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On standard locally catenative L schemes

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Abstract

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A standard locally catenative L scheme extracts the essential feature of the locally catenative property. We investigate conditions under which a standard locally catenative L scheme has multiple locally catenative L systems.

1. Introduction and preliminary results

A D0L scheme is a pair $S=\langle \Sigma, h\rangle$ where Σ is a finite nonempty set and h is a homomorphism from Σ^* into itself, called the generation mapping of S. A D0L system is a triplet $G=\langle \Sigma, h, w\rangle$ where $\langle \Sigma, h\rangle$ is a D0L scheme and w is a string in $\Sigma^{\dagger}$ called the axiom of G, while $\langle \Sigma, h\rangle$ is called the underlying scheme of G. In a D0L system G, we are not only interested in the generated language $L(G)=\{h^i(w)\,|\,i\geqslant 0\}$ but also the generated sequence $E(G)=w,\ h(w),\ h^2(w),\ldots.$ One intriguing problem on D0L sequences is about the locally catenative property: A sequence is locally catenative if all the strings except some initial ones can be written as a concatenation of previously appeared strings in the same way.

Definition. Let $G=\langle \Sigma, h, w\rangle$ be a D0L system. G is said to be $\langle i_1, i_2,\ldots, i_k\rangle$ locally catenative (abbreviated as l.c.) with cut p if $w_q=w_{q-i_1}w_{q-i_2}\ldots w_{q-i_k}$ for any $q\geqslant p$,

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where $k \geq 2$, p, i_1, $i_2, \ldots, i_k \geq 1$. $\langle i_1, i_2, \ldots, i_k \rangle$ is called an l.c. formula of G. Cut and/or l.c. formula may be omitted when their values are immaterial.

Observe that if an l.c. formula is once realized at some point of a D0L sequence, then all the strings thereafter fulfil the same formula and thus the D0L system is l.c. [3]. If a D0L system $G = \langle \Sigma, h, w \rangle$ is $\langle i_1, i_2, \ldots, i_k \rangle$ l.c. with cut p, then G is $\langle i_1, i_2, \ldots, i_k \rangle$ l.c. with cut p' where $p' \geq p$, and also G is l.c. with infinitely many l.c. formulas. For example, $G = \langle \{a, b\}, h, a \rangle$ where $h(a) = b$, $h(b) = ab$, is $\langle 2, 1 \rangle$ l.c. with cut 2. It is $\langle 2, 1 \rangle$ l.c. with cut 3, $\langle 2, 3, 2 \rangle$ l.c. with cut 3, $\langle 4, 3, 3, 2 \rangle$ l.c. with cut 4, and so on.

Definition. For a positive integer m, let Σ_m denote the set of integers 0 through $m-1$: $\Sigma_m = \{0, 1, \ldots, m-1\}$. The standard $\langle i_1, i_2, \ldots, i_k \rangle$ l.c. L scheme is a D0L scheme $S = \langle \Sigma_n, h \rangle$ where

$$n = \max\{i_1, i_2, \ldots, i_k\}, \qquad h(i) = i + 1 \text{ (for } i = 0, 1, \ldots, n-2) \quad \text{and}$$

$$h(n-1) = (n - i_1)(n - i_2) \cdots (n - i_k).$$

We sometimes denote it by $S(n, \delta)$ where $\delta = (n - i_1)(n - i_2) \cdots (n - i_k)$ as h is apparent from n and δ. Note that there is at least one 0 in δ by definition. We call the L system $G = \langle \Sigma_n, h, 0 \rangle$ the primary L system of S.

We have the following relationship between an l.c. L system and a standard l.c. L scheme.

Theorem 1.1 (Kobuchi [1]). *A* D0L *system* $G = \langle \Sigma, h, w \rangle$ *is* $\langle i_1, i_2, \ldots, i_k \rangle$ *l.c. if and only if there exist a standard* $\langle i_1, i_2, \ldots, i_k \rangle$ *l.c. L scheme* $S = \langle \Sigma_n, h' \rangle$ *and a* λ*-free homomorphism* $\gamma : \Sigma_n \to \Sigma^{\dagger}$ *such that for any x in* $E(G')$ $h(\gamma(x)) = \gamma(h'(x))$, *where* $G' = \langle \Sigma_n, h', 0 \rangle$.

That is, if a D0L system G is $\langle i_1, i_2, \ldots, i_k \rangle$ l.c., then its generated sequence $E(G)$ embodies the structure of the generated sequence of the primary L system of the standard $\langle i_1, i_2, \ldots, i_k \rangle$ l.c. L scheme. Thus we can say that the primary L system of a standard l.c. L scheme extracts the essential feature of the l.c. property.

For a given L scheme, we can have infinitely many L systems choosing distinct axiom strings. Even if an L system G is l.c., another L system with the same underlying L scheme as G is not necessarily l.c. If there is more than one intrinsically different l.c. L system with the same underlying L scheme, what kind of property does this L scheme have? What kind of condition must the axioms satisfy in that case? What is the relationship among the l.c. formulas? We would like to investigate these problems. To do so, we will consider only standard l.c. L schemes in this paper since their primary L systems typify l.c. L systems.

Proposition 1.2 (Seki [5]). *For any standard l.c. L scheme, its generation mapping is injective.*

Proposition 1.3 (Rozenberg et al. [4, 5]). *If a* D0L *system is* $\langle i_1, i_2, \ldots, i_k\rangle$ *l.c., then it is* $\langle i_1, i_2, \ldots, i_k\rangle$ *l.c. with cut* $n = \max\{i_1, i_2, \ldots, i_k\}$ *whenever the generation mapping of its underlying L scheme is injective.*

We are interested in standard l.c. L schemes which have l.c. L systems other than primary L systems. We have the following proposition about the axioms of l.c. L systems with a standard l.c. L scheme as an underlying L scheme.

Proposition 1.4 (Seki et al. [6]). *If a* D0L *system* $G = \langle \Sigma_n, h, w_0\rangle$ *is l.c., where* $\langle \Sigma_n, h\rangle$ *is a standard l.c. L scheme, then* w_0 *is a substring of a string in* $E(G')$ *where* $G' = \langle \Sigma_n, h, 0\rangle$.

Proof. Let G be $\langle j_1, j_2, \ldots, j_r\rangle$ l.c. and $E(G) = w_0, w_1, w_2, \ldots$. By Propositions 1.2 and 1.3, G is l.c. with cut $p = \max\{j_1, j_2, \ldots, j_r\}$. That is, w_p contains w_0 as a substring. Let $w_0 = b_1 b_2 \ldots b_s$ where $b_i \in \Sigma_n$ for $1 \leq i \leq s$. For any $q \geq 0$, $h^q(b_1)$ is a prefix of w_q. If we take q such that $|h^q(b_1)| \geq |w_p|$ and that $q - p$ is a multiple of j_1, $h^q(b_1)$ has w_p as a prefix. Thus w_0 appears as a substring of $h^q(b_1)$, which is an element of $E(G')$. □

If w_0 in Proposition 1.4 does not have 0, we can apply h^{-1} to w_0 repeatedly until we get $w = h^{-i}(w_0)$ that contains 0. Then the D0L system $\langle \Sigma_n, h, w\rangle$ is still l.c. with the same cut and l.c. formula. Hence we are basically interested in axioms that contain 0 and are substrings of strings of the primary L systems.

We also do not have to worry about reversed L schemes since the following proposition holds readily.

Proposition 1.5. *Let* $S(n, \delta)$ *be a standard l.c. L scheme. If the L system* $\langle S(n, \delta), w\rangle$ *is* $\langle i_1, i_2, \ldots, i_k\rangle$ *l.c., then the L system* $\langle S(n, \delta^R), w^R\rangle$ *is* $\langle i_k, i_{k-1}, \ldots, i_1\rangle$ *l.c., where* x^R *is the reversed string of* x.

2. Parallel decomposable L schemes

Here we give one sufficient condition for a standard l.c. L scheme to have other l.c. L systems than its primary L system.

Definition. A string $w \in \Sigma^+$ is said to be parallel decomposable in a D0L scheme $S = \langle \Sigma, h\rangle$ if $w = w_1 w_2 \ldots w_k$ ($w_i \in \Sigma^+$, $k \geq 2$) and if there exists i_0 ($1 \leq i_0 \leq k$) such that $h^{q_i}(w_{i_0}) = w_i$ for some $q_i \geq 0$ ($1 \leq i \leq k$). We call w_i and q_i a component (of parallel decomposition of w) and the depth of w, respectively.

Theorem 2.1 (Seki [5]). *Let* G *be the primary L system of a standard l.c. L scheme* $S = \langle \Sigma_n, h\rangle$. *Let* $E(G) = 0, 1, \ldots, n-1, w_n, w_{n+1}, \ldots$. *If* w_p ($p \geq n$) *in* $E(G)$ *is parallel* *decomposable in S, then a* D0L *system* $G' = \langle \Sigma_n, h, \alpha\rangle$, *where* α *is a component of a parallel decomposition of* w_p, *is also l.c.*

Proof. Let $w_p = \alpha_1\alpha_2 \dots \alpha_m$ $(\alpha_i \in \Sigma_n^\dagger)$ where there exist i_0 $(1 \leq i_0 \leq m)$ and $q_i (\geq 0)$ such that $h^{q_i}(\alpha_{i_0}) = \alpha_i$ for any i $(1 \leq i \leq m)$. For any $u \geq p$,

$$\begin{aligned} w_u &= h^{u-p}(\alpha_1) \dots h^{u-p}(\alpha_{i_0}) \dots h^{u-p}(\alpha_m) \\ &= h^{u-p+q_1}(\alpha_{i_0}) \dots h^{u-p}(\alpha_{i_0}) \dots h^{u-p+q_m}(\alpha_{i_0}). \end{aligned}$$

We will show that $G' = \langle \Sigma_n, h, \alpha_{i_0}\rangle$ is l.c. Then a D0L system $\langle \Sigma_n, h, \alpha_i\rangle$ is also l.c. for any i $(1 \leq i \leq m)$. Let $\alpha_{i_0} = a_1 a_2 \dots a_t$ $(a_j \in \Sigma_n)$. Then,

$$\begin{aligned} h^p(\alpha_{i_0}) &= h^p(a_1)h^p(a_2) \dots h^p(a_t) \\ &= w_{p+a_1} w_{p+a_2} \dots w_{p+a_t} \\ &= h^{a_1+q_1}(\alpha_{i_0}) \dots h^{a_1+q_m}(\alpha_{i_0}) h^{a_2+q_1}(\alpha_{i_0}) \dots h^{a_2+q_m}(a_{i_0}) \\ &\quad \dots \\ &\quad h^{a_t+q_1}(\alpha_{i_0}) \dots h^{a_t+q_m}(\alpha_{i_0}). \end{aligned}$$

Thus G' is $\langle p-a_1-q_1, \dots, p-a_1-q_m, \dots, p-a_t-q_1, \dots, p-a_t-q_m\rangle$ l.c. □

Note that

$$\begin{aligned} w_p &= h^{q_1}(\alpha_{i_0}) h^{q_2}(\alpha_{i_0}) \dots h^{q_m}(\alpha_{i_0}) \\ &= h^{a_1+q_1}(0) \dots h^{a_t+q_1}(0) h^{a_1+q_2}(0) \dots h^{a_t+q_2}(0) \dots h^{a_1+q_m}(0) \dots h^{a_t+q_m}(0). \end{aligned}$$

That is, G is $\langle p-a_1-q_1, \dots, p-a_t-q_1, \dots, p-a_1-q_m, \dots, p-a_t-q_m\rangle$ l.c. with cut p. Thus, the above mentioned l.c. formula for G' is a permutation of an l.c. formula of G. Note also that if w_p in $E(G)$ is parallel decomposable in S, then, for any $u \geq p$, w_u is parallel decomposable in S. The possible smallest value for p is n. It is the case where, in a standard D0L scheme $S(n, \delta)$, δ is parallel decomposable. See the following example.

Example. Consider the standard l.c. L scheme $S(5, 2301) = \langle \Sigma_5, h\rangle$. Then the D0L system $G' = \langle \Sigma_5, h, 01\rangle$ is also l.c.

0	01
1	12
2	23
3	34
4	42301
2301	23013412
...	...
the primary L system of S	$E(G')$
$\langle 3, 2, 5, 4\rangle$ l.c.	$\langle 3, 5, 2, 4\rangle$ l.c.

3. Cyclic L schemes

Here we will state another sufficient condition for a standard l.c. L scheme to have an l.c. L system besides the primary one.

Definition. For a positive integer n, a string w in $\Sigma_n^\dagger$ is said to be (n, c)-cyclic $(0 \leq c < n)$ if $w \in \Sigma_n$ or $w = a_1 a_2 \ldots a_s$ $(s \geq 2,\ a_i \in \Sigma_n$ for $1 \leq i \leq s)$ such that $a_i + c \equiv a_{i+1} \pmod n$ for $1 \leq i < s$. A D0L system $G = \langle \Sigma_n, h, w_0 \rangle$ is said to be (n, c)-cyclic if every string in $E(G)$ is (n, c)-cyclic.

Lemma 3.1 (Seki [5]). *Let G be the primary L system of a standard l.c. L scheme $S(n, \delta) = \langle \Sigma_n, h \rangle$. Then the following two conditions are equivalent.*

(1) *δ is (n, c)-cyclic, and δ starts and ends with* 0.

(2) *G is (n, c)-cyclic.*

Proof. (1)⇒(2): Let $E(G)$ be $0, 1, \ldots, n-1, w_n, w_{n+1}, \ldots$. We will show that if w_p is (n, c)-cyclic, then w_{p+1} is also (n, c)-cyclic for $p \geq n$. Let $w_p = a_1 \ldots a_s$ where $a_i + c \equiv a_{i+1} \pmod n$. If none of the a_i's is equal to $n-1$, then $w_{p+1} = (a_1 + 1) \ldots (a_s + 1)$ is also (n, c)-cyclic. Assume that $a_j = n-1$ $(1 \leq j \leq s)$. Then $a_{j-1} = n - 1 - c$ and $a_{j+1} = n - 1 + c \pmod n$. Since $h(a_{j-1}) = n - c$, $h(a_{j+1}) = c$, and $h(a_j) = \delta$ starts and ends with 0, w_{p+1} is also (n, c)-cyclic.

(2)⇒(1): Obvious. □

In this section, we consider standard l.c. L schemes $S(n, \delta)$ of this type. That is, δ always satisfies condition (1) in Lemma 3.1 and h denotes the generation mapping of S. Then δ has the following property.

Lemma 3.2. *Let $\delta \in \Sigma_n^\dagger$ be (n, c)-cyclic and assume that it starts and ends with* 0. *Then the following holds, where $m = \mathrm{GCD}(n, c)$. If an integer I is a multiple of m in Σ_n, then I occurs in δ, and vice versa.*

Proof. Let $c = mc'$ and $n = mn'$. Then c' and n' are relatively prime. There exist integers x and y such that $xc' + yn' = 1$. So $xmc' + ymn' = m$, that is, $xc + yn = m$. Let $I = im$ for $0 \leq i < n'$. $I = ixc + iyn$, that is, $I = ixc \pmod n$ so I occurs in δ. Conversely, if I appears in δ, then it is a multiple of c with mod n. Thus it is a multiple of m. □

From the discussion above, there are n' distinct integers that occur in δ. Thus the structure of δ is

$$\underbrace{0 \cdot c \cdot 2c \cdot \ldots}_{n'} \cdot \underbrace{0 \cdot c \cdot 2c \cdot \ldots}_{n'} \cdot \ldots \cdot \underbrace{0 \cdot c \cdot 2c \cdot \ldots}_{n'} \cdot 0.$$

And the length of δ can be written as $rn' + 1$ for some $r \geq 1$.

Now, we have the main theorem for this section as follows.

Theorem 3.3. *Let $S(n, \delta) = \langle \Sigma_n, h \rangle$ be a standard l.c. L scheme such that δ is (n, c)-cyclic, starting and ending with 0. Then a* D0L *system $G' = \langle \Sigma_n, h, w_0 \rangle$ is l.c. if w_0 is (n, c)-cyclic.*

To prove the theorem, we need the following lemma.

Lemma 3.4. *Consider $S(n, \delta) = \langle \Sigma_n, h \rangle$ as in Theorem 3.3. Let $x \in \Sigma_n^\dagger$ contain 0 and be (n, c)-cyclic. Assume that the length $|x|$ is a multiple of n' which is defined by $n = mn'$ and $m = \text{GCD}(n, c)$. Then $|h^{jm}(x)| = (r+1)^j |x|$ for any $j \geq 0$ where $|\delta| = rn' + 1$.*

Proof. Let $|x| = tn'$. When $j = 0$, $|h^{jm}(x)| = |x|$ and we are done. As any n' consecutive symbols in x contain all the multiples of m in Σ_n and nothing else, in m steps, exactly one symbol in it expands to δ. If $|h^{jm}(x)| = (r+1)^j |x|$, there are $(r+1)^j t$ strings of length n' without overlapping. Then $|h^{(j+1)m}(x)| = |h^{jm}(x)| + (r+1)^j trn' = (r+1)^{j+1} |x|$. □

Proof of Theorem 3.3. Let $w_0 = b_1 b_2 \ldots b_s$ ($b_i \in \Sigma_n$ for $1 \leq i \leq s$) and $E(G') = w_0, w_1, w_2, \ldots$. Without loss of generality, we can assume that w_0 contains 0. This means that all b_i's are multiples of m ($0 \leq b_i < n'm$). With the same argument as that in the proof of Lemma 3.1, G' is (n, c)-cyclic since w_0 is (n, c)-cyclic. Notice that, for $0 \leq j < n$, all w_j's start with distinct n elements in Σ_n. Notice also that w_n starts with b_1 and ends with b_s. We will prove that G' is l.c. with cut n; that is, w_n can be written as a concatenation of strings in $\{w_0, w_1, \ldots, w_{n-1}\}$. The first component of w_n is w_0. As w_0 ends with $b_s = b_1 + (s-1)c \pmod{n}$, the second component of w_n must start with $b_1 + sc \pmod{n}$. It must be w_q, where $q = sc \pmod{n}$. The third component must start with $b_1 + 2sc \pmod{n}$ and must be w_q, where $q = 2sc \pmod{n}$. Let $\text{GCD}(s, n') = g$, $s = gs'$, and $n' = gn''$. Then $n''sc \equiv 0 \pmod{n}$ and the $(n''+1)$-st component of w_n must be w_0. The components may repeat the same sequence of strings. To construct w_n, we need only n'' strings $w_0, w_{gm}, w_{2gm}, \ldots, w_{(n''-1)gm}$. (Note that $gm = \text{GCD}(sc, n)$.) The initial segment of w_n is α which is a concatenation of all these n'' strings in the proper order starting with w_0. That is, $|\alpha| = |w_0| + |w_{gm}| + \cdots + |w_{(n''-1)gm}|$, and α starts with b_1 and ends with $b_1 - c \pmod{n}$. We would like to show that $w_n = \alpha^l w_0$ for some $l \geq 1$. As we know about (n, c)-cyclic property of α, what we have to examine is about its length. We will prove that $|w_n| - |w_0|$ is a multiple of $|\alpha|$. Note that the primary L system G of S is

$$\underbrace{\langle \underbrace{n, n-c, n-2c, \ldots,}_{n'} \underbrace{n, n-c, \ldots,}_{n'} \underbrace{n, \ldots,}_{\ldots} n}_{r\text{ times}} \rangle \text{l.c.}$$

Let $G_i = \langle \Sigma_n, h, b_i \rangle$ for $1 \leq i \leq s$. Then G_i satisfies the same l.c. formula as G. Every string in $E(G')$ is a concatenation of strings of $E(G')$ for $1 \leq i \leq s$ at the same level.

Hence

$$|w_q| = |w_{q-n}| + |w_{q-n+c}| + \cdots + |w_{q-n}| + |w_{q-n+c}| + \cdots + |w_{q-n}| \quad \text{for } q \geq n.$$

That is,

$$|w_n| = \underbrace{\underbrace{|w_0| + |w_m| + \cdots + |w_{(n'-1)m}|} + \underbrace{|w_0| + \cdots + |w_{(n'-1)m}|} + \cdots + \underbrace{|w_0| + \cdots + |w_{(n'-1)m}|}}_{r\ \text{times}} + |w_0|$$

We will show that $|w_0| + |w_m| + \cdots + |w_{(n'-1)m}|$ is a multiple of $|\alpha| = |w_0| + |w_{gm}| + \cdots + |w_{(n''-1)gm}|$. Remember that α is (n, c)-cyclic and that α starts with b_1 and ends with $b_1 - c \pmod n$. This means that $|\alpha|$ is a multiple of n'. For $0 \leq i < n''$ and $0 \leq j < g$, $w_{igm+jm} = h^{jm}(w_{igm})$. So, for $0 \leq j < g$,

$$|w_{jm}| + |w_{gm+jm}| + \cdots + |w_{(n''-1)gm+jm}| = |h^{jm}(\alpha)|.$$

By Lemma 3.4, $|h^{jm}(\alpha)| = (r+1)^j |\alpha|$. Then

$$|w_0| + |w_m| + \cdots + |w_{(n'-1)m}| = \sum_{j=0}^{g-1} (|w_{jm}| + |w_{gm+jm}| + \cdots + |w_{(n''-1)gm+jm}|)$$

$$= ((r+1)^g - 1)/r \cdot |\alpha|.$$

That is, $|w_n| = |\alpha|((r+1)^q - 1) + |w_0|$, which means $w_n = \alpha^{((r+1)^q - 1)} w_0$.

If s is a multiple of n', then $g = n'$ and $n'' = 1$. In this particular case, we have $\alpha = w_0$ and G' is

$$\langle \underbrace{n, n, \ldots, n}_{(r+1)^{n'}\ \text{times}} \rangle \text{ l.c.} \qquad \square$$

Example. Consider the standard l.c. L scheme $S(8, 02460) = \langle \Sigma_8, h \rangle$. Then the D0L system $G' = \langle \Sigma_8, h, 024 \rangle$ is also l.c.

0	024
1	135
2	246
3	357
4	4602460
5	5713571
6	602460246024602
7	713571357135713
02460	0246024602460246024602460246024
...	...
the primary L system of S	$E(G')$
$\langle 8, 6, 4, 2, 8 \rangle$ l.c.	$\langle 8, 2, 4, 6, 8 \rangle$ l.c.

4. Semicyclic L systems

In Sections 2 and 3, we have presented two sufficient conditions for a standard l.c. L scheme to have other l.c. L systems than its primary L system. They look very natural for the l.c. property [2]. At first, we thought those two conditions also make necessary conditions. But it turned out that this is not the case. Here we give another sufficient condition which is rather complicated and looks slightly odd.

Theorem 4.1 (Seki et al. [6]). *For a standard* $\langle b_0, b_1, b_1+c, \ldots, b_1+rc, b_2, b_2+c, \ldots, b_2+rc, \ldots, b_p, b_p+c, \ldots, b_p+rc\rangle$ *l.c. L scheme* $S=\langle\Sigma_n, h\rangle$, *where* $r\geq 1$ *and there exist* $t_0, t_1, \ldots, t_p$ (≥ 0) *such that*

$$b_0-b_1+c=t_0b_0,$$

$$b_i-b_{i+1}+(r+1)c=t_ib_0 \quad \textit{for } 1\leq i\leq p-1, \quad \textit{and}$$

$$b_p+rc=t_pb_0,$$

the L system $G=\langle\Sigma_n, h, (rc)\cdot(r-1)c\cdot\cdots\cdot c\cdot 0\rangle$ *is l.c.*

Let $E(G)=w_0, w_1, \ldots$. For the sake of simplicity, we denote δ by $(n-b_0)\delta'$ and also denote $(n-b_i)(n-b_i-c)\cdots(n-b_i-rc)$ by $\underline{n-b_i}$ for $1\leq i\leq p$. With this notation, we can write

$$h(n-1)=(n-b_0)\delta'=(n-b_0)(\underline{n-b_1})(\underline{n-b_2})\cdots(\underline{n-b_p}).$$

This L scheme has the properties found in the following lemmas.

Lemma 4.2 (Seki et al. [6]). *For an integer* $q\geq n-1$,

$$w_q=h^{q-n+1+rc}(n-1)h^{q-n+1+(r-1)c}(n-1)\ldots h^{q-n+1+c}(n-1)h^{q-n+1}(n-1).$$

Lemma 4.3 (Seki et al. [6]). *For* i $(1\leq i\leq p)$ *and* m $(\geq b_i+rc)$,

$$h^m(\underline{n-b_i})=h^{m-b_i+1}(n-1)h^{m-b_i+1-c}(n-1)\ldots h^{m-b_i+1-rc}(n-1).$$

We will call a string w in $\Sigma_n^\dagger$ decomposable (in G) if $w=w_{j_1}w_{j_2}\ldots w_{j_l}$ for some l (≥ 1) and $j_1, j_2, \ldots, j_l$ (≥ 0). Thus "w is decomposable" means that w is a concatenation of strings in $E(G)$. Lemma 4.3 says that $h^m(\underline{n-b_i})$ and hence $h^{m'}(\delta')$ for a large m' are decomposable by Lemma 4.2. The generation mapping h has the following property.

Lemma 4.4 (Seki et al. [6]). *For* t (≥ 0) *and* s $(0\leq s\leq b_0)$,

$$h^{tb_0+s}(n-1)=h^s(n-1)h^{b_0+s-1}(\delta')h^{2b_0+s-1}(\delta')\ldots h^{tb_0+s-1}(\delta').$$

Proof

$$\begin{aligned} h^{tb_0+s}(n-1) &= h^{tb_0+s-1}((n-b_0)\delta') \\ &= h^{(t-1)b_0+s-1}((n-b_0)\delta')h^{tb_0+s-1}(\delta') \\ &= h^{(t-1)b_0+s}(n-1)h^{tb_0+s-1}(\delta') \\ &= h^{s}(n-1)h^{b_0+s-1}(\delta')\dots h^{(t-1)b_0+s-1}(\delta')h^{tb_0+s-1}(\delta'). \qquad \square \end{aligned}$$

Lemma 4.5 (Seki et al. [6]). *Consider a string* $h^{m_1}(n-1)h^{m_2}(n-1)\dots h^{m_{r+1}}(n-1)$ *where there exist* t_i'*'s* ($\geqslant 0$) *such that* $m_{i+1}-m_i+c=t_i'b_0$ *for* $1\leqslant i\leqslant r$. *Then it is decomposable.*

Proof. First we will show that $h^{m_1}(n-1)h^{m_1-c}(n-1)\dots h^{m_1-(r-1)c}(n-1)h^{m_{r+1}}(n-1)$ is decomposable. There exists t ($\geqslant 0$) such that $m_{r+1}-m_1+rc=tb_0$;

$$\begin{aligned} &h^{m_1}(n-1)h^{m_1-c}(n-1)\dots h^{m_1-(r-1)c}(n-1)h^{m_{r+1}}(n-1) \\ &\quad = h^{m_1}(n-1)h^{m_1-c}(n-1)\dots h^{m_1-(r-1)c}(n-1)h^{m_1-rc}(n-1) \\ &\quad\quad h^{b_0+m_1-rc-1}(\delta')h^{2b_0+m_1-rc-1}(\delta')\dots h^{m_{r+1}-1}(\delta'). \end{aligned}$$

By Lemmas 4.2 and 4.3, this is decomposable. Now we will show that for r' ($1\leqslant r'<r$),

$$\begin{aligned} &h^{m_1}(n-1)h^{m_1-c}(n-1)\dots h^{m_1-(r'-1)c}(n-1)h^{m_{r'+1}}(n-1) \\ &\quad h^{m_{r'+2}}(n-1)\dots h^{m_{r+1}}(n-1) \end{aligned}$$

is decomposable assuming that $h^{m_1}(n-1)h^{m_1-c}(n-1)\dots h^{m_1-r'c}(n-1)h^{m_{r'+2}}(n-1)\dots h^{m_{r+1}}(n-1)$ is decomposable. There exists t' ($\geqslant 0$) such that $m_{r'+1}-m_1+r'c=t'b_0$.

$$\begin{aligned} &h^{m_1}(n-1)h^{m_1-c}(n-1)\dots h^{m_1-(r'-1)c}(n-1)h^{m_{r'+1}}(n-1)\dots h^{m_{r+1}}(n-1) \\ &\quad = h^{m_1}(n-1)h^{m_1-c}(n-1)\dots h^{m_1-(r'-1)c}(n-1)h^{m_1-r'c}(n-1) \\ &\quad\quad h^{b_0+m_1-r'c-1}(\delta')h^{2b_0+m_1-r'c-1}(\delta')\dots h^{m_{r'+1}-1}(\delta')h^{m_{r'+2}}(n-1)\dots h^{m_{r+1}}(n-1) \\ &\quad = h^{m_1}(n-1)h^{m_1-c}(n-1)\dots h^{m_1-(r'-1)c}(n-1)h^{m_1-r'c}(n-1) \\ &\quad\quad h^{b_0+m_1-r'c-1}(\underline{n-b_1})\dots h^{b_0+m_1-r'c-1}(\underline{n-b_p})\dots \\ &\quad\quad h^{m_{r'+1}-1}(\underline{n-b_1})\dots h^{m_{r'+1}-1}(\underline{n-b_p}) \\ &\quad\quad h^{m_{r'+2}}(n-1)\dots h^{m_{r+1}}(n-1) \\ &\quad = h^{m_1}(n-1)h^{m_1-c}(n-1)\dots h^{m_1-r'c}(n-1) \\ &\quad\quad h^{b_0+m_1-r'c-b_1}(n-1)h^{b_0+m_1-r'c-b_1-c}(n-1)\dots \\ &\quad\quad h^{b_0+m_1-r'c-b_1-rc}(n-1)h^{b_0+m_1-r'c-b_2}(n-1) \\ &\quad\quad \dots h^{b_0+m_1-r'c-b_p}(n-1)h^{b_0+m_1-r'c-b_p-c}(n-1)\dots h^{b_0+m_1-r'c-b_p-rc}(n-1) \\ &\quad\quad h^{2b_0+m_1-r'c-b_1}(n-1)\dots h^{2b_0+m_1-r'c-b_1-rc}(n-1)\dots \\ &\quad\quad h^{m_{r'+1}-b_p}(n-1)h^{m_{r'+1}-b_p-c}(n-1)\dots h^{m_{r'+1}-b_p-rc}(n-1)h^{m_{r'+2}}(n-1) \\ &\quad\quad \dots h^{m_{r+1}}(n-1). \end{aligned}$$

By the hypothesis and $b_0-b_1+c=t_0b_0$, the initial segment from $h^{m_1}(n-1)$ through $h^{b_0+m_1-r'c-b_1-(r-r'-1)c}(n-1)$ is decomposable. For i $(1\leq i<p)$ and j $(1\leq j\leq t')$, by the hypothesis and $b_i-b_{i+1}+(r+1)c=t_ib_0$, a segment from $h^{jb_0+m_1-r'c-b_i-(r-r')c}(n-1)$ through $h^{jb_0+m_1-r'c-b_{i+1}-(r-r'-1)c}(n-1)$ is decomposable. For j $(1\leq j<t')$, by $b_0-b_1+c=t_0b_0$ and the hypothesis, a segment from $h^{jb_0+m_1-r'c-b_p-(r-r')c}(n-1)$ through $h^{(j+1)b_0+m_1-r'c-b_1-(r-r'-1)c}(n-1)$ is decomposable. By $m_{r'+2}-m_{r'+1}+c=t'_{r'+1}b_0$, $b_p+rc=t_pb_0$, and the hypothesis, the last segment from $h^{t'b_0+m_1-r'c-b_p-(r-r')c}(n-1)$ through $h^{m_{r+1}}(n-1)$ is decomposable. Thus the whole string is decomposable. □

Proof of Theorem 4.1. For a large enough q,

$$\begin{aligned}
w_q &= h^{q-n+1+rc}(n-1)h^{q-n+1+(r-1)c}(n-1)\dots h^{q-n+1}(n-1)\\
&= h^{q-n+1+rc-b_0}(n-1)h^{q-n+rc}(\underline{n-b_1})\dots h^{q-n+rc}(\underline{n-b_p})\\
&\quad h^{q-n+1+(r-1)c}(n-1)\dots h^{q-n+1}(n-1)\\
&= h^{q-n+1+rc-b_0}(n-1)h^{q-n+1+rc-b_1}(n-1)\dots h^{q-n+1-b_1}(n-1)\\
&\quad h^{q-n+1+rc-b_2}(n-1)\dots h^{q-n+1-b_2}(n-1)\dots\\
&\quad h^{q-n+1+rc-b_p}(n-1)\dots h^{q-n+1-b_p}(n-1)\\
&\quad h^{q-n+1+(r-1)c}(n-1)\dots h^{q-n+1}(n-1)\\
&= h^{q-n+1+rc-b_0}(n-1)h^{q-n+1+rc-b_1}(n-1)\dots h^{q-n+1+c-b_1}(n-1)\\
&\quad h^{q-n+1-b_1}(n-1)h^{q-n+1+rc-b_2}(n-1)\dots h^{q-n+1+c-b_2}(n-1)\\
&\quad \dots\\
&\quad h^{q-n+1-b_{p-1}}(n-1)h^{q-n+1+rc-b_p}(n-1)\dots h^{q-n+1+c-b_p}(n-1)\\
&\quad h^{q-n+1-b_p}(n-1)h^{q-n+1+(r-1)c}(n-1)\dots h^{q-n+1}(n-1).
\end{aligned}$$

By Lemma 4.5, a segment on each line of the last expression is decomposable. This means that w_q satisfies some l.c. formula, and thus G is l.c. □

Example. Consider the standard l.c. L scheme $S(3,2210)=\langle\Sigma_3,h\rangle$. Then the D0L system $G'=\langle\Sigma_3,h,210\rangle$ is $\langle 1,3,3,5,4,3,3,5,5,7,6,5,4,4,6,5,4,3,3,5,4,3\rangle$ l.c. with cut 7.

In Theorem 4.1, the condition $b_p+rc=t_pb_0$ is unnecessary when $r=1$.

Theorem 4.6 (Seki et al. [6]). *For a standard* $\langle b_0,b_1,b_1+c,b_2,b_2+c,\dots,b_p,b_p+c\rangle$ *l.c.* L *scheme* $S=\langle\Sigma_n,h\rangle$, *where there exist* $t_0,t_1,\dots,t_{p-1}$ (≥ 0) *such that*

$$b_0-b_1+c=t_0b_0 \quad \textit{and}$$

$$b_i-b_{i+1}+2c=t_ib_0 \quad \textit{for } 1\leq i\leq p-1,$$

the L *system* $G=\langle\Sigma_n,h,c\cdot 0\rangle$ *is l.c.*

We present a substitute for Lemma 4.5.

Lemma 4.7 (Seki et al. [6]). *A string* $h^{m_1}(n-1)h^{m_2}(n-1)$ *is decomposable if there exists* t (≥ 0) *such that* $m_2 - m_1 + c = tb_0$.

Proof. $h^{m_1}(n-1)h^{m_2}(n-1) = h^{m_1}(n-1)h^{m_1-c}(n-1)h^{b_0+m_1-c-1}(\underline{n-b_1})\ldots h^{b_0+m_1-c-1}(\underline{n-b_p})\ldots h^{m_2-1}(\underline{n-b_1})\ldots h^{m_2-1}(\underline{n-b_p})$. □

Proof of Theorem 4.6. By Lemmas 4.3 and 4.4, for a large enough m,

$$\begin{aligned} h^m(n-1) &= h^{m-b_0}(n-1)h^{m-1}(\delta') \\ &= h^{m-b_0}(n-1)h^{m-1}(\underline{n-b_1})\ldots h^{m-1}(\underline{n-b_p}) \\ &= h^{m-b_0}(n-1)h^{m-b_1}(n-1)h^{m_1-b_1-c}(n-1)h^{m-b_2}(n-1)\ldots \\ &\quad h^{m-b_{p-1}-c}(n-1)h^{m-b_p}(n-1)h^{m-b_p-c}(n-1). \end{aligned}$$

By Lemma 4.7, $h^m(n-1)$ can be written as $\alpha h^{m-t'(b_p+c)}(n-1)$ where $t' \geq 1$ and α is decomposable.

For a large enough q,

$$\begin{aligned} w_q &= h^{q-n+1+c}(n-1)h^{q-n+1}(n-1) \\ &= h^{q-n+1+c-b_0}(n-1)h^{q-n+c}(\delta')h^{q-n+1}(n-1) \\ &= h^{q-n+1+c-b_0}(n-1)h^{q-n+1+c-b_1}(n-1) \\ &\quad h^{q-n+1-b_1}(n-1)h^{q-n+1+c-b_2}(n-1) \\ &\quad \ldots \\ &\quad h^{q-n+1-b_{p-1}}(n-1)h^{q-n+1+c-b_p}(n-1) \\ &\quad h^{q-n+1-b_p}(n-1)h^{q-n+1}(n-1). \end{aligned}$$

By Lemma 4.7, each line except the last one of the last expression is decomposable. And the last line can be written as $\alpha h^{q-n+1-(t'+1)b_p-t'c}(n-1)h^{q-n+1}(n-1)$. If we choose t' properly such that $(t'+1)(b_p+c)$ is a multiple of b_0, then the last line is also decomposable. Thus G is l.c. □

The D0L scheme in the following example satisfies the condition of Theorem 4.6 but not that of Theorem 4.1.

Example. Consider the standard l.c. L scheme $S(5, 210) = \langle \Sigma_5, h \rangle$. Then the D0L system $G' = \langle \Sigma_5, h, 10 \rangle$ is $\langle 3, 8, 13, 15, 17, 14, 11, 8, 5 \rangle$ l.c. with cut 17.

5. Concluding remarks

We have presented three sufficient conditions for a standard l.c. L scheme to have multiple l.c. L systems. In the case of parallel decomposable L schemes and cyclic

L schemes, the l.c. formula and cut of a new l.c. L system are predictable. But for semicyclic L schemes the l.c. formula is complicated and the smallest cut can be very large. In all the cases, if the axiom of a new l.c. L system is a substring of some element w of the primary L system, w is a concatenation of axioms of l.c. L systems, not all of which are the primary L systems. Regarding semicyclic schemes, we have presented Theorem 4.6 which does not require one condition in Theorem 4.1 for $r=1$. This condition may be eliminated for any r. Furthermore, conditions in Theorem 3.3 and Theorem 4.1 are satisfied by many common L schemes. Combining these theorems may lead to a sufficient and necessary condition.

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