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On the minimum rank of the join of graphs and decomposable graphs

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Abstract

For a given undirected graph G, the minimum rank of G is defined to be the smallest possible rank over all real symmetric matrices A whose (i, j)th entry is nonzero whenever $i \neq j$ and $\{i, j\}$ is an edge in G. In this work we consider joins and unions of graphs, and characterize the minimum rank of such graphs in the case of 'balanced inertia'. Several consequences are provided for decomposable graphs, also known as cographs.

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1. Introduction

We are interested in studying the general issue of the ranks of all symmetric matrices associated to a fixed graph.

All matrices are considered real, and all graphs are simple, i.e., no loops or multiple edges. If $A \in M_n$ is a fixed symmetric matrix, then the *graph of* A, denoted by $G(A)$, has vertex set $\{1, \dots, n\}$ and edges consisting of the unordered pairs $\{i, j\}$ such that $a_{ij} \neq 0$ with $i \neq j$. Graphs

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G of the form $G = G(A)$ do not have loops or multiple edges, and the diagonal of A is ignored in the determination of $G(A)$. For a given graph $G = (V, E)$, we let $S(G) = \{A \mid G(A) = G\}$.

Suppose that G is a graph on n vertices. Then the *minimum rank of* G is defined to be

$$\mathrm{mr}(G) = \min\{\mathrm{rank}\ A : G(A) = G\}.$$

It is not difficult to verify that $\mathrm{mr}(G) = n - M(G)$, where $M(G)$ is the *maximum multiplicity of* G, and is defined to be

$$M(G) = \max\{\mathrm{mult}_A(\lambda) : \lambda \in \sigma(A) \text{ and } G(A) = G\}.$$

Here $\sigma(A)$ denotes the spectrum of A and $\mathrm{mult}_A(\lambda)$ is the multiplicity of $\lambda \in \sigma(A)$. For a fixed $m \times n$ matrix A, $R(A)$ denotes the range of A. The complement of any graph G will be denoted by $\overline{G}$.

An interesting and still unresolved problem is to characterize $\mathrm{mr}(G)$ for a given graph G. Naturally, there have been numerous results, which take on many different forms. For more information consult [6], where it is proved that $\mathrm{mr}(G) = n - 1$ if and only if G is the path on n vertices (P_n); and [3], where all graphs on n vertices that satisfy $\mathrm{mr}(G) = 2$ are characterized. Related results for general trees can be found in [9], and for unicyclic graphs see [1,2].

We close this introductory section with a key definition and an outline of the paper.

For a given graph G, we call a matrix A an *optimal matrix for* G, if $A \in S(G)$ and $\mathrm{rank}(A) = \mathrm{mr}(G)$.

The remainder of this paper is divided up as follows. Section 2 develops some relevant terminology and key results from indefinite inner product spaces, which serve as foundations for the next topic in Section 3 called inertia-balanced graphs. In Section 3 we establish our main observations on the minimum rank of the join of an arbitrary number of inertia-balanced graphs. In Section 4, we specialize to the case of decomposable graphs and characterize the minimum rank of decomposable graphs.

2. Indefinite inner products and the Rotation Lemma

Let H be a fixed $k \times k$ nonsingular symmetric matrix. The function $[\cdot, \cdot]_H$ from $\mathbb{R}^k \times \mathbb{R}^k$ to $\mathbb{R}$, defined by $[\mathbf{x}, \mathbf{y}]_H = \mathbf{y}^{\mathrm{T}} H\mathbf{x}$, is called a *nondegenerate inner product* on $\mathbb{R}^k$. For a given subspace $\mathscr{W} \subseteq \mathbb{R}^k$, we define

$$\mathscr{W}^{\perp} = \{\mathbf{x} \in \mathbb{R}^k \mid [\mathbf{x}, \mathbf{y}]_H = 0 \text{ for all } \mathbf{y} \in \mathscr{W}\}.$$

The following is a basic result along these lines.

Theorem 2.1 [11, §2.1]. *For each* $\mathscr{W} \subseteq \mathbb{R}^k$,

(i) $\dim \mathscr{W} + \dim \mathscr{W}^{\perp} = k$;
(ii) $\mathbb{R}^k = \mathscr{W} \oplus \mathscr{W}^{\perp}$ *if and only if* $\mathscr{W} \cap \mathscr{W}^{\perp} = 0$.

We say that a subspace $\mathscr{W}$ is H-*positive* if the restriction of the quadratic form $\mathbf{x}^{\mathrm{T}} H\mathbf{x}$ on $\mathscr{W}$ is positive definite. Subspaces called H-*negative*, H-*nonnegative*, H-*nonpositive*, and H-*indefinite* are defined in a similar way. The nondegenerate inner product itself is said to be *indefinite* if $\mathbb{R}^k$ is an H-indefinite (trivial) subspace, namely, if there exists $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$ with $[\mathbf{x}, \mathbf{x}]_H > 0$, $[\mathbf{y}, \mathbf{y}]_H < 0$.

Theorem 2.2 [5, pp. 24–26; 11, §2.1]. *It is always possible to write* $\mathbb{R}^k = \mathscr{W} \oplus \mathscr{W}^{\perp}$, *where* $\mathscr{W}$ *is* H*-positive, and* $\mathscr{W}^{\perp}$ *is* H*-negative. In addition,* $\dim \mathscr{W} = i_+(H)$, *the number of positive eigenvalues of* H.

A $k \times k$ matrix P is called H*-unitary* (or H*-orthogonal*) if $P^{\mathrm{T}}HP = H$. Naturally, H-unitary matrices are associated to the so-called H*-isometries*, since $[P\mathbf{x}, P\mathbf{y}]_H = [\mathbf{x}, \mathbf{y}]_H$ for all $\mathbf{x}, \mathbf{y}$.

We now restrict our discussion to the nondegenerate inner product on $\mathbb{R}^k$ defined by the symmetric matrix

$$H_k = \begin{bmatrix} I_{\lceil k/2 \rceil}; & 0 \\ 0 & -I_{\lfloor k/2 \rfloor} \end{bmatrix}.$$

The nondegenerate inner product defined by $[\mathbf{x}, \mathbf{y}]_{H_k}$ will simply be denoted by $[\mathbf{x}, \mathbf{y}]_k$, or even shortened to $[\mathbf{x}, \mathbf{y}]$, when there is no risk of confusion. For two matrices M, N, both having k rows, we define $[M, N]_k = N^{\mathrm{T}} H_k M$. If $\mathbf{v}$ is a fixed nonzero vector, we let $\langle \mathbf{v} \rangle$ denote the subspace spanned by $\mathbf{v}$.

The following is a key technical result which will be used in the next section.

Rotation Lemma 2.3. *Let each of the matrices* $M_1, \ldots, M_r$ *have* $k \geqslant 3$ *rows and no zero columns. Then there exist* H_k*-unitary matrices* $P_1, \ldots, P_r$ *such that, for each distinct* i, j *in* $\{1, \ldots, r\}$, *the matrix* $[P_i M_i, P_j M_j]_k$ *has no zero entries.*

Proof. Let $M_1 = [\mathbf{w}_1 \cdots \mathbf{w}_s]$, $M_2 = [\mathbf{w}_{s+1} \cdots \mathbf{w}_{s+t}]$. Since $\mathbf{w}_i \neq 0$ for each i, by Theorem 2.1 we have $\dim \langle \mathbf{w}_i \rangle^{\perp} = k - 1$. Let $\mathscr{C} = \bigcup_1^{s+t} \langle \mathbf{w}_i \rangle^{\perp}$. Since $\mathscr{C}$ has no interior, we can find $\mathbf{u} \in \mathbb{R} \setminus \mathscr{C}$ with $[\mathbf{u}, \mathbf{u}] > 0$. Again by Theorem 2.1, we can write $\mathbb{R}^k = \langle \mathbf{u} \rangle \oplus \langle \mathbf{u} \rangle^{\perp}$. In particular, we can complete $\{\mathbf{u}\}$ to an orthogonal basis $\mathscr{B}$ of $\mathbb{R}^k$. By Theorem 2.2, or by Sylvester's Inertia Law (see [8, Theorem 4.5.8]), since $i_+(H_k) \geqslant 2$, there must be $\mathbf{v} \in \mathscr{B}$ with $[\mathbf{v}, \mathbf{v}] > 0$. Let $\pi = \mathrm{Span}\{\mathbf{u}, \mathbf{v}\}$, thus $\pi^{\perp} = \mathrm{Span}(\mathscr{B} \setminus \{\mathbf{u}, \mathbf{v}\})$, and $\mathbb{R}^k = \pi \oplus \pi^{\perp}$.

For each i, write $\mathbf{w}_i = \mathbf{w}_i' + \mathbf{w}_i''$, where $\mathbf{w}_i' \in \pi$, $\mathbf{w}_i'' \in \pi^{\perp}$. Note that, for each i, $\mathbf{w}_i' \neq 0$, since $[\mathbf{w}_i, \mathbf{u}] \neq 0$, by the definition of $\mathscr{C}$. Since π is a positive subspace, there exist H-isometries on π with no fixed points. In addition, for any $\varepsilon > 0$, we can find an H-isometry ϕ on π such that,

$$0 < |[\mathbf{w}_i', \phi(\mathbf{w}_j')] - [\mathbf{w}_i', \mathbf{w}_j']| < \varepsilon \tag{1}$$

for each $i \leqslant s$, $j > s$. In particular, let

$$\varepsilon = \min\{|[\mathbf{w}_i, \mathbf{w}_j]| : i \leqslant s < j, [\mathbf{w}_i, \mathbf{w}_j] \neq 0\}$$

(if the previous set is empty, let $\varepsilon = 1$). Extend ϕ to an H_k-isometry on $\mathbb{R}^k$ by requiring ϕ to act as the identity on $\pi^{\perp}$. Let P_2 be the standard matrix of ϕ. Clearly P_2 is H_k-unitary. Note that

$$[\mathbf{w}_i, P_2\mathbf{w}_j] - [\mathbf{w}_i, \mathbf{w}_j] = [\mathbf{w}_i', P_2\mathbf{w}_j'] - [\mathbf{w}_i', \mathbf{w}_j']. \tag{2}$$

Let $i \leqslant s < j$. By (1) and (2) we have $0 < |[\mathbf{w}_i, P_2\mathbf{w}_j] - [\mathbf{w}_i, \mathbf{w}_j]| < \varepsilon$. Since either $[\mathbf{w}_i, \mathbf{w}_j] = 0$ or $|[\mathbf{w}_i, \mathbf{w}_j]| \geqslant \varepsilon$, we conclude $[\mathbf{w}_i, P_2\mathbf{w}_j] \neq 0$, for all $i \leqslant s < j$, that is, $[M_1, P_2M_2]_k$ has no zero entries.

We now repeat the same process on the two matrices $M_1' = [M_1 \quad P_2M_2]$ and M_3, determining a H_k-unitary matrix P_3 such that $[M_1', P_3M_3]_k$ has no zero entries; that is, both $[M_1, P_3M_3]_k$ and $[P_2M_2, P_3M_3]_k$ have no zero entries. We can proceed in the same manner with $M_4, \ldots, M_r$, and by setting $P_1 = I_k$, the proof is complete. $\square$

3. Inertia-balanced graphs

The relative position of high multiplicity eigenvalues has been of interest previously (see [10]). We are also concerned with this notion, particularly with the eigenvalue 0. To this end, we say that a symmetric matrix is said to have *balanced inertia* (or to be *inertia-balanced*) if $i_-(A) \leqslant i_+(A) \leqslant i_-(A) + 1$. Recall that for a fixed symmetric matrix X, the *inertia of* X is the triple of numbers $(i_-(X), i_0(X), i_+(X))$, consisting of the number of negative, zero, and positive eigenvalues of X, respectively. Similarly, a graph G is said to be *inertia-balanced* if there exists an optimal matrix for G having balanced inertia. In other words a graph is inertia-balanced if there exists a matrix $A \in S(G)$ such that $\operatorname{rank}(A) = \operatorname{mr}(G)$ and A has balanced inertia.

Note that K_n, the complete graph on n vertices, is also inertia-balanced, since J, the matrix of all ones, is both optimal for K_n and has balanced inertia. In Theorem 3.3 we prove that all trees are inertia-balanced.

Proposition 3.1. *Let A be an $n \times n$ symmetric matrix of rank k. Then A is inertia-balanced if and only if $A = [M, M]_k$ for some $k \times n$ matrix M.*

Proof. If A is inertia-balanced, by Sylvester's Inertia Law, we can write $A = S^{\mathrm{T}} HS$ where

$$H = \begin{bmatrix} I_{\lceil k/2 \rceil} & 0 & 0 \\ 0 & -I_{\lfloor k/2 \rfloor} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It suffices to define M as the submatrix of S consisting of the first k rows, to obtain $A = [M, M]_k$. Conversely, if $A = M^{\mathrm{T}} H_k M$ and rank $A = k$, then M has linearly independent rows. Complete M to a nonsingular $n \times n$ matrix $\begin{bmatrix} M \\ N \end{bmatrix}$. Then

$$A = \begin{bmatrix} M^{\mathrm{T}} & N^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} H_k & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix}.$$

By applying Sylvester's Inertia Law a second time, we arrive at the desired conclusion. □

We will now discuss the behavior of balanced inertia in the case of union and join of graphs. Recall that, if G_1 and G_2 are disjoint graphs, the *union* and the *join* of G_1 and G_2, denoted respectively by $G_1 \cup G_2$ and $G_1 \vee G_2$, are the graphs defined by

$$\begin{aligned} V(G_1 \cup G_2) &= V(G_1 \vee G_2) = V(G_1) \cup V(G_2); \\ E(G_1 \cup G_2) &= E(G_1) \cup E(G_2); \\ E(G_1 \vee G_2) &= E(G_1) \cup E(G_2) \cup E, \end{aligned}$$

where E consists of all the edges (u, v) with $u \in V(G_1)$, $v \in V(G_2)$. A union or a join of r graphs is defined inductively by

$$\bigcup_{i=1}^{r} G_i = \left(\bigcup_{i=1}^{r-1} G_i \right) \cup G_r, \qquad \bigvee_{i=1}^{r} G_i = \left(\bigvee_{i=1}^{r-1} G_i \right) \vee G_r.$$

Matrices with graph $\bigcup_{i=1}^r G_i$ or $\bigvee_{i=1}^r G_i$, can be written in the form

$$\begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_r \end{bmatrix}, \qquad \begin{bmatrix} A_1 & C_{1,2} & \cdots & C_{1,r} \\ C_{1,2}^{\mathrm{T}} & A_2 & \cdots & C_{2,r} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1,r}^{\mathrm{T}} & C_{2,r}^{\mathrm{T}} & \cdots & A_r \end{bmatrix},$$

respectively, where, for each i, j, $G(A_i) = G_i$, while $C_{i,j}$ has no zero entries. The matrix above on the left will also be denoted by $\bigoplus_{i=1}^r A_i$.

Proposition 3.2. *Let $G = \bigcup_{i=1}^r G_i$, $r > 1$, where each G_i is inertia-balanced. Then G is inertia-balanced, and* $\mathrm{mr}(G) = \sum_{i=1}^r \mathrm{mr}(G_i)$.

Proof. Let $k = \mathrm{mr}(G)$, $k_i = \mathrm{mr}(G_i)$, $i = 1, \ldots, r$. It is well-known that $k = \sum_{i=1}^r k_i$. By reordering the G_i's, we can assume that $k_1, \ldots, k_s$ are odd, while $k_{s+1}, \ldots, k_r$ are even, for some $s \geqslant 0$. In addition, for each i, let A_i be an optimal inertia-balanced matrix for G_i. In particular, we can write $A_i = [M_i, M_i]_{k_i}$. We claim that $A = \bigoplus_{i=1}^r (-1)^{i-1} A_i$ is an optimal inertia-balanced matrix for G. Indeed, we see that rank $A = \mathrm{mr}(G)$, and $G(A) = G$. Concerning inertia, let $M = \bigoplus_{i=1}^r M_i$, and $D = \bigoplus_{i=1}^r (-1)^{i-1} H_{k_i}$. We clearly have $A = M^{\mathrm{T}} D M$. Since $D = P^{\mathrm{T}} H_k P$ for a suitable permutation matrix P, we finally obtain $A = [PM, PM]_k$, that is, A has balanced inertia. □

We are now in a position to establish that all trees are inertia-balanced.

Theorem 3.3. *Each acyclic graph is inertia-balanced.*

Proof. By Proposition 3.2 it is enough to consider the connected case. First note that, for any n, the path on n vertices P_n is inertia-balanced. Indeed, if A is a matrix with graph P_n, then A has n distinct eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_n$. We can easily see that the matrix $A' = A - \lambda_{\lceil n/2 \rceil} I_n$ has rank $n-1$ and is inertia balanced. Since $\mathrm{mr}(P_n) = n-1$, A' is an optimal inertia-balanced matrix for P_n.

Let T be a tree on n vertices. As shown in [9], we can determine a suitable number of vertices $v_1, \ldots, v_q \in V(T)$ such that the graph $\widetilde{T}$ obtained from T by removing $v_1, \ldots, v_q$ is the union of p disjoint paths, where $p - q = M(T)$. Note that

$$|\widetilde{T}| = n - q, \qquad M(\widetilde{T}) = p, \qquad \mathrm{mr}(\widetilde{T}) = n - p - q.$$

In particular, $\widetilde{T}$ is inertia-balanced by Proposition 3.2. Let $\tilde{A}$ be an optimal inertia-balanced matrix for $\widetilde{T}$. To the matrix $\widetilde{A}$, we append q rows and q columns, in such a way that we obtain a matrix A with graph T. We then have

$$\operatorname{rank} A \leqslant \operatorname{rank} \widetilde{A} + 2q = n - p + q = n - M(T) = \mathrm{mr}(T),$$

so that A is optimal for T. By applying the Cauchy interlacing inequalities, we deduce that A is also inertia-balanced. Thus T is inertia balanced, as desired. □

If $G = \bigcup_{i=1}^r G_i$, where each G_i is connected, the subgraph $\breve{G} = \bigcup_{|G_i|>1} G_i$ is called the *core* of G, while $\ddot{G} = \bigcup_{|G_i|=1} G_i$ is called the *isolated part* of G. Note that, if $r = 1$, i.e., G is connected, then $G = \ddot{G}$ if and only if $|G| = 1$.

It is also immediate to deduce the following result.

Corollary 3.4. *For each graph* G, $\mathrm{mr}(G) = \mathrm{mr}(\breve{G})$. *Moreover*, G *is inertia-balanced if and only if* $\breve{G}$ *is inertia-balanced*.

We now consider join of graphs. To this end, we define the *join minimum rank* of G as $\mathrm{jmr}(G) = \mathrm{mr}(K_1 \vee G)$. In order to study the relationship between $\mathrm{mr}(G)$ and $\mathrm{jmr}(G)$, we need to recall the following result on the minimum rank of a graph after deleting a vertex.

Lemma 3.5 [1, Lemma 2.2]. *Let* G *be the graph obtained from a graph* G' *by removing a vertex* v (*say* $v = 1$) *and all the edges incident to* v, *and let*

$$\mathscr{M} = \left\{ B \mid B = \begin{bmatrix} c & \mathbf{b}^{\mathrm{T}} \\ \mathbf{b} & A \end{bmatrix}; G(B) = G';\ \mathbf{b} \in R(A) \right\}.$$

Then

(i) $\mathrm{mr}(G') = \mathrm{mr}(G)$ *if and only if* $\min_{B\in\mathscr{M}}\{\mathrm{rank}\, A\} = \mathrm{mr}(G)$;
(ii) $\mathrm{mr}(G') = \mathrm{mr}(G) + 1$ *if and only if* $\min_{B\in\mathscr{M}}\{\mathrm{rank}\, A\} = \mathrm{mr}(G) + 1$;
(iii) $\mathrm{mr}(G') = \mathrm{mr}(G) + 2$ *otherwise*.

Proposition 3.6. *For any graph* G

$$\mathrm{jmr}(G) = \begin{cases} \mathrm{mr}(G) & \textit{if and only if } |\ddot{G}| = 0, \\ \mathrm{mr}(G) + 1 & \textit{if and only if } |\ddot{G}| = 1, \\ \mathrm{mr}(G) + 2 & \textit{if and only if } |\ddot{G}| \geqslant 2. \end{cases}$$

Proof. As noted in the proof of Corollary 3.4, the optimal matrices for G are exactly those matrices obtained by bordering with zero rows and columns any optimal matrix for $\breve{G}$. We note further that, given a symmetric matrix A, there exist vectors $\mathbf{b} \in R(A)$ with no zero components if and only if A has no zero rows. Therefore, if in Lemma 3.5 we consider $G' = K_1 \vee G$, and we define $\mathscr{N} = \{A \mid G(A) = G,\ A \text{ has no zero rows}\}$, we then have

$$\min_{B\in\mathscr{M}}\{\mathrm{rank}\, A\} = \min_{A\in\mathscr{N}}\{\mathrm{rank}\, A\} = \mathrm{mr}(G) + |\ddot{G}|.$$

By applying Lemma 3.5, the proof is complete. □

In order to prove our main result on the minimum rank of a join of graphs, we also need the following fact.

Lemma 3.7. *Let* G *be an inertia-balanced graph on* n *vertices*. *Then*, *for any* $m \geqslant \mathrm{jmr}(G)$, *there exists an* $m \times n$ *matrix* M *with no zero columns such that the matrix* $A = [M, M]_m$ *has graph* G.

Proof. Let $k = \mathrm{mr}(G)$, $l = \mathrm{jmr}(G)$. By Corollary 3.4, $\breve{G}$ is inertia-balanced. Let $\breve{A}$ be an optimal inertia-balanced matrix for $\breve{G}$ (if $\breve{G} = \emptyset$, let $\breve{A}$ be the 0×0 empty matrix). By Proposition 3.1, we can write $\breve{A} = [\breve{M}, \breve{M}]_k$ for some matrix $\breve{M}$. Furthermore, since $\breve{A}$ has no zero rows and columns, necessarily $\breve{M}$ has no zero columns. Write $\breve{M} = \left[\begin{smallmatrix} B \\ C \end{smallmatrix}\right]$ where B has $\lceil k/2 \rceil$ rows. In particular, $\breve{A} = B^{\mathrm{T}}B - C^{\mathrm{T}}C$.

Case I: $l = k$. By Proposition 3.6, we have $G = \breve{G}$. Define

$$M = \begin{bmatrix} B \\ 0_{m-l} \\ C \end{bmatrix},$$

where 0_{m-l} is a zero matrix with $m - l$ rows and a suitable number of columns. If we set $A = [M, M]_m$, an easy check shows that $A = M^{\mathrm{T}} H_m M = B^{\mathrm{T}} B - C^{\mathrm{T}} C = \breve{A}$, so that $G(A) = G$.

Case II: $l = k + 1$. By Proposition 3.6, we have $G = \breve{G} \cup K_1$. Define

$$M = \begin{bmatrix} B & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & 1 \\ 0_{m-l} & \mathbf{0} \\ C & \mathbf{0} \end{bmatrix}.$$

Now $A = M^{\mathrm{T}} H_m M = \left[\begin{smallmatrix} \breve{A} & 0 \\ 0 & \delta \end{smallmatrix}\right]$, where $\delta = -1$ if k is odd and $m = k + 1$, $\delta = +1$ otherwise. In either case $G(A) = G$.

Case III: $l = k + 2$. We now have $G = \breve{G} \cup (\bigcup_1^r K_1)$, for some $r \geqslant 2$. Define

$$M = \begin{bmatrix} B & 0_{\lceil k/2 \rceil} \\ \mathbf{0}^{\mathrm{T}} & \mathbf{1}^{\mathrm{T}} \\ 0_{m-l} & 0_{m-l} \\ \mathbf{0}^{\mathrm{T}} & \mathbf{1}^{\mathrm{T}} \\ C & 0_{\lfloor k/2 \rfloor} \end{bmatrix},$$

where $\mathbf{1}$ is the vector in $\mathbb{R}^r$ all of whose entries are equal to 1. Then $A = M^{\mathrm{T}} H_m M = \left[\begin{smallmatrix} \breve{A} & 0 \\ 0 & 0 \end{smallmatrix}\right]$, and again $G(A) = G$. □

Theorem 3.8. *Let $G = \bigvee_{i=1}^r G_i$, $r > 1$, where each G_i is inertia-balanced. If* $\max\{\mathrm{jmr}(G_i)\} \geqslant 3$, *then G is inertia-balanced and* $\mathrm{mr}(G) = \max\{\mathrm{jmr}(G_i)\}$.

Proof. Let $m = \max\{\mathrm{jmr}(G_i)\}$. We first prove that $\mathrm{mr}(G) \geqslant m$. Indeed, assume that the maximum of $\mathrm{jmr}(G_i)$ is attained at G_1. Let G' be the subgraph induced by G_1 and by any other vertex v in $G \setminus G_1$. Then $\mathrm{mr}(G) \geqslant \mathrm{mr}(G') = \mathrm{mr}(G_1 \vee K_1) = \mathrm{jmr}(G_1) = m$. To prove the opposite inequality, by virtue of Lemma 3.7, we can construct matrices $M_1, \ldots, M_r$ with m rows and no zero columns such that, for each i, $A_i = [M_i, M_i]_m$ has graph G_i. By applying the Rotation Lemma, there exist H_m-unitary matrices $P_1, \ldots, P_r$ such that, for $i \neq j$, $[P_i M_i, P_j M_j]_m$ has no zero entries. Define $M = [P_1 M_1 P_2 M_2 \cdots P_r M_r]$, and $A = [M, M]_m$. Clearly $\operatorname{rank} A \leqslant m$. We now verify that $G(A) = G$. Indeed

$$A = \begin{bmatrix} A_1 & [P_2 M_2, P_1 M_1]_m & \cdots & [P_r M_r, P_1 M_1]_m \\ [P_1 M_1, P_2 M_2]_m & A_2 & \cdots & [P_r M_r, P_2 M_2]_m \\ \vdots & \vdots & \ddots & \vdots \\ [P_1 M_1, P_r M_r]_m & [P_2 M_2, P_r M_r]_m & \cdots & A_r \end{bmatrix},$$

where $G(A_i) = G_i$, while the off diagonal blocks have no zero entries. Hence it follows that $G(A) = G$. □

Theorem 3.8 characterizes the minimum rank of a join of inertia-balanced graphs under the condition that $\mathrm{jmr}(G_i) \geqslant 3$ for at least one i. Such a condition is essential, as, for the inertia-balanced graph $G = K_{3,3,3} = \overline{K}_3 \vee \overline{K}_3 \vee \overline{K}_3$, we have $3 = \mathrm{mr}(G) \neq \max\{\mathrm{jmr}(G_i)\} = 2$. We will be able to provide a complete description of the minimum rank for the join of inertia-balanced graphs in Corollary 4.8.

We close this section with some remarks on the existence of non-inertia balanced graphs. Currently, we have not constructed a graph that does not have the property of being inertia-balanced. Such an issue is of interest to the authors, and will be a topic considered in a subsequent paper.

4. Decomposable graphs

A graph is said to be *decomposable* if it can be expressed as a sequence of joins and unions of isolated vertices (see [13]). An example of a decomposable graph is given in Fig. 1. Here we write $k \cup H$ or $k \vee H$ to mean the graph obtained from H by either a union of H and an isolated vertex (labeled k) or H joined to an isolated vertex (labeled k). For the graph G in Fig. 1 we can write, $G = (1 \cup 2) \vee ((4 \vee (3 \cup 5)) \cup (6 \vee 7))$.

Recently, Royle [14] has worked out the rank of the adjacency matrix for a decomposable graph. We note here that Royle and others call decomposable graphs *cographs*.

To each decomposable graph, we can associate a root tree called the *composition tree*, as presented in [7]. Such a tree depends on the order in which the operations of join and union are used to build the graph. Fig. 2 shows the composition tree of the graph in Fig. 1.

The upper vertex in the composition tree is called the *root*. Note that for a connected decomposable graph, the root will always be a join. The *height*, $h(G)$, of a decomposable graph G is the number of edges of a longest path having the root as an endpoint. In the previous example $h(G) = 4$, obtained by joining the root to either vertex 3 or vertex 5.

When writing $G = \bigvee_{i=1}^{r} G_i$, we will assume that none of the G_i's can be further decomposed as a join of proper subgraphs. The G_i's will be called the *primary constituents* of G. With regard to the graph in Fig. 1, the primary constituents are $1 \cup 2$ and $(4 \vee (3 \cup 5)) \cup (6 \vee 7)$. Similarly, when writing $G = \bigcup_{i=1}^{r} G_i$, we will assume that each G_i is connected. In this case the G_i's are called the *components* of G. In particular, if $G = \bigvee_{i=1}^{r} G_i$ is a decomposable graph, then each primary constituent is either K_1 or is the union of two or more components, which will be called the *secondary constituents* of G. Again, referring to Fig. 1, the secondary constituents are 1, 2, $4 \vee (3 \cup 5)$, and $6 \vee 7$.

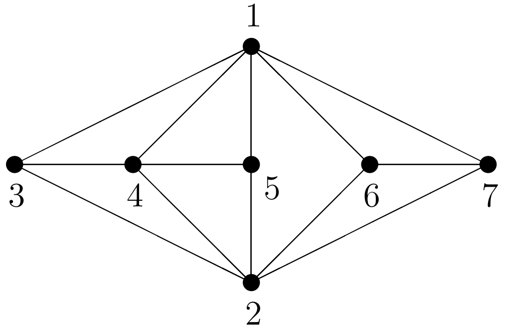

Fig. 1. Decomposable graph G.

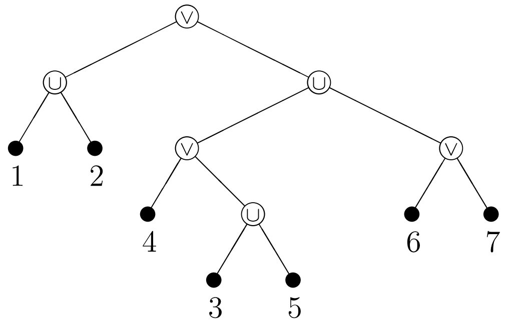

Fig. 2. Composition tree of G.

Remark 4.1. If H is a secondary constituent of $G = \bigvee_{i=1}^{r} G_i$, that is, H is a component of G_i, for some i, then H cannot be the only component of G_i. This fact easily implies

$$\operatorname{jmr}(G_i) \geqslant \operatorname{mr}(H) + 1. \tag{3}$$

An interesting characterization of decomposable graphs is the following fact.

Proposition 4.2 [12, Thm. 9.32]. *A graph is decomposable if and only if it does not have P_4 as an induced subgraph.*

We now prove that all decomposable graphs are inertia balanced. This fact is somehow expected, given Proposition 3.2 and Theorem 3.8 and the fact that K_1 is inertia-balanced. Still, some attention is required when all the primary constituents have join minimum rank smaller than 3.

Lemma 4.3. *Let G be a connected decomposable graph of height $h(G) \leqslant 1$. Then G is inertia-balanced and* $\operatorname{mr}(G) = h(G)$.

Proof. If $h(G) = 0$, then $G = K_1$ and the claim follows. If $h(G) = 1$, then $G = K_n$, $n \geqslant 2$. In particular, the $n \times n$ matrix all of whose entries are equal to 1 is an optimal inertia-balanced matrix for G, and $\operatorname{mr}(G) = 1$. □

Let $G = \bigvee_{i=1}^{r} G_i$ be a decomposable graph. Then G is said to be *anomalous* if

(i) for each i, $\operatorname{jmr}(G_i) \leqslant 2$; and
(ii) $K_{3,3,3} = \overline{K}_3 \vee \overline{K}_3 \vee \overline{K}_3$ is a subgraph of G.

In particular, in a non-anomalous graph $G = \bigvee_{i=1}^{r} G_i$ there are at most two i for which $|G_i| \geqslant 3$ and $G_i = \ddot{G}_i$.

Theorem 4.4. *Let $G = \bigvee_{i=1}^{r} G_i$, $r > 1$, be a decomposable graph of height $h(G) \leqslant 3$. Then G is inertia-balanced, and*

$$\operatorname{mr}(G) = \begin{cases} \max_i\{\operatorname{jmr}(G_i)\} & \text{if } G \text{ is not anomalous;} \\ 3 & \text{if } G \text{ is anomalous.} \end{cases}$$

Proof. By Lemma 4.3, it is sufficient to consider $h(G) \geqslant 2$. Since G is not a clique, we have $\mathrm{mr}(G) \geqslant 2$. Note that the secondary constituents of G have height at most 1, so that, by Lemma 4.3, they are inertia-balanced. Thus, each G_i is inertia-balanced by either Proposition 3.2, or trivially, in the event $G_i = K_1$. Therefore, if $\max_i\{\mathrm{jmr}(G_i)\} \geqslant 3$, the claim follows directly from Theorem 3.8.

Now assume $\max_i\{\mathrm{jmr}(G_i)\} = 2$. Let $S = \{i \mid G_i = \ddot{G}_i,\ |G_i| \geqslant 3\}$. For each i we define m_i and n_i as follows:

- if $i \in S$, then let $m_i = |G_i|, n_i = 0$. In particular, note that $G_i = \overline{K}_{m_i}$.
- if $i \notin S$, since $\mathrm{jmr}(G_i) \leqslant 2$, G_i is the union of at most two cliques. Define m_i and n_i so that $G = K_{m_i} \cup K_{n_i}$.

Case I: G is not anomalous, that is, $|S| \leqslant 2$. For each i select $\alpha_i \in \mathbb{Z}$ satisfying the following conditions:

- for each $i \neq j, \alpha_i \neq \alpha_j$;
- for each $i \in S, \alpha_i \in \{-1, +1\}$;
- for each $i \notin S, \alpha_i \notin \{-1, +1\}$.

Finally, let

$$M_i = \begin{bmatrix} \mathbf{1}_{m_i}^{\mathrm{T}} & \alpha_i \mathbf{1}_{n_i}^{\mathrm{T}} \\ \alpha_i \mathbf{1}_{m_i}^{\mathrm{T}} & \mathbf{1}_{n_i}^{\mathrm{T}} \end{bmatrix}; \qquad M = \begin{bmatrix} M_1 & \cdots & M_r \end{bmatrix}; \qquad A = [M, M]_2.$$

Clearly A is inertia-balanced and rank $A \leqslant 2$. It is routine to verify that $G(A) = G$, so that $\mathrm{mr}(G) \leqslant 2$. Since we know that $\mathrm{mr}(G) \geqslant 2$, and $\max_i\{\mathrm{jmr}(G_i)\} = 2$, we conclude that G is inertia-balanced and $\mathrm{mr}(G) = \max\{\mathrm{jmr}(G_i)\}$.

Case II: G is anomalous, that is $|S| \geqslant 3$. As we already noted, G contains $K_{3,3,3}$ as induced subgraph. Since, as shown in [3], $\mathrm{mr}(K_{3,3,3}) = 3$, it follows that $\mathrm{mr}(G) \geqslant 3$. Define

$$M_i = \begin{bmatrix} \mathbf{1}_{m_i}^{\mathrm{T}} \\ \mathbf{0}_{m_i}^{\mathrm{T}} \\ \mathbf{1}_{m_i}^{\mathrm{T}} \end{bmatrix}, \quad i \in S; \qquad M_i = \begin{bmatrix} \mathbf{1}_{m_i}^{\mathrm{T}} & \mathbf{0}_{n_i}^{\mathrm{T}} \\ \mathbf{0}_{m_i}^{\mathrm{T}} & \mathbf{0}_{n_i}^{\mathrm{T}} \\ \mathbf{0}_{m_i}^{\mathrm{T}} & \mathbf{1}_{n_i}^{\mathrm{T}} \end{bmatrix}, \quad i \notin S.$$

Note that, if $A_i = [M_i, M_i]_3$, then $G(A_i) = G_i$. Since all the matrices M_i do not have any zero columns, by the Rotation Lemma we can find H_3-unitary matrices $P_1, \ldots, P_r$ such that $[P_1 M_1, P_r M_r]_3$ has no zero entries, for $i \neq j$. Therefore, by defining $M = [P_1 M_1 \quad \cdots \quad P_r M_r]$, and $A = [M, M]_3$, we have $G(A) = G$. Since A is inertia-balanced and rank $A \leqslant 3$, we conclude that G is inertia-balanced and $\mathrm{mr}(G) = 3$. $\square$

This result can be extended to any decomposable graph as follows.

Theorem 4.5. *Let $G = \bigvee_{i=1}^{r} G_i$, $r > 1$, be a connected decomposable graph. Then G is inertia-balanced, and*

$$\mathrm{mr}(G) = \begin{cases} \max_i\{\mathrm{jmr}(G_i)\} & \text{if } G \text{ is not anomalous;} \\ 3 & \text{if } G \text{ is anomalous.} \end{cases}$$

Proof. The proof is by induction on $h(G)$. If $h(G) \leqslant 3$ the result has been proved in Theorem 4.4. Thus, let $h(G) \geqslant 4$. By the inductive hypothesis, all the secondary constituents of G are inertia-balanced; therefore so are all the G_i's. In addition, there exists a secondary constituent H with $h(H) \geqslant 2$. In particular H is not a clique, so that $\operatorname{mr}(H) \geqslant 2$. By (3) we then obtain $\max_i\{\operatorname{jmr}(G_i)\} \geqslant \operatorname{mr}(H) + 1 \geqslant 3$. We now apply Theorem 3.8 to complete the proof. □

Corollary 4.6. *For each connected decomposable graph* $G \neq K_1$, $\operatorname{mr}(G) \geqslant \lfloor h(G)/2 \rfloor + 1$, *and this inequality is sharp*.

Proof. The proof is by induction on $h(G)$. If $h(G) = 1$, then $G = K_m, m > 1$, and $\operatorname{mr}(G) = 1 = \lfloor h(G)/2 \rfloor + 1$. If $h(G) = 2$, then $G \neq K_m$, and so $\operatorname{mr}(G) \geqslant 2 = \lfloor h(G)/2 \rfloor + 1$. If $h(G) \geqslant 3$, then G has a secondary constituent H such that $h(H) = h(G) - 2$, $\operatorname{mr}(H) \geqslant \lfloor h(H)/2 \rfloor + 1$. By (3) we then have

$$\operatorname{mr}(G) \geqslant \max_i\{\operatorname{jmr}(G_i)\} \geqslant \operatorname{mr}(H) + 1 \geqslant \lfloor h(H)/2 \rfloor + 2 = \lfloor h(G)/2 \rfloor + 1.$$

The inequality is sharp by defining inductively $\Gamma_0 = K_1, \Gamma_1 = K_1 \vee K_1, \Gamma_n = (\Gamma_{n-2} \cup K_1) \vee K_1$. Indeed, $h(\Gamma_n) = n$ for each n. Furthermore, $\operatorname{mr}(\Gamma_1) = 1, \operatorname{mr}(\Gamma_2) = 2, \operatorname{mr}(\Gamma_n) = \operatorname{mr}(\Gamma_{n-2}) + 1$, for $n > 2$. Hence $\operatorname{mr}(\Gamma_n) = \lfloor n/2 \rfloor + 1$, for $n \geqslant 1$. □

A graph on n vertices is called *degree antiregular* if the collection of vertex degrees coincides with $\{1, 2, \ldots, n-1\}$. It is not difficult to verify (see [13]) that there is exactly one degree antiregular graph on n vertices, namely Γ_n, the same graph as constructed in Corollary 4.6 with $\operatorname{mr}(\Gamma_n) = \operatorname{mr}(\Gamma_{n-2}) + 1$.

Recently all graphs with minimum rank equal to two have been characterized in [3] (infinite field case) and in [4] (finite field case), where the approach taken was to characterize all the possible forbidden subgraphs (including P_4). Their proof, in the real case (see [3]), is rather long and involves numerous cases. A shorter method to obtain a complete characterization of the connected graphs whose minimum rank is two is a consequence of Theorem 4.5. The connection between graphs with $\operatorname{mr}(G) = 2$ and decomposable graphs boils down to the graph P_4, the path on four vertices. As noted in the introduction $\operatorname{mr}(P_4) = 3$, so if G is a graph with $\operatorname{mr}(G) = 2$, then G cannot contain P_4 as an induced graph. By Proposition 4.2 any graph that does not contain P_4 as an induced subgraph is decomposable.

Corollary 4.7. *A connected graph G has minimum rank* 2 *if and only if* $G = \bigvee_{i=1}^{r} G_i$, $r > 1$, *where either*

(a) $G_i = \overline{K_{m_i}} \cup K_{n_i}$, *for suitable* $m_i \geqslant 1$, $n_i \geqslant 0$, *or*
(b) $G_i = \overline{K_{m_i}}$, *for a suitable* $m_i \geqslant 3$;

and option (b) *occurs at most twice*.

Proof. *Sufficiency*: Since G is not a clique, $\operatorname{mr}(G) \geqslant 2$. Furthermore, by assumption G is decomposable, not anomalous, and $\operatorname{jmr}(G_i) \leqslant 2$ for each i. Therefore the claim follows from Theorem 4.5.

Necessity: If $\operatorname{mr}(G) = 2$, then G does not contain P_4 as induced subgraph. By Proposition 4.2, G is decomposable, so that $G = \bigvee_{i=1}^{r} G_i$. We then have $\operatorname{jmr}(G_i) \leqslant 2$ for each i. Therefore, either

$G_i = K_{m_i} \cup K_{n_i}$, for some $m_i \geqslant 1$, $n_i \geqslant 0$, or $G = \overline{K}_{m_i}$, $m_i \geqslant 3$. Finally, since $\mathrm{mr}(G) \neq 3$, G is not anomalous, namely, $G_i = \overline{K}_{m_i}$, $m_i \geqslant 3$ can occur at most twice. $\square$

Finally, we close with a complete description of the minimum rank for the join of inertia-balanced graphs.

Corollary 4.8. *Let $G = \bigvee_{i=1}^{r} G_i$, $r > 1$, where each G_i is inertia-balanced. Then G is inertia-balanced, and*

$$\mathrm{mr}(G) = \begin{cases} \max_i\{\mathrm{jmr}(G_i)\} & \text{if } G \text{ is not anomalous;} \\ 3 & \text{if } G \text{ is anomalous.} \end{cases}$$

Proof. If $\max\{\mathrm{jmr}(G_i)\} \geqslant 3$, the result follows by Theorem 3.8. Therefore, for each i, we may then assume $\mathrm{jmr}(G_i) \leqslant 2$. In particular, $\mathrm{mr}(G_i) \leqslant 2$, so that G_i cannot contain P_4 as induced subgraph. By Corollary 4.2, all G_i are decomposable, and hence so is G. We may now apply Corollary 4.7 to obtain the desired conclusion. $\square$

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