \subset_! \Sigma_X\}$ contains a club too.*

Proof. The proof of previous proposition works in the present situation. The only modification is that for each $U \in \Sigma_X \setminus \mathcal{B}$ we assign a countable family $\mathcal{A}_U \subset \mathcal{B}$ of pairwise disjoint co-zero subsets of U such that $\operatorname{cl}\bigcup \mathcal{A}_U = \operatorname{cl} U$. Such $\mathcal{A}_U$ exists. For example, any maximal disjoint family of elements from $\mathcal{B}$ which are contained in U can serve as $\mathcal{A}_U$. The new club is the family

$$\mathcal{C}' = \bigl\{A \cup \mathcal{Q}\colon \mathcal{Q} \in \mathcal{C} \text{ and } A \in [\Sigma_X]^{\leqslant\omega} \text{ with } \mathcal{A}_U \subset \mathcal{Q} \text{ for all } U \in A\bigr\},$$

where $\mathcal{C} \subset \{\mathcal{P} \in [\mathcal{B}]^{\leqslant\omega}\colon \mathcal{P} \subset_! \mathcal{B}\}$ is a club. □

3. Inverse systems with d-open bounding maps

Recall some facts from [5]. Let $\mathcal{P}$ be an open family in a topological space X and $x, y \in X$. We say that $x \sim_{\mathcal{P}} y$ if and only if $x \in V \Leftrightarrow y \in V$ for every $V \in \mathcal{P}$. The family of all sets $[x]_{\mathcal{P}} = \{y\colon y \sim_{\mathcal{P}} x\}$ is denoted by $X/\mathcal{P}$. There exists a mapping $q : X \to X/\mathcal{P}$ defined by $q[x] = [x]_{\mathcal{P}}$. The set $X/\mathcal{P}$ is equipped with the topology $\mathcal{T}_{\mathcal{P}}$ generated by all images $q(V)$, $V \in \mathcal{P}$.

Lemma 3.1. *([5, Lemma 1]) The mapping $q : X \to X/\mathcal{P}$ is continuous provided $\mathcal{P}$ is an open family X which is closed under finite intersection. Moreover, if $X = \bigcup \mathcal{P}$, then the family $\{q(V)\colon V \in \mathcal{P}\}$ is a base for the topology $\mathcal{T}_{\mathcal{P}}$.* □

Lemma 3.2. *Let a space X be the limit of an inverse system $\{X_\sigma, \pi^\sigma_\varrho, \Sigma\}$ with surjective projections $\pi_\sigma : X \to X_\sigma$. Then the bonding maps π^σ_ϱ are d-open if and only if each π_σ is d-open.*

Proof. Assume all π^σ_ϱ are d-open. We are going to prove that any projection π_ρ is d-open. It suffices to show that $\pi_\rho((\pi_\sigma)^{-1}(U))$ is dense in some open subset of X_ρ for any open $U \subset X_\sigma$, where $\sigma \geqslant \rho$. Since π^σ_ρ is d-open and $\pi_\rho((\pi_\sigma)^{-1}(U)) = \pi^\sigma_\rho(U)$, π_ρ is d-open. Conversely, if the limit projections are d-open, then, by Lemma 2.3, the bonding maps are also d-open. □

Theorem 3.3. *A Hausdorff space X is very* I*-favorable if and only if $X = \text{a-}\varprojlim S$, where $S = \{X_A, q^A_B, \mathcal{C}\}$ is a σ-complete inverse system such that all X_A are (not-necessarily Hausdorff) spaces with countable weight and the bonding maps q^A_B are d-open and onto.*

Proof. Suppose $(X, \mathcal{T})$ is very I-favorable. By Corollary 2.6, there exists a club $\mathcal{C} \subset \{\mathcal{P} \in [\mathcal{T}]^{\leqslant\omega}\colon \mathcal{P} \subset_! \mathcal{T}\}$ satisfying conditions (i)–(iii). For every $A \in \mathcal{C}$ consider the space $X_A = X/A$ and the map $q_A : X \to X_A$. Since each A is a cover of X closed under finite intersections, by Lemma 3.1, q_A is a continuous surjection and $\{q_A(U)\colon U \in A\}$ is a countable base for X_A. Moreover, $q_A^{-1}(q_A(U)) = U$ for all $U \in A$, see [5]. This, according to Remark 2.2, implies that each q_A is d-open (recall that $A \subset_! \mathcal{T}$). If $A, B \in \mathcal{C}$ with $B \subset A$, then there exists a map $q^A_B : X_A \to X_B$ which is continuous because $(q^A_B)^{-1}(q_B(U)) = q_A(U)$ for every $U \in B$. The maps q^A_B are also d-open, see Lemma 3.2. In this way we obtained the inverse system $S = \{X_A, q^A_B, \mathcal{C}\}$

consisting of spaces with countable weight and d-open bonding maps. Since $\mathcal{C}$ is closed under increasing chains, S is σ-complete. It remains to show that the map $h: X \to \varprojlim S$, $h(x) = (q_A(x))_{A\in\mathcal{C}}$, is an embedding. Let $\pi_A: \varprojlim S \to X_A$, $A \in \mathcal{C}$, be the limit projections of S. The family $\{\pi_A^{-1}(q_A(U))\colon U \in A,\ A \in \mathcal{C}\}$ is a base for the topology of $\varprojlim S$. Since $h^{-1}(\pi_A^{-1}(q_A(U))) = U$ for any $U \in A \in \mathcal{C}$, h is continuous and $h(X)$ is dense in $\varprojlim S$. Because $\mathcal{C}$ satisfies condition (ii) (see Corollary 2.6), h is one-to-one. Finally, since $h(U) = h(X) \cap \pi_A^{-1}(q_A(U))$ for any $U \in A \in \mathcal{C}$ (see [5, the proof of Theorem 11]) and $\mathcal{C}$ contains a base for $\mathcal{T}$, h is an embedding.

Suppose now that $X = \text{a-}\varprojlim S$, where $S = \{X_A, q_B^A, \mathcal{C}\}$ is a σ-complete inverse system such that all X_A are spaces with countable weight and the bonding maps q_B^A are d-open and onto. Then, by Lemma 3.2, all limit projections $\pi_A: \varprojlim S \to X_A$, $A \in \mathcal{C}$, are d-open. Since X is dense in $\varprojlim S$, any restriction $q_A = \pi_A|X: X \to X_A$ is also d-open. Moreover, all q_A are surjective (see the definition of a-$\varprojlim$). Then, according to Proposition 2.1, $\{q_A^{-1}(U)\colon U \in \mathcal{T}_A\} \subset_! \mathcal{T}$, where $\mathcal{T}_A$ is the topology of X_A. Consequently, if $\mathcal{B}_A$ is a countable base for $\mathcal{T}_A$, we have $\mathcal{P}_A = \{q_A^{-1}(U)\colon U \in \mathcal{B}_A\} \subset_! \mathcal{T}$. The last relation implies $\mathcal{P}_A \subset_! \mathcal{B}$ with $\mathcal{B} = \bigcup\{\mathcal{P}_A\colon A \in \mathcal{C}\}$ being a base for $\mathcal{T}$. Let us show that $\mathcal{P} = \{\mathcal{P}_A\colon A \in \mathcal{C}\}$ is a club in $\{\mathcal{Q} \in [\mathcal{B}]^{\leqslant\omega}\colon \mathcal{Q} \subset_! \mathcal{B}\}$. Since S is σ-complete, the supremum of any increasing sequence from $\mathcal{C}$ is again in $\mathcal{C}$. This implies that $\mathcal{P}$ is closed under increasing chains. So, it remains to prove that for every countable family $\{U_j\colon j = 1, 2, \ldots\} \subset \mathcal{B}$ there exists $A \in \mathcal{C}$ with $U_j \in \mathcal{P}_A$ for all $j \geqslant 1$. Because every U_j is of the form $q_{A_j}^{-1}(V_j)$ for some $A_j \in \mathcal{C}$ and $V_j \in \mathcal{B}_{A_j}$, there exists $A \in \mathcal{C}$ with $A > A_j$ for each j. It is easily seen that $\mathcal{P}_A$ contains the family $\{U_j\colon j \geqslant 1\}$ for any such A. Therefore, $\mathcal{P}$ is a club in $\{\mathcal{Q} \in [\mathcal{B}]^{\leqslant\omega}\colon \mathcal{Q} \subset_! \mathcal{B}\}$. Finally, according to Proposition 2.7, the family $\{\mathcal{Q} \in [\mathcal{T}]^{\leqslant\omega}\colon \mathcal{Q} \subset_! \mathcal{T}\}$ also contains a club. Hence, X is very I-favorable. □

It follows from Theorem 3.3 that every dense subset of a space from each of the following classes is very I-favorable: products of first countable spaces, κ-metrizable compacta. More generally, by [13, Theorem 2.1(iv)], every space with a lattice of d-open maps is very I-favorable.

The next theorem provides another examples of very I-favorable spaces.

Theorem 3.4. *Let $f: X \xrightarrow{onto} Y$ be a perfect map with X, Y being regular spaces. Then Y is very* I-*favorable, provided so is X.*

Proof. This theorem was established in [2] when X and Y are compact. The same proof works in our more general situation. □

Corollary 3.5. *Every continuous image under a perfect map of a space possessing a lattice of d-open maps is very* I-*favorable.* □

4. Very I-favorable spaces with respect to the co-zero sets

We say that a space X is *very* I-*favorable with respect to the co-zero sets* if there exists a strong winning strategy $\sigma: \bigcup\{\Sigma_X^n\colon n \geqslant 0\} \to \Sigma_X$, where Σ_X denotes the collection of all co-zero sets in X. By Proposition 2.5, this is equivalent to the existence of a club in the family $\{\mathcal{P} \in [\Sigma_X]^{\leqslant\omega}\colon \mathcal{P} \subset_! \Sigma_X\}$.

A completely regular space X is d-*openly generated* if X is the almost limit of a σ-complete inverse system $S = \{X_\sigma, \pi_\varrho^\sigma, \Gamma\}$ consisting of separable metric spaces X_σ and d-open surjective bonding maps π_ϱ^σ.

Theorem 4.1. *A completely regular space X is very* I-*favorable with respect to the co-zero sets if and only if X is* d-*openly generated.*

Proof. Suppose X is very I-favorable with respect to the co-zero sets and $\sigma: \bigcup\{\Sigma_X^n\colon n \geqslant 0\} \to \Sigma_X$ is a strong winning strategy in Σ_X. We place X as a C^*-embedded subset of a Tychonoff cube $\mathbb{I}^A$. If $B \subset A$, let $\pi_B: \mathbb{I}^A \to \mathbb{I}^B$ be the natural projection and p_B be restriction map $\pi_B|X$. Let also $X_B = p_B(X)$. If $U \subset X$ we write $B \in k(U)$ to denote that $p_B^{-1}(p_B(U)) = U$.

Claim 1. *For every $U \in \Sigma_X$ there exists a countable $B_U \subset A$ such that $B_U \in k(U)$ with $p_{B_U}(U)$ being a co-zero set in X_{B_U}.*

For every $U \in \Sigma_X$ there exists a continuous function $f_U: X \to [0, 1]$ with $f_U^{-1}((0, 1]) = U$. Next, extend f_U to a continuous function $g: \mathbb{I}^A \to [0, 1]$ (recall that X is C^*-embedded in $\mathbb{I}^A$). Then, there exists a countable set $B_U \subset A$ and a function $h: \mathbb{I}^{B_U} \to [0, 1]$ with $g = h \circ \pi_{B_U}$. Obviously, $U = p_{B_U}^{-1}(h^{-1}((0, 1]) \cap p_{B_U}(X))$, which completes the proof of the claim.

Let $\mathcal{B} = \{U_\alpha\colon \alpha < \tau\}$ be a base for the topology of X consisting of co-zero sets such that for each α there exists a finite set $H_\alpha \subset A$ with $H_\alpha \in k(U_\alpha)$. For any finite set $C \subset A$ let γ_C be a fixed countable base for X_C.

Claim 2. *For every countable $B \subset A$ there exists a countable set $\Gamma \subset A$ containing B and a countable family $\mathcal{U}_\Gamma \subset \Sigma_X$ satisfying the following conditions*:

(i) *$\mathcal{U}_\Gamma$ is closed under σ and finite intersections*;

(ii) $\Gamma \in k(U)$ *for all* $U \in \mathcal{U}_\Gamma$;
(iii) $\mathcal{B}_\Gamma = \{p_\Gamma(U)\colon U \in \mathcal{U}_\Gamma\}$ *is a base for* $p_\Gamma(X)$.

We construct by induction a sequence $\{C(m)\}_{m\geqslant 0}$ of countable subsets of A, and a sequence $\{\mathcal{V}_m\}_{m\geqslant 0}$ of countable subfamilies of Σ_X such that:

- $C_0 = B$ and $\mathcal{V}_0 = \{p_B^{-1}(V)\colon V \in \mathcal{B}_B\}$, where $\mathcal{B}_B$ is a base for X_B;
- $C(m+1) = C(m) \cup \bigcup\{B_U\colon U \in \mathcal{V}_m\}$;
- $\mathcal{V}_{3m+1} = \mathcal{V}_{3m} \cup \{\sigma(U_1,\ldots,U_n)\colon U_1,\ldots,U_n \in \mathcal{V}_{3m},\ n \geqslant 1\}$;
- $\mathcal{V}_{3m+2} = \mathcal{V}_{3m+1} \cup \bigcup\{p_C^{-1}(\gamma_C)\colon C \subset C(3m+1)$ is finite$\}$;
- $\mathcal{V}_{3m+3} = \mathcal{V}_{3m+2} \cup \{\bigcap_{i=1}^{i=n} U_i\colon U_1,\ldots,U_n \in \mathcal{V}_{3m+2},\ n \geqslant 1\}$.

It is easily seen that the set $\Gamma = \bigcup_{m=0}^{\infty} C_m$ and the family $\mathcal{U}_\Gamma = \bigcup_{m=0}^{\infty} \mathcal{V}_m$ satisfy the conditions (i)–(iii) from Claim 2.

Claim 3. *The map* $p_\Gamma : X \to X_\Gamma$ *is a d-open map.*

It follows from (ii) that $\mathcal{U}_\Gamma = \{p_\Gamma^{-1}(V)\colon V \in \mathcal{B}_\Gamma\}$. According to Lemma 2.4, $\mathcal{U}_\Gamma \subset_! \Sigma_X$. Consequently, $\mathcal{U}_\Gamma \subset_! \mathcal{T}_X$. Therefore, we can apply Proposition 2.1 to conclude that p_Γ is d-open.

Now, consider the family Λ of all $\Gamma \in [A]^{\leqslant\omega}$ such that there exists a countable family $\mathcal{U}_\Gamma \subset \Sigma_X$ satisfying the conditions (i)–(iii) from Claim 2. We consider the inverse system $S = \{X_\Gamma, p_\Theta^\Gamma, \Lambda\}$, where $\Theta \subset \Gamma \in \Lambda$ and $p_\Theta^\Gamma : X_\Gamma \to X_\Theta$ is the restriction of the projection $\pi_\Theta^\Gamma : \mathbb{I}^\Gamma \to \mathbb{I}^\Theta$ on the set X_Γ. Since $p_\Theta = p_\Theta^\Gamma \circ p_\Gamma$ and both p_Γ and p_Θ are d-open surjections, p_Θ^Γ is also d-open (see Lemma 2.3). Moreover, the union of any increasing chain in Λ is again in Λ. So, Λ, equipped the inclusion order, is σ-complete. Finally, by Claim 2, Λ covers the set A. Therefore, the limit of S is a subset of $\mathbb{I}^A$ containing X as a dense subset. Hence, X is d-openly generated.

Suppose that X is d-openly generated. So, $X = \text{a-}\varprojlim S$, where $S = \{X_\sigma, p_\varrho^\sigma, \Gamma\}$ is a σ-complete inverse system consisting of separable metric spaces X_σ and d-open surjective bonding maps p_ϱ^σ. Let $p_\sigma : \varprojlim S \to X_\sigma$, $\sigma \in \Gamma$, be the limit projections and $q_\sigma = p_\sigma | X$. As in the proof of Theorem 3.3, we can show that $\mathcal{P} = \{\mathcal{P}_\sigma\colon \sigma \in \Gamma\}$ is a club in the family $\{\mathcal{Q} \in [\mathcal{B}_X]^{\leqslant\omega}\colon \mathcal{Q} \subset_! \mathcal{B}_X\}$, where $\mathcal{B}_X = \bigcup\{\mathcal{P}_\sigma\colon \sigma \in \Gamma\}$ and $\mathcal{P}_\sigma = \{q_\sigma^{-1}(V)\colon V \in \mathcal{B}_\sigma\}$ with $\mathcal{B}_\sigma$ being a countable base for the topology of X_σ. Since $\mathcal{B}_X$ consists of co-zero sets, by Corollary 2.8, the family $\{\mathcal{Q} \in [\Sigma_X]^{\leqslant\omega}\colon \mathcal{Q} \subset_! \Sigma_X\}$ contains also a club. Hence, X is very I-favorable with respect to the co-zero sets. □

We say that a space $X \subset Y$ is regularly embedded in Y is there exists a function $\mathrm{e} : \mathcal{T}_X \to \mathcal{T}_Y$ satisfying the following conditions for any $U, V \in \mathcal{T}_X$:

- $\mathrm{e}(\emptyset) = \emptyset$;
- $\mathrm{e}(U) \cap X = U$;
- $\mathrm{e}(U) \cap \mathrm{e}(V) = \emptyset$ provided $U \cap V = \emptyset$.

Theorem 4.1 and [13, Theorem 2.1(ii)] yield the following external characterization of very I-favorable spaces with respect to the co-zero sets (I-favorable spaces with respect to the co-zero sets have a similar external characterization, see [14, Theorem 1.1]).

Corollary 4.2. *A completely regular space is very* I*-favorable with respect to the co-zero sets if and only if every* C^**-embedding of* X *in any Tychonoff space* Y *is regular.*

The next corollary provides an answer of a question from [14] whether there exists a characterization of κ-metrizable compacta in terms a game between two players.

Corollary 4.3. *A compact Hausdorff space is very* I*-favorable with respect to the co-zero sets if and only if* X *is* κ*-metrizable.*

Proof. A compact Hausdorff space is κ-metrizable spaces iff X is the limit space of a σ-complete inverse system consisting of compact metric spaces and open surjective bonding maps, see [11] and [10]. Since every d-open surjective map between compact Hausdorff spaces is open, this corollary follows from Theorem 4.1. □

Recall that a normal space is called perfectly normal if every open set is a co-zero set. So, any perfectly normal spaces is very I-favorable if and only if it is very I-favorable with respect to the co-zero sets. Thus, we have the next corollary.

Corollary 4.4. *Every perfectly normal very* I*-favorable space is d-openly generated.*

Lemma 4.5. *Let $(X, \mathcal{T})$ be a completely regular space. If there is a strong winning strategy $\sigma' : \bigcup\{\mathcal{T}^n\colon n \geqslant 0\} \to \mathcal{T}$, then there is a strong winning strategy $\sigma : \bigcup\{\mathcal{R}^n\colon n \geqslant 0\} \to \mathcal{R}$, where $\mathcal{R}$ consists of all regular open subset of X.*

Proof. Assume that $\sigma' : \bigcup\{\mathcal{T}^n\colon n \geqslant 0\} \to \mathcal{T}$ is a strong winning strategy. We define a strong winning strategy on $\mathcal{R}$. Let $\sigma(\varnothing) = \operatorname{Int}\operatorname{cl}\sigma'(\varnothing)$. We define by induction $\sigma((V_0, V_1, \ldots, V_k))$, $V_{k+1} \subset \sigma((V_0, V_1, \ldots, V_k))$, by

$$\sigma\big((V_0, V_1, \ldots, V_{n+1})\big) = \operatorname{Int}\operatorname{cl}\sigma'\big(\big(V'_0, V'_1, \ldots, V'_{n+1}\big)\big),$$

where $V'_{k+1} = V_{k+1} \cap \sigma'((V'_0, V'_1, \ldots, V'_k))$.

Let us show that $\mathcal{F} = \{V_n\colon n \in \omega\} \cup \{\sigma((V_0, V_1, \ldots, V_{n+1}))\colon n \in \omega\} \subset_! \mathcal{R}$. If $\mathcal{S} \subset \mathcal{F}$ and $x \notin \operatorname{cl}\bigcup\mathcal{S}$, let

$$\mathcal{F}' = \big\{V'_n\colon n \in \omega\big\} \cup \big\{\sigma'\big(\big(V'_0, V'_1, \ldots, V'_{n+1}\big)\big)\colon n \in \omega\big\}$$

and

$$\mathcal{S}' = \big\{W' \in \mathcal{F}'\colon W \in \mathcal{S}\big\}.$$

Note that $\bigcup\mathcal{S}' \subset \bigcup\mathcal{S}$, hence $x \notin \operatorname{cl}\bigcup\mathcal{S}'$. So, there is $W' \in \mathcal{S}'$ such that $W' \cap U' = \emptyset$ for all $U' \in \mathcal{F}'$. Assume that $W' = V_{k+1} \cap \sigma'((V'_0, V'_1, \ldots, V'_k))$ and $U' = V_{i+1} \cap \sigma'((V'_0, V'_1, \ldots, V'_i))$. Then we infer that

$$V_{k+1} \cap \operatorname{Int}\operatorname{cl}\sigma'\big(\big(V'_0, V'_1, \ldots, V'_k\big)\big) \cap V_{i+1} \cap \operatorname{Int}\operatorname{cl}\sigma'\big(\big(V'_0, V'_1, \ldots, V'_i\big)\big) = \emptyset.$$

Since $V_{k+1} \subset \sigma((V_0, V_1, \ldots, V_k)) = \operatorname{Int}\operatorname{cl}\sigma'((V'_0, V'_1, \ldots, V'_k))$ and $V_{i+1} \subset \sigma((V_0, V_1, \ldots, V_i)) = \operatorname{Int}\operatorname{cl}\sigma'((V'_0, V'_1, \ldots, V'_i))$, we get $V_{k+1} \cap V_{i+1} = \emptyset$. Suppose $W' = V_{k+1} \cap \sigma'((V'_0, V'_1, \ldots, V'_k))$ and $U' = \sigma'((V'_0, V'_1, \ldots, V'_i))$. Then

$$V_{k+1} \cap \operatorname{Int}\operatorname{cl}\sigma'\big(\big(V'_0, V'_1, \ldots, V'_k\big)\big) \cap \operatorname{Int}\operatorname{cl}\sigma'\big(\big(V'_0, V'_1, \ldots, V'_i\big)\big) = \emptyset.$$

So, $W \cap U = \emptyset$. Similarly, we obtain $W \cap U = \emptyset$ if $W' = \sigma'((V'_0, V'_1, \ldots, V'_k))$ and $U' = \sigma'((V'_0, V'_1, \ldots, V'_i))$. This completes the proof. □

We say that a topological space X is *perfectly κ-normal* if for every open and disjoint subset U, V there are open F_σ subset W_U, W_V with $W_U \cap W_V = \emptyset$ and $U \subset W_U$ and $V \subset W_V$. It is clear that a space X is perfectly κ-normal if and only if that each regular open set in X is F_σ.

Proposition 4.6. *If a normal perfectly κ-normal space is a continuous image of a very* I*-favorable space under a perfect map, then X is d-openly generated.*

Proof. Every open F_σ-subset of a normal space is a co-zero set, see [3]. So, every regular open subset of a normal and perfectly κ-normal space is a co-zero set. Consequently, if X is the image of very I-favorable space and X is normal and perfectly κ-normal, then X is very I-favorable (see Theorem 3.4). Hence, according to Lemma 4.5, X is a very I-favorable with respect to the co-zero sets. Finally, Theorem 4.1 implies that X is d-openly generated. □

Corollary 4.7. *If the image of a compact Hausdorff very* I*-favorable space under a continuous map is perfectly κ-normal, then X is κ-metrizable.*

Corollary 4.7 implies the following result of Shchepin [11, Theorem 18] which has been proved by different methods: If the image of a κ-metrizable compact Hausdorff space X under a continuous map is perfectly κ-normal, then X is κ-metrizable too.

Let us also mention that, according to Shapiro's result [9], continuous images of κ-metrizable compacta have special spectral representations. This result implies that any such an image is I-favorable.

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