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Variational Problems with Differential Constraints: A Geometric Approach without Lagrange Multipliers

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A geometric reformulation of variational problems with differential constraints leads to a characterization of the constaints in terms of a closed differential ideal $\mathscr{C}$ of the exterior algebra of differential forms on an appropriately chosen manifold. The collection of all vertical isovector fields of the ideal C is shown to provide a continuous parameterization of all solutions of the constraints through imbedding in a maximal Lie group of point transformations that preserves the constraints. This parameterization provides direct access to necessary and sufficient conditions for stationarity of an action or penalty integral in the presence of the given constraints. Specific examples of control problems with second-order partial differential constraints are given. The penalty functional is allowed to depend on the state variables, the control variables and their first partial derivatives, through both volume integrals and boundary integrals. © 1986 Academic Press, Inc.

1. INTRODUCTION

The classical approach to variational problems with differential constraints is through the use of Lagrange multipliers. Thus, if there are m state variables and $r < m$ differential constraints, r Lagrange multipliers are introduced so that the constrained variational problem with m state variables is converted into an unconstrained variational problem with $m + r$ state variables. Solutions of the resulting system of $m + r$ Euler–Lagrange equations must then be obtained followed by an elimination of the r Lagrange multipliers, to obtain solutions of the original constrained variational problem. For m and r of moderate size, putting in and then taking out the r Lagrange multipliers is a formidable task that is often compounded by the nonlinear manner in which the Lagrange multipliers and their derivatives enter into the system of $m + r$ Euler-Lagrange equations.

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The Lagrange multiplier method works because the multipliers become new variables whose introduction allows a relaxion of the differential constraints during the variation process. Solution of the $m+r$ Euler–Lagrange equations followed by an elimination of the Lagrange multipliers then reimposes the constraints and solves the constrained problem.

An exception to this involved process occurs when the differential constraints are completely integrable. In this case, the r constraints can be solved explicitly for r of the state variables in terms of the remaining $m-r$ state variables. The r explicit solutions of the differential constraints can then be used to obtain a new free variational problem that involves only $m-r$ unconstrained state variables. Solution of the new unconstrained variational problem in $m-r$ state variables is equivalent to solving the original constrained variational problem because the $m-r$ new state variables provide a complete parameterization of the set of all solutions of the completely integrable differential constraints.

The case of completely integrable differential constraints points up a particularly important, but often overlooked aspect of constrained variational problems. Suppose that a complete parameterization of the set of all solutions of the differential constraints can be obtained. In this event, all variations of the state variables that preserve the differential constraints can be computed directly by considering variations of the state variables that are induced by variation of the parameters of the solutions to the constraints. This is the approach taken in this paper.

Complete parameterization of the solution set of the constraints is obtained as follows. The works of Cartan [1] show that any finite system of differential constraints gives rise to a finitely generated closed ideal $\mathscr{C}$ of the algebra of exterior differential forms over an appropriately chosen manifold K, such that $\mathscr{C}$ is the maximal closed ideal that is annihilated by all solutions of the differential constraints. Accordingly, deformations that carry $\mathscr{C}$ into $\mathscr{C}$ are deformations that preserve the constraints.

A fundamental paper by Harrison and Estabrook [2] introduced the concept of an *isovector field of an ideal*; namely, a vector on the manifold K whose flow transports any element of the ideal into an element of the ideal. Now, the collection of all isovector fields of the ideal $\mathscr{C}$ forms a Lie group of automorphisms of K (the *isogroup* of $\mathscr{C}$) that is the maximal Lie group of automorphisms that transports elements of $\mathscr{C}$ into elements of $\mathscr{C}$. This implies that composition of any solution S of the constraints with all elements of the isogroup of $\mathscr{C}$ generates *all* solutions of the constraints that are continuously connected to S [3]. Accordingly, the isogroup of $\mathscr{C}$ serves to parameterize the solution set of the differential constraints and thus determines the connectivity of this solution set. Further, and of greater importance, the parameterization obtains without actually solving the system of constraints. This follows from the fact that the isovectors of an ideal are determined solely by the structure of the ideal and do not depend in any way on whether the ideal has or has not been "solved."

The geometric ideas just set forth provide a direct method of approach to variational problems with differential constraints. Once the isogroup of the constraint ideal $\mathscr{C}$ has been computed, questions of stationarity or optimality of an action or penelty functional become accessible through study of the deformations (variations) that are generated by the isogroup of $\mathscr{C}$; that is, the action functional is varied within the set of solutions of the differential constraints.

An analysis of questions of stationarity of variational problems with differential constraints is given for an arbitrary but smooth action functional that includes boundary terms. As it turns out, the resulting conditions for stationarity are uniquely determined by the structure of the isogroup of the constrained ideal. We take particular note of the fact that an isogroup of finite dimension leads to finitely many integral conditions rather than the usual system of Euler–Lagrange field equations. Examples of the various possibilities are given and the paper concludes with two examples of control problems for fields that satisfy second order partial differential equations.

2. Geometric Formulation of the Problem

Let M_n be the n-dimensional manifold of independent variables. We assume that M_n is orientable and is referred to a system of local coordinates $\{x^i \mid 1 \leqslant i \leqslant n\}$. The tangent space of M_n is denoted by $T(M_n)$ and the algebra of exterior differential forms on M_n is denoted by $\Lambda(M_n)$.

The volume element of M_n is

$$\mu = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n \tag{2.1}$$

and constitutes a basis for $\Lambda^n(M_n)$. Since $\{\partial_i = \partial/\partial x^i \mid 1 \leqslant i \leqslant n\}$ is a basis for $T(M_n)$, $\{\mu_i = \partial_i \rfloor \mu \mid 1 < i < n\}$ is a conjugate basis for $\Lambda^{n-1}(M_n)$ and $\{\mu_{ji} = \partial_j \rfloor \mu_i \mid 1 \leqslant j < i \leqslant n\}$ is a conjugate basis for $\Lambda^{n-2}(M_n)$. Here, $\rfloor$ denotes the operation of inner multiplication and we have [3]

$$\begin{aligned} d\mu_i &= 0, \qquad dx^j \wedge \mu_i = \delta_i^j \mu, \\ d\mu_{ij} &= 0, \qquad dx^k \wedge \mu_{ij} = \delta_i^k \mu_j - \delta_j^k \mu_i, \end{aligned} \tag{2.2}$$

where $\wedge$ denotes the exterior product.

Suppose that there are m state variables $\{\phi^\alpha(x^j) \mid 1 \leqslant \alpha \leqslant m\}$ and that the problem involves only first and second partial derivatives of these state variables. The appropriate underlying space for the problem is then an $(n+m+nm)$-dimensional *kinematic space* K [3, Chap. 2; 6, Chaps. 6, 7] with local coordinate functions $\{x^i, q^\alpha, y_i^\alpha \mid 1 \leqslant i \leqslant n,\ 1 \leqslant \alpha \leqslant m\}$ and contact 1-forms

$$C^\alpha = dq^\alpha - y_i^\alpha\, dx^i, \qquad 1 \leqslant \alpha \leqslant m. \tag{2.3}$$

Let B be an arcwise connected, simply connected subset of M_n with nonzero volume measure ($\int_B \mu \neq 0$) and smooth boundary, ∂B. The collection of *regular maps* of B into K is defined by

$$R(B) = \{\Phi\colon B \to K \mid \Phi^*\mu \neq 0,\ \Phi^* C^\alpha = 0,\ 1 \leqslant \alpha \leqslant m\}. \tag{2.4}$$

The condition $\Phi^*\mu \neq 0$ shows that we may choose a representation for any regular map Φ of the form

$$\Phi \mid x^i = x^i, \qquad q^\alpha = \phi^\alpha(x^j), \qquad y_i^\alpha = \phi_i^\alpha(x^j)$$

without loss of generality.

The relations (2.3) then show that satisfaction of the conditions $\Phi^* C^\alpha = 0$, $1 \leqslant \alpha \leqslant m$, gives $\Phi^* y_i^\alpha = \partial_i \Phi^* q^\alpha = \partial_i \phi^\alpha$. Thus, any regular map Φ has a representation of the form

$$\Phi \mid x^i = x^i, \qquad q^\alpha = \phi^\alpha(x^j), \qquad y_i^\alpha = \partial_i \phi^\alpha(x^j). \tag{2.5}$$

The space K is thus the space of graphs of the state variables and their first partial derivatives that are realized by regular maps $\Phi\colon B \to K$.

We assume that functions l and $\{l^i \mid 1 < i < n\}$ are given and that the action or penelty functional is given by

$$a[\phi^\alpha] = \int_B l(x^i, \phi^\alpha(x^j), \partial_i \phi^\alpha(x^j))\, \mu + \int_{\partial B} l^k(x^i, \phi^\alpha(x^j), \partial_i \phi^\alpha(x^j))\, \mu_k. \tag{2.6}$$

Functions L and L^i may be defined on K by

$$\begin{aligned} L(x^i, q^\alpha, y_i^\alpha) &= l(x^i, q^\alpha, y_i^\alpha), \\ L^k(x^i, q^\alpha, y_i^\alpha) &= l^k(x^i, q^\alpha, y_i^\alpha), \end{aligned} \tag{2.7}$$

in which case (2.5) shows that

$$A[\Phi] = \int_B \Phi^*(L\mu) + \int_{\partial B} \Phi^*(L^i \mu_i) = a[\phi^\alpha] \tag{2.8}$$

for any regular map Φ. Now $A[\cdot]$ may be viewed as a map from the class of regular maps $R(B)$ into $\mathbb{R}$. Thus, since (2.8) gives $A[\Phi] = a[\phi^\alpha]$ for any regular map Φ, the action functional $a[\phi^\alpha]$ may be replaced by the map A: $R(B) \to \mathbb{R}$.

Let $\{\omega_a \mid 1 \leqslant a \leqslant r\}$ be a given system of r exterior differential forms on K. These serve to define a system of differential constraints under satisfaction of the conditions

$$\Phi^* \omega_a = 0, \qquad 1 \leqslant a \leqslant r, \qquad \Phi \in R(B). \tag{2.9}$$

For example, suppose that we have the system of second-order constraints

$$w^{ij}_{a\alpha}(x^k, \phi^\beta, \partial_k \phi^\beta)\, \partial_i \partial_j \phi^\alpha + w_a(x^k, \phi^\beta, \partial_k \phi^\beta) = 0, \qquad 1 \leqslant a \leqslant r. \tag{2.10}$$

If we define the functions $W^{ij}_{a\alpha}$ and W_a on K by

$$W^{ij}_{a\alpha} = w^{ij}_{a\alpha}(x^k, q^\beta, y^\beta_k), \qquad W_a = w_a(x^k, q^\beta, y^\beta_k),$$

then

$$\omega_a = W^{ij}_{a\alpha}\, dy^\alpha_i \wedge \mu_j + W_a \mu \tag{2.11}$$

are n-forms on K. When the relations $dx^j \wedge \mu_i = \delta^j_i \mu$ are used (see (2.2)), it follows that

$$\Phi^* \omega_a = \{w^{ij}_{a\alpha}(x^k, \phi^\beta, \partial_k \phi^\beta)\, \partial_i\, \partial_j \phi^\alpha + w_a(x^k, \phi^\beta, \partial_k \phi^\beta)\}\, \mu \tag{2.12}$$

for any $\Phi \in R(B)$, and hence $\Phi^* \omega_a = 0$, $1 \leqslant a \leqslant r$, reproduces the constraints (2.10).

We started these considerations with the class of regular maps $R(B)$, and hence all such maps satisfy the constraints $\Phi^* \mu \neq 0$, $\Phi^* C^\alpha = 0$, $1 \leqslant \alpha \leqslant m$. Thus, the given constraints $\Phi^* \omega_a = 0$, will be satisfied only for the class of *constraint maps*

$$R_c(B) = \{\Phi \in R(B) \mid \Phi^* \omega_a = 0, 1 \leqslant a \leqslant r\}. \tag{2.13}$$

It is explicitly assumed that the class $R_c(B)$ of constraint maps is non-vacuous; that is, the system of constraints $\Phi^* \omega_a = 0$, $\Phi^* C^\alpha = 0$, $\Phi^* \mu \neq 0$ is consistent.

If Φ is any constraint map, the action map $A[\Phi]$ is well defined and takes its value in $\mathbb{R}$. Thus, as Φ ranges over $R_c(B)$, $A[\Phi]$ ranges over some subset of $\mathbb{R}$. The variational problem that we wish to solve may now be stated as follows. *Find all critical points of the action map $A[\cdot]$: $R_c(B) \to \mathbb{R}$.* It should be carefully noted that we require the critical points of the action map $A[\cdot]$ as a map from $R_c(B)$ into $\mathbb{R}$ rather than as a map from $R(B)$ into $\mathbb{R}$. Thus, the action map is to be scrutinized only on the set $R_c(B)$. It is thus essential that any element of $R_c(B)$ belong to at least a 1-parameter family of elements of $R_c(B)$. If this were not the case, the constraints would be satisfied only for a disconnected set of regular maps and questions of stationarity of the action map become mute. As it turns out, the method to be developed in this paper provides a simple and direct test for this possibility. The method will also provide means for handling situations in which the connectivity of $R_c(B)$ is that of a finitely generated group, in which case the stationarity conditions are expressed by a finite system of integral conditions as opposed to a system of Euler–Lagrange partial differential equations.

3. Isovector Fields and Variations

The problem at hand is that of characterizing the connectivity of the class of constraint maps $R_c(B)$. Since each $\Phi \in R_c(B)$ has B as domain and an n-dimensional subset of K as range, the connectivity of $R_c(B)$ becomes accessible by study of smooth deformations of K that carry elements of $R_c(B)$ into elements of $R_c(B)$. Now, smooth deformations of K may be thought of as arising from the action of a continuous family of automorphisms of K that contains the identity map, and these in turn may be realized in terms of transport along orbits of vector fields on K.

A general vector field $V \in T(K)$ on K has the representation

$$V = v^i \partial_i + v^\alpha \partial_\alpha + v_i^\alpha \partial_\alpha^i, \tag{3.1}$$

where $\{v^i, v^\alpha, v_i^\alpha\}$ is a system of smooth functions on K and $\partial_\alpha = \partial/\partial q^\alpha$, $\partial_\alpha^i = \partial/\partial y_i^\alpha$. Let $T_V(s)\colon K \to K$ denote the 1-parameter family of automorphisms of K that is generated by transport along the orbits of $V \in T(K)$; that is,

$$\begin{aligned} T_V(s)\colon \quad K \to K \mid {}'x^i = \exp(sV)\, x^i, \\ {}'q^\alpha = \exp(sV)\, q^\alpha, \qquad {}'y_i^\alpha = \exp(sV)\, y_i^\alpha. \end{aligned} \tag{3.2}$$

If Φ is any map of B into K, then $T_V(s)$ can be composed with Φ to yield the 1-parameter family of maps

$$\Phi_V(s) = T_V(s) \circ \Phi \tag{3.3}$$

of B into K. It then follows from (3.2) that $\Phi_V(s)$ has the realization

$$\begin{aligned} {}'x^i &= x^i + sv^i(x^j, \phi^\beta(x^j), \partial_k \phi^\beta(x^j)) + o(s), \\ {}'\phi^\alpha(x^j) &= \phi^\alpha(x^j) + sv^\alpha(x^j, \phi^\beta(x^j), \partial_k \phi^\beta(x^j)) + o(s), \\ {}'y_i^\alpha(x^j) &= y_i^\alpha(x^j) + sv_i^\alpha(x^j, \phi^\beta(x^j), \partial_k \phi^\beta(x^j)) + o(s). \end{aligned}$$

It is clear from this realization that we must set $v^i(x^j, q^\alpha, y_i^\alpha) = 0$, for otherwise the automorphisms $T_V(s)$ of K will generate deformations of the manifold M_n of independent variables. (For problems in the calculus of variations in the large with constraints, we would have $v^i \neq 0$ because the domain B would also be subject to variation and the nonzero v^i would lead in a natural manner to transversality conditions for the constrained problem.) We therefore restrict attention to the collection of *vertical* vector fields ($V \lrcorner \alpha = 0$, $\forall \alpha \in \Lambda(M_n)$),

$$TV(K) = \{ V \in T(K) \mid V = v^\alpha \partial_\alpha + v_i^\alpha \partial_\alpha^i \}. \tag{3.4}$$

If Φ is a map from M_n to K and $V \in TV(K)$, then $\Phi_V(s) = T_V(s) \circ \Phi$ is a map from M_n to K that induces the pullback map

$$\Phi_V(s)^*: \quad \Lambda(K) \to \Lambda(M_n). \tag{3.5}$$

However, $\Phi_V(s)^* = (T_V(s) \circ \Phi)^* = \Phi^* \circ T_V(s)^*$ and [3] $T_V(s)^*\alpha = \exp(s£_V)\,\alpha$ for any $\alpha \in \Lambda(K)$, where $£_V\beta$ is the Lie derivative of β with respective to the vector field V. We therefore have

$$\Phi_V(s)^*\alpha = \Phi^* \exp(s£_V)\,\alpha \quad \forall \alpha \in \Lambda(K). \tag{3.6}$$

By definition, a map Φ: $M_n \to K$ belongs to the class of constraint maps $R_c(K)$ if and only if

$$\Phi^*\omega_a = 0, \qquad \Phi^*C^\alpha = 0, \qquad \Phi^*\mu \neq 0. \tag{3.7}$$

Thus, when (3.6) is used, $\Phi \in R_c(B)$ will imply $\Phi_V(s) \in R_c(B)$ if and only if

$$\begin{gathered} \Phi^* \exp(s£_V)\,\omega_a = 0, \qquad \Phi^* \exp(s£_V)\,C^\alpha = 0, \\ \Phi^* \exp(s£_V)\,\mu \neq 0. \end{gathered} \tag{3.8}$$

Now, $\Phi^* \exp(s£_V)\,\mu \neq 0$ for all s in a neighborhood of $s = 0$ if $\Phi^*\mu \neq 0$, so we may disregard the condition $\Phi^* \exp(s£_V)\,\mu \neq 0$ provided we restrict s to a sufficiently small neighborhood of $s = 0$.

Any $\Phi \in R_c(B)$ is such that Φ^* annihilates each C^α and each ω_a. Thus, since Φ^* commutes with exterior differentiation, Φ^* also annihilates each dC^α and each $d\omega_a$. Accordingly, Φ^* annihilates the *constraint ideal*

$$\mathscr{C} = I\{C^\alpha, \omega_a, dC^\alpha, d\omega_a \mid 1 \leqslant \alpha \leqslant m, 1 \leqslant a \leqslant r\} \tag{3.9}$$

of the exterior algebra $\Lambda(K)$ for any $\Phi \in R_c(B)$. Conversely, the ideal $\mathscr{C}$ is the largest ideal of $\Lambda(K)$ with generators of degree not exceeding n that is annihilated by $R_c(B)^*$. The conditions (3.8) will thus be satisfied for s in a sufficiently small neighborhood of $s = 0$ if

$$\exp(s£_V)\,C^\alpha \equiv 0 \bmod \mathscr{C}, \qquad \exp(s£_V)\,\omega_a \equiv 0 \bmod \mathscr{C}; \tag{3.10}$$

that is,

$$£_V C^\alpha \equiv 0 \bmod \mathscr{C}, \qquad £_V \omega_a \equiv 0 \bmod \mathscr{C}. \tag{3.11}$$

Now, Lie differentiation and exterior differentiation commute, and hence (3.11) imply $£_V dC^\alpha \equiv 0 \bmod \mathscr{C}$, $£_V d\omega_a \equiv 0 \bmod \mathscr{C}$. Thus, (3.8) will be satisfied if and only if $£_V \mathscr{C} \subset \mathscr{C}$; that is, if and only if $V \in TV(K)$ is an isovector [2, 3] of the constraint ideal. The collection of all *vertical isovectors of the constraint ideal* is denoted by $T_c(K)$,

$$T_c(K) = \{ V \in TV(K) \mid £_V \mathscr{C} \subset \mathscr{C} \}. \tag{3.12}$$

If $\Phi \in R_c(K)$, then $\Phi_V(s) = T_V(s) \circ \Phi \in R_c(K)$ for any $V \in T_c(K)$ and all s in a neighborhood of $s = 0$, $\Phi_V(0) = \Phi$, and $T_c(K)$ is the largest subset of $TV(K)$ for which $\Phi_V(s) \in R_c(B)$ for all s in a neighborhood of $s = 0$.

This result shows that the connectiviety of $R_c(B)$ is exactly that generated by the flows $\{T_V(s)\}$ for all vertical isovector fields V of the constraint ideal. Thus, if the constraints are to remain satisfied, *the only changes that are permitted in any $\Phi \in R_c(B)$ are those that are generated by composition with $T_V(s)$ for some $V \in T_c(K)$*. This, however, exactly serves our purposes, for we now know the most general change in a $\Phi \in R_c(B)$ for which the constraints will remain satisfied. Accordingly variations that are generated by flows of isovector fields of the constraint ideal will not require the use of Lagrange multipliers in order to accomodate the constraints! To see this directly, simply note that $\Phi \in R_c(B)$ implies $\Phi_V(s) \in R_c(B)$ for $V \in T_c(K)$, and hence $A[\Phi]$ and $A[\Phi_V(s)]$ are the values of the action functional for states that satisfy the constraints. Accordingly,

$$\Delta_V(s)\, A[\Phi] = A[\Phi_V(s)] - A[\Phi] \tag{3.13}$$

is the *finite variation* of the action functional in the presence of the constraints that is generated by $V \in T_c(K)$. *The* (infinitesimal) *variation of the action functional in the presence of constraints*, that is generated by $V \in T_c(K)$, is defined by

$$\delta_V A[\Phi] = \lim_{s \to 0} \left(\frac{1}{s} \Delta_V(s)\, A[\Phi] \right). \tag{3.14}$$

$\delta_V A[\Phi]$ is easily computed, for $\Phi_V(s) = T_V(s) \circ \Phi$ gives $\Phi_V(s)^* \alpha = \Phi^* \circ T_V(s)^* \alpha = \Phi^* \exp(s £_V)\, \alpha$. Thus, for the action functional (2.8), (3.13). and (3.14) yield

$$\delta_V A[\Phi] = \int_B \Phi^* £_V(L\mu) + \int_{\partial B} \Phi^* £_V(L^i \mu_i). \tag{3.15}$$

These considerations lead directly to the following definitions of constrained stationarity. A map Φ: $M_n \to K$ is said to render the action functional $A[\cdot]$ *stationary in the presence of constraints* if and only if $\Phi \in R_c(B)$ and

$$\delta_V A[\Phi] = 0 \tag{3.16}$$

for all $V \in T_c(K)$. Thus, *Φ renders the action functional* (2.8) *stationary in the presence of constraints if and only if*

$$\Phi^* C^\alpha = 0, \qquad \Phi^* \omega_a = 0, \qquad \Phi^* \mu \neq 0, \tag{3.17}$$

and

$$\int_B \Phi^* \pounds_V(L\mu) + \int_{\partial B} \Phi^* \pounds_V(L^i \mu_i) = 0 \tag{3.18}$$

for all $V \in T_c(K)$.

4. Characterization of $T_c(K)$

It is clear from the results of the previous section that everything depends on the structure of the collection $T_c(K)$ of all vertical isovectors of the constraint ideal. By definition (see (3.12)), $V \in T_c(K)$ if and only if $V \in TV(K)$ and $\pounds_V \mathscr{C} \subset \mathscr{C}$. Thus, since [3, 4]

$$\pounds_{fU+gV} \mathscr{C} = f\pounds_U \mathscr{C} + g\pounds_V \mathscr{C} + df \wedge U \,\lrcorner\, \mathscr{C} + dg \wedge V \,\lrcorner\, \mathscr{C},$$

$T_c(K)$ is a vector subspace of $TV(K)$ over $\mathbb{R}$ but not a submodule. Further, $\pounds_{[U,V]} \mathscr{C} = (\pounds_U \pounds_V - \pounds_V \pounds_U)\mathscr{C}$, and hence *$T_c(K)$ is a Lie subalgebra of $TV(K)$ over* $\mathbb{R}$ with the Lie product $[U, V] f = U(Vf) - V(Uf)$.

The constraint ideal, $\mathscr{C}$, is generated by $\{C^\alpha, \omega_a, dC^\alpha, d\omega_a\}$; and hence $V \in T_c(K)$ if and only if $\pounds_V C^\alpha \equiv 0 \mod \mathscr{C}$, $\pounds_V \omega_a \equiv 0 \mod \mathscr{C}$, $\pounds_V dC^\alpha \equiv 0 \mod \mathscr{C}$, $\pounds_V d\omega_a \equiv 0 \mod \mathscr{C}$. Thus, since $\mathscr{C}$ is a closed ideal and $\pounds_V$ and d commute, $\pounds_V C^\alpha \equiv 0 \mod \mathscr{C}$, $\pounds_V \omega_a \equiv 0 \mod \mathscr{C}$ imply $\pounds_V dC^\alpha \equiv 0 \mod \mathscr{C}$, $\pounds_V d\omega_a \equiv 0 \mod \mathscr{C}$. *A vector field $V \in TV(K)$ is an isovector of $\mathscr{C}$ if and only if* (see [6, Lemma 4-6.2]),

$$\pounds_V C^\alpha \equiv 0 \mod \mathscr{C}, \qquad 1 \leqslant \alpha \leqslant m, \tag{4.1}$$

$$\pounds_V \omega_a \equiv 0 \mod \mathscr{C}, \qquad 1 \leqslant a \leqslant r. \tag{4.2}$$

Next, we note that (2.3) gives $dq^\alpha = y_i^\alpha \, dx^i + C^\alpha$ and hence

$$dq^\alpha \equiv y_i^\alpha \, dx^i \mod \mathscr{C}, \qquad 1 \leqslant \alpha \leqslant m. \tag{4.3}$$

Thus, since the C^{α}'s and the ω_a's may be assumed to be independent without loss of generality, we may use (4.3) to eliminate all dq^{α}'s from the ω_a's. Accordingly, the ω_a's may be written in such a way that they do not contain terms with dq^{α}-factors. Finally, if any of the ω_a's are of degree zero, say $g_1,\ldots, g_t$, then they may be replaced by the equivalent n-forms $g_1\mu,\ldots, g_t\mu$. We may accordingly assume that $\deg(\omega_a) \geqslant 1$, $1 \leqslant a \leqslant r$.

If V is an arbitrary element of $TV(K)$, we have

$$V = v^{\alpha}\,\partial_{\alpha} + v_i^{\alpha}\,\partial_{\alpha}^{i},$$

and hence (2.3) gives $£_V C^{\alpha} = dv^{\alpha} - v_i^{\alpha}\,dx^i$. Noting that $\deg(C^{\alpha}) = 1$ for all α and $\deg(\omega_a) \geqslant 1$ for all a, (4.1) and (4.2) will be satisfied if and only if $(A_{\alpha}^{\beta}, B^{\alpha a}, F_{a\beta}, G_{a\beta}, H_a^b, K_a^b) \in \Lambda(K)$ can be found such that (see [6, Lemma 4-6.3]),

$$dv^{\alpha} - v_i^{\alpha}\,dx^i = A_{\beta}^{\alpha} C^{\beta} + B^{\alpha a}\omega_a, \qquad 1 \leqslant \alpha \leqslant m, \tag{4.4}$$

$$£_V \omega_a = F_{a\beta} \wedge C^{\beta} + G_{a\beta} \wedge dC^{\beta} + H_a^b \wedge \omega_b + K_a^b \wedge d\omega_b, \qquad 1 \leqslant a \leqslant r. \tag{4.5}$$

Here, it is understood that $B^{\alpha a} = 0$ if $\deg(\omega_a) \neq 1$ and that $F_{a\beta}$, $G_{a\beta}$, H_a^b, K_a^b are such that the degree of each term on the right-hand side of (4.5) is the same as the degree of the term on the left-hand side of (4.5). The Lie Algebra $T_c(K)$ is then determined by resolving (4.4), (4.5) on the basis elements of $\Lambda(K)$, solving for and then eliminating the quantities (A_{β}^{α}, $B^{\alpha a}$, $F_{a\beta}$, $G_{a\beta}$, H_a^b, K_a^b), and then securing satisfaction of the remaining equations. Although the resulting equations will always be linear, the calculations are often of monumental proportions. For example, if we have 4 independent and 4 state variables, then $\dim(K) = 24$. Thus, with one constraint form of degree 4 (a second order partial differential equation), the underlying vector space is $\Lambda^4(K)$ and $\dim(\Lambda^4(K)) = \binom{24}{4} = 10{,}626$. Programs for direct computer assistence with such computations, available in the REDUCE-2 symbolic language environment [3, 5], reduce computation of the generating equations for the isovectors to a morning's work at a computer terminal. Determinations of the isovector fields of a constraint ideal thus falls within the realm of the possible even for "large" problems.

The solution procedure described above is critically dependent upon the structure of the constraint forms $\{\omega_a\}$. Thus, each problem must be worked out anew. There are, however, situations in which a partial solution may be obtained. If all of the constraint forms ω_a are of degree greater then 1, then all of the $B^{\alpha a}$'s vanish in (4.4);

$$dv^{\alpha} - v_i^{\alpha}\,dx^i = A_{\beta}^{\alpha} C^{\beta} = (dq^{\beta} - y_i^{\beta}\,dx^i). \tag{4.6}$$

The general solution of (4.6) is a special case of the general solution of $\pounds_V C^\alpha = A^\alpha_\beta C^\beta$ with the conditions $v^i = 0$, that is given elsewhere [3],

$$v^\alpha = f^\alpha(x^j, q^\gamma), \qquad v^\alpha_i = (\partial_i + y^\beta_i \partial_\beta) f^\alpha(x^j, q^\gamma). \tag{4.7}$$

Accordingly, when (4.7) is substituted into (4.5), we obtain a system of equations for the determination of the remaining free functions $\{f^\alpha(x^j, q^\gamma) \mid 1 \leqslant \alpha \leqslant m\}$. Now, this determination may result in $V = v^\alpha \partial_\alpha + v^\alpha_i \partial_\alpha$ that is a finite dimensional subspace of $TV(K)$ over $\mathbb{R}$, or in a infinite dimensional subspace of $TV(K)$ over $\mathbb{R}$. In the latter case, there will be arbitrary functions that occur in the specification of $\{v^\alpha, v^\alpha_i\}$, while in the former there will be only scalar multiples of fixed functions of $\{x, q^\alpha, y^\alpha_i\}$:

The number of arbitrary functions and the number of arbitrary constants that occur in the general solution of (4.4), (4.5) will turn out to be highly significant in the process of securing stationarization of the action functional. We therefore give the following definitions. The *finite dimension*, $\dim(T_c(K))$, of the Lie algebra $T_c(K)$ is the number of arbitrary scalar parameters that occurs in the specification of an element of $T_c(K)$ in general position. The *transfinite dimension*, $\mathrm{Dim}_\infty(T_c(K))$, of $T_c(K)$ is the number of arbitrary functions that occurs in the specification of an element of $T_c(K)$ in general position. For example, if there are no constraint forms $\{\omega_a\}$, (4.7) shows that there are m arbitrary functions $\{f^\alpha(x^j, q^\beta) \mid 1 \leqslant \alpha \leqslant m\}$ that occur in $V \in T_c(K)$ and hence $\dim(T_c(K)) = 0$, $\mathrm{Dim}_\infty(T_c(K)) = m$. This is the case of problems in the calculus of variations without imposed constraints $\{\omega_a\}$. On the other hand, suppose that we find that the general element of $T_c(K)$ looks like

$$V = aq^1 x^3 \partial_{q^1} + f \partial_{q^2} + b \partial_{y^2_2} - (\partial_{x^2} f) \partial_{y^2_1},$$

where a and b are arbitrary scalar parameters and f is an arbitrary function of the x's. We would then have $\dim(T_c(K)) = 2$, $\mathrm{Dim}_\infty(T_c(K)) = 1$.

There is one very important point that should be noted at this juncture. *If $T_c(K)$ consists only of the zero vector*, $\dim(T_c(K)) = 0$, $\mathrm{Dim}_\infty(T_c(K)) = 0$, *then $R_c(B)$ consists solely of isolated elements.* Stationarization of the action functional in the presence of constraints is thus impossible; the action functional has whatever value it assumes at the isolated map $R_c(K)$.

5. Stationarity Conditions

We saw at the end of Section 3 that a map Φ: $M_n \to K$ renders the action functional stationary in the pressure of constraints if and only if $\Phi^* C^\alpha = 0$, $\Phi^* \omega_a = 0$, $\Phi^* \mu \neq 0$, and

$$\delta_V A[\Phi] = \int_B \Phi^* \pounds_V(L\mu) + \int_{\partial B} \Phi^* \pounds_V(L^i \mu_i) = 0 \tag{5.0}$$

for all $V \in T_c(K)$. The necessary and sufficient conditions for satisfaction of (5.0) for all $V \in T_c(K)$ are referred to as the *stationary conditions* and are the subject of this section.

It is clear from (5.0) that everything hinges on obtaining explicit evaluations of $\pounds_V(L\mu)$ and $\pounds_V(L^i\mu_i)$. Since the latter is the simpler, it will be taken up first. By definition, $\pounds_V\alpha = V \rfloor d\alpha + d(V \rfloor \alpha)$, and hence

$$\pounds_V(L^i\mu_i) = V \rfloor d(L^i\mu_i) + d(V \rfloor L^i\mu_i). \tag{5.1}$$

However, any $V \in T_c(K)$ belongs to $TV(K)$ so that

$$V \rfloor dx^i = 0, \qquad V \rfloor \mu = 0, \qquad V \rfloor \mu_i = 0. \tag{5.2}$$

Thus, (5.1) gives

$$\pounds_V(L^i\mu_i) = (v^\alpha\, \partial_\alpha L^i + v^\alpha_j\, \partial^j_\alpha L^i)\, \mu_i. \tag{5.3}$$

The computation of $\pounds_V(L\mu)$ is equally direct, but not of particular use in its raw form. We therefore note that any $V \in T_c(K)$ is a vertical isovector field of the constraint ideal while any $\Phi \in R_c(B)$ is such that Φ^* annihilates the constraint ideal $\mathscr{C}$. We therefore have $\Phi^*\pounds_V(L\mu) = \Phi^*\pounds_V(L\mu + \rho)$ for any n-form ρ that belongs to the constraint ideal and for any $V \in T_c(K)$. Now,

$$J = C^\alpha \wedge \partial^i_\alpha L\mu_i \in \mathscr{C} \tag{5.4}$$

and a straightforward calculation shows that

$$d(L\mu + J) = C^\alpha \wedge E_\alpha, \tag{5.5}$$

where

$$E_\alpha = \partial_\alpha L\mu - d(\partial^i_\alpha L\mu_i), \qquad 1 \leqslant \alpha \leqslant m \tag{5.6}$$

are the Euler–Lagrange n-forms. In fact, $\Phi^*E_\alpha = 0$ are the Euler–Lagrange equations if there are no constraints. These considerations allow us to replace $L\mu$ by the Cartan n-form $L\mu + J$ in the computations which will avoid the usual problem of having to perform integrations by parts of a number of terms.

Direct application of the definition of $\pounds_V$ gives

$$\pounds_V(L\mu + J) = V \rfloor d(L\mu + J) + d(V \rfloor (L\mu + J)). \tag{5.7}$$

Use of (5.2), (5.4) and (5.5) thus gives

$$\pounds_V(L\mu + J) = (V \rfloor C^\alpha)\, E_\alpha + d\{(V \rfloor C^\alpha)\, \partial^i_\alpha L\mu_i\} \bmod \mathscr{C}. \tag{5.8}$$

However, $V \,\lrcorner\, C^\alpha = v^\alpha$ for any $V \in TV(K)$ and hence

$$£_V(L\mu + J) = v^\alpha E_\alpha + d\{v^\alpha\, \partial^i_\alpha L\mu_i\} \bmod \mathscr{C}. \tag{5.9}$$

Noting that $\Phi^*\mathscr{C} = 0$ for any $\Phi \in R_c(B)$, a substitution of (5.3) and (5.9) into (5.1) and use of Stokes' theorem shows that

$$\delta_V A[\Phi] = \int_B \Phi^*(v^\alpha E_\alpha) + \int_{\partial B} \Phi^*\{v^\alpha(\partial^i_\alpha L + \partial_\alpha L^i) + v^\alpha_j\, \partial^j_\alpha\, L^i\}\, \mu_i. \tag{5.10}$$

A map $\Phi \in R_c(K)$ renders $A[\Phi]$ stationary in the presence of constraints if and only if

$$\int_B \Phi^*(v^\alpha E_\alpha) + \int_{\partial B} \Phi^*\{v^\alpha(\partial^i_\alpha L + \partial_\alpha L^i) + v^\alpha_j\, \partial^j_\alpha L^i\}\, \mu_i = 0 \tag{5.11}$$

is satisfied for all $V \in T_c(K)$.

Particular note should be made of the fact that $\partial^j_\alpha L^i \neq 0$ implies requirements on the boundary values of the functions v^α_j that determine the variations of the derivatives. On the other hand, if $\partial^j_\alpha L^i = 0$, evaluation of (5.11) will only require knowledge of the functions $v^\alpha(x^j, q^\beta, y^\beta_j)$ that appear in

$$V = v^\alpha\, \partial_\alpha + v^\alpha_i\, \partial^i_\alpha.$$

We therefore confine our attention, from now on, to problems for which $\partial^j_\alpha L^i = 0$; that is $L^i = L^i(x^j, q^\beta)$. Accordingly, it is sufficient to give only the part

$$V_q = v^\alpha\, \partial_\alpha$$

of any $V \in T_c(K)$ in order to evaluate the stationarity conditions (5.11). Incidentally, if $\deg(\omega_a) > 1$, $1 \leqslant a \leqslant r$, then V_q determines V by [6, Chap. 6],

$$V = V_q + Z_i(V_q \,\lrcorner\, C^\alpha)\, \partial^i_\alpha, \qquad Z_i = \partial_i + y^\beta_i\, \partial_\beta.$$

Progress past this point is inseparably linked to complete specification of $T_c(K)$, and hence a general procedure can not be given. Each problem must be analyzed in its own right. We therefore give three examples in the case $m = 2$ that are, to a certain extent, typical.

The first example is that for which $T_c(K)$ is specified by

$$V_q = f(x^j)\, \partial_{q^1} + (q^2 - x^2)\, f(x^j)\, \partial_{q^2}, \tag{5.12}$$

where $f(x^i)$ is an arbitrary function. Thus, $\text{Dim}_\infty(T_c(K))=1$, $\dim(T_c(K))=0$. When (5.12) is substituted into (5.11), we have

$$\int_B f\Phi^*(E_1+(q^2-x^2)\,E_2)$$

$$=-\int_{\partial B} f\Phi^*\{(\partial_1^i L+\partial_{q^1}L^i)+(q^2-x^2)(\partial_2^i L+\partial_{q^2}L^i)\}\,\mu_i \qquad (5.13)$$

for all $f(x^j)$. In order that this shall hold for all $f(x^j)$, it must hold for those $f(x^j)$ that vanish on the boundary. In this event, (5.13) reduces to $\int_B f\Phi^*(E_1+(q^2-x^2)\,E_2)=0$ and the fundamental lemma of the calculus of variations gives

$$\Phi^*(E_1+(q^2-x^2)\,E_2)=0. \qquad (5.14)$$

When $\Phi\in R_c(K)$ is such that (5.14) is satisfied, (5.13) reduces to

$$\int_{\partial B} f\Phi^*\{(\partial_1^i L+\partial_{q^1}L^i)+(q^2-x^2)(\partial_2^i L+\partial_{q^2}L^i)\}\,\mu_i=0. \qquad (5.15)$$

Thus, the stationarity conditions are (5.14) subject to any boundary data that satisfied (5.15).

For the second example, $T_c(K)$ is specified by

$$V_q=aF(x^j,q^\beta,y_i^\beta)\,\partial_{q^1}+aG(x^j,q^\beta,y_j^\beta)\,\partial_{q^2}, \qquad (5.16)$$

where F and G are given specific functions of their indicated arguments. Thus, $\text{Dim}_\infty(T_c(K))=0$, $\dim(T_c(K))=1$. We now substitute (5.16) into (5.11) to obtain

$$a\int_B \Phi^*(FE_1+GE_2)$$

$$=-a\int_{\partial B}\Phi^*\{F(\partial_1^i L+\partial_{q^1}L^i)+G(\partial_2^i L+\partial_{q^2}L^i)\}\,\mu_i. \qquad (5.17)$$

In order that (5.17) hold for all values of a, it is necessary and sufficient that $\Phi\in R_c(K)$ satisfy the single integral condition

$$\int_B \Phi^*(FE_1+GE_2)$$

$$=-\int_{\partial B}\Phi^*\{F(\partial_1^i L+\partial_{q^1}L^i)+G(\partial_2^i L+\partial_{q^2}L^i)\}\,\mu_i. \qquad (5.18)$$

In fact, a little reflection will show that $\mathrm{Dim}_\infty(T_c(K)) = 0$ will result in $\dim(T_c(K))$ integral conditions rather than a system of Euler–Lagrange equations.

The third example is where $T_c(K)$ is specified by

$$V_q = f(x^j)(q^1 - y_2^2)\,\partial_{q^1} + aF(x^j, q^\beta, y_j^\beta)\,\partial_{q^2}, \tag{5.19}$$

where $f(x^j)$ is an arbitrary function of the x^i's and F is a given specific function of its indicated arguments. An analysis similar to that given above shows that the stationarity conditions are

$$\Phi^*\{(q^1 - y_2^2)\,E_1\} = 0, \tag{5.20}$$

$$\int_B \Phi^*\{FE_2\} = -\int_{\partial B} \Phi^*\{F(\partial_2^i L - \partial_{q^2} L^i)\}\,\mu_i, \tag{5.21}$$

subject to any system of boundary conditions for which

$$\int_{\partial B} f\Phi^*\{(q^1 - y_2^2)(\partial_1^i L - \partial_{q^1} L^i)\}\,\mu_i = 0. \tag{5.22}$$

6. Applications to Control Problems

Application to control problems with differential constraints is immediate. All that is required is to take the realization of Φ: $M_n \to K$ so that $\Phi^* q^\alpha = \phi^\alpha(x^j)$ for $1 \leqslant \alpha \leqslant m_1 < m$, $\Phi^*(q^{m_1+\beta}) = u^\beta$ for $1 < \beta < m - m_1$, where $\{\phi^\alpha(x^j)\}$ are the state variables and $\{u^\beta(x^j)\}$ are the control variables. The constraints $\{\omega_a \mid 1 \leqslant a < r\}$ then describe the evolution of the state in the presence of control and $A[\Phi]$ is the penalty functional for the control process. The admissible set of control laws for the process is obtained by solving the conditions of stationarity of $A[\Phi]$ in the presence of constraints. If this is done in the manner discussed above, there are no Lagrange multipliers and the equations governing the admissible set of control laws are directly obtained without having to solve for and then eliminate the Lagrange multipliers.

The simplest way of seeing what is involved is to look at specific examples. We therefore consider problems with two independent variables ($n = 2$), one state variable ϕ, and one control variable u ($m = 2$ with $\Phi^* q^1 = \phi$, $\Phi^* q^2 = u$). In such cases, it is simplest to use the variables (ϕ, u) rather than (q^1, q^2). Thus, (y_1^1, y_2^1) represent the first derivatives of ϕ when Φ^* acts, while (y_1^2, y_2^2) represent the first derivatives of the control variable u.

A reasonable control problem is that for which the constraint, $\Phi^*(\omega_1)=0$, is

$$\frac{\partial^2\phi}{\partial x^2}=\frac{\partial u}{\partial t}\frac{\partial\phi}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial\phi}{\partial t}, \tag{6.1}$$

that is,

$$\omega_1 = du \wedge d\phi - dt \wedge dy_1^1. \tag{6.2}$$

A specific physical realization of (6.1) obtains when ϕ undergoes 1-dimensional convective diffusion in a compressible fluid medium with density ρ, 1-dimensional fluid velocity v_x, and

$$\rho = -\partial u/\partial x, \qquad \rho v_x = \partial u/\partial t$$

so that the continuity equation $\partial_t\rho+\partial_x(\rho v_x)=0$ is satisfied. The control u is thus realizable through appropriate realizations of the 1-dimensional compressible fluid flow.

The first thing we note is that the constraint form ω_1 is a 2-form. Accordingly, (4.6) and (4.7) show that $V=v^\alpha\,\partial_\alpha+v_i^\alpha\,\partial_\alpha^i$ must be such that $v_i^\alpha=(\partial_i+y_i^\beta\,\partial_\beta)\,v^\alpha(x,t,\phi,u)$ to secure satisfaction of $£_V C^\alpha\subset\mathscr{C}$. Here, $C^1=d\phi-y_1^1\,dx-y_2^1\,dt$, $C^2=du-y_1^2\,dx-y_2^2\,dt$ and $\mu=dx\wedge dt$, $\mu_1=dt$, $\mu_2=-dx$. Straightforward but lengthy calculation shows that $£_V\omega_1\subset\mathscr{C}$ if and only if

$$V_q=(a\phi+b)\,\partial_\phi+f(\phi)\,\partial_u, \tag{6.3}$$

where (a,b) are arbitrary constants and $f(\phi)$ is an arbitrary function of the variable ϕ. Thus $\dim(T_c(K))=2$ and $\mathrm{Dim}_\infty(T_c(K))=1$.

It is now simply a matter of substituting (6.3) into (5.11) and recall that $\{L^i\}$ are assumed such that $\partial_\alpha^j L^i=0$, to obtain the following stationary conditions:

$$\int_B \phi\Phi^*E_\phi+\int_{\partial B}\phi\Phi^*(\partial_1^i L+\partial_\phi L^i)\,\mu_i=0 \tag{6.4}$$

from the parameter a,

$$\int_B \Phi^*E_\phi+\int_{\partial B}\Phi^*(\partial_1^i L+\partial_\phi L^i)\,\mu_i=0 \tag{6.5}$$

from the parameter b, and

$$\int_B f(\phi)\,\Phi^*E_u+\int_{\partial B}f(\phi)\,\Phi^*(\partial_2^i L+\partial_u L^i)\,\mu_i=0 \tag{6.6}$$

for all smooth functions $f(\phi)$. Thus, if we set $f(\phi)=\sum c_n\phi^n$, (6.6) gives

$$\int_B \phi^n \Phi^* E_u + \int_{\partial B} \phi^n \Phi^*(\partial_2^i L + \partial_u L^i)\,\mu_i = 0, \qquad n = 0, 1, 2, \ldots \tag{6.7}$$

Here L and L^i are the functions that serve to determine the penalty functional $A[\Phi]=\int_B \Phi^*(L\mu)+\int_{\partial B}\Phi^*(L^i\mu_i)$ and (E_ϕ, E_u) are the Euler–Lagrange 2-forms for the pair (ϕ, u), respectively (see (5.6)). *The problem of stationarizing* $A[\Phi]$ *in the presence of the constraint* (6.1) *is thus solved by finding all pairs* $(\phi(x,t), u(x,t))$ *that satisfy the constraint* (6.1) *and the integral conditions* (6.4), (6.5), *and* (6.7). It is interesting to note in this regard that (6.7) will be identically satisfied if (L, L^i) do not depend explicitly on the control variable, as is often the case. In this event, we would then have only the constraint (6.1) and the two integral conditions (6.4) and (6.5).

The reason why such direct results obtain is that we have been able to characterize all continuous deformations of K that preserve satisfaction of the constraint equation (6.1), even though explicit solution of the constraint equation has not been obtained. Comparison for purposes of stationarization of $A[\Phi]$ then occurs only in this class of deformations, for only this class guarantees that the constraints will be satisfied for $\Phi_V(s)$ if they are satisfied for $\Phi_V(0)=\Phi$. Accordingly, since (6.3) shows that all such deformations are generated by $\phi\partial_\phi$, ∂_ϕ, and $f(\phi)\,\partial_u$ for all smooth functions $f(\phi)$, integral conditions rather than field equations result; there are no arbitrary functions of (x, t) for which the fundamental lemma of the calculus of variations may be used.

For a second example, consider the situation in which the constraint equation is

$$\frac{\partial}{\partial x}\left(u\frac{\partial\phi}{\partial x}\right)+\frac{\partial}{\partial t}\left(u\frac{\partial\phi}{\partial t}\right)=1$$

$$(\omega_1 = d(uy_1^1\,dt - uy_2^1\,dx) - dx \wedge dt) \tag{6.8}$$

subject to the Dirichlet boundary data

$$\phi|_{\partial B}=0. \tag{6.9}$$

Here, ϕ is the state variable, u is the control variable, and the notation is the same as that in the previous example. Disregarding for the moment the boundary data (6.9), a lengthy but straighforward calculation shows that

$V_q = R(x, t, \phi, u)\,\partial_\phi + S(x, t, \phi, u)\,\partial_u$ generates an element of $T_c(K)$ if and only if

$$\begin{aligned} \partial_u R = \partial_x R = \partial_t R = 0, &\qquad \partial_\phi S + u\,\partial_\phi\,\partial_\phi R = 0, \\ u\,\partial_u S - S = 0, &\qquad \partial_x S + 2u\,\partial_\phi \partial_x R = 0, \\ \partial_t S + 2u\,\partial_\phi\,\partial_t R = 0, &\qquad u\,\partial_\phi R + u^2(\partial_x\,\partial_x + \partial_t\,\partial_t)\,R + S = 0. \end{aligned}$$

Thus, all elements of $T_c(K)$ are generated by

$$V_q = f(\phi)\,\partial_\phi - u\,\frac{df(\phi)}{d\phi}\,\partial_u, \tag{6.10}$$

where $f(\phi)$ is an arbitrary smooth (C^∞) function of its indicated argument. We therefore have $\dim(T_c(K)) = 0$, $\mathrm{Dim}_\infty(T_c(K)) = 1$. Indeed, (6.10) defines an infinite dimensional Lie algebra, for

$$[X(f), X(g)] = X\left(f\frac{dg}{d\phi} - g\frac{df}{d\phi}\right),$$

where $X(f)$ is the continuum of operators

$$X(f) = f\,\partial_\phi - u\,\frac{df}{d\phi}\,\partial_u.$$

The deformations generated by (6.10) are all deformations of K that preserve satisfaction of the differential constraint (6.8). Accordingly, if we require

$$f(0) = 0, \tag{6.11}$$

then all such deformations will also preserve satisfaction of the Dirichlet boundary data (6.9). Thus, (6.10) and (6.11) define all deformations that preserve the constraints (6.8) and (6.9).

The stationarity conditions now follow directly from (5.11) for any $A[\Phi]$ for which $\partial^i_\alpha L^j = 0$:

$$\begin{aligned} 0 = &\int_B \left\{f(\phi)\,\Phi^* E_\phi - u\,\frac{df(\phi)}{d\phi}\,\Phi^* E_u\right\} \\ &+ \int_{\partial B} \{f(\phi)\,\Phi^*(\partial^i_1 L + \partial_\phi L^i) \\ &- u\,\frac{df(\phi)}{d\phi}\,\Phi^*(\partial^i_2 L + \partial_u L^i)\}\,\mu_i. \end{aligned} \tag{6.12}$$

If we formally write $f(\phi)=\sum_1^\infty C_n\phi^n$, in view of (6.11), then (6.12) and the boundary conditions (6.9) give

$$0=\int_B \{\phi\Phi^*E_\phi - u\Phi^*E_u\} - \int_{\partial B} u\Phi^*(\partial_2^i L + \partial_u L^i)\,\mu_i \qquad (6.13)$$

for C_1 and

$$0=\{\phi^n\Phi^*E_\phi - nu\phi^{n-1}\Phi^*E_u\}, \qquad n=2,3,..., \qquad (6.14)$$

for $C_2, C_3,...$. Again, we obtain an infinite system of integral conditions rather than Euler–Lagrange field equations with Lagrange multipliers.

In addition to providing access to the stationarity conditions, knowledge of $T_c(K)$ provides a significant amount of information about the nature of solutions of the constraint equations. If we take $f(\phi)=\phi^{n+1}$ for n a positive integer, an isovector field of the constraints (6.8), (6.9) is generated by

$$V_q=\phi^{n+1}\,\partial_\phi-(n+1)\,u\phi^n\,\partial_u.$$

The flow of this isovector field is obtained by solving $d\Phi(s)/ds=\Phi(s)^{n+1}$, $dU(s)/ds=-(n+1)\,U(s)\,\Phi(s)^n$ subject to the initial data $\Phi(0)=\phi$, $U(0)=u$;

$$\Phi=\phi(1-ns\phi^n)^{-1/n}, \qquad U=u|1-ns\phi^n|^{(n+1)/n}. \qquad (6.15)$$

Thus, if $(\phi(x,t), u(x,t))$ is a solution of (6.8) subject to the Dirichlet data (6.9), then (6.15) defines a 1-parameter family of solutions $(\Phi(x,t;s), U(x,t;s))$ of (6.8), (6.9). The pathology inherent in solutions of the constraints is in clear evidence from the first of (6.15), although the second shows that $U(x,t;s)$ remains bounded for bounded $(\phi(x,t), u(x,t))$.

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