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An Iterative Recurrence Formula

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The recurrence $G(0)=0$, $G(n)=n-G(G(n-1))$ $(n \geqslant 1)$, is shown to have the simple solution $G(n)=[(n+1)a]$, where $a=(\sqrt{5}-1)/2$. Generalizations are disscussed. © 1986 Academic Press, Inc.

The following recurrence formula appears in "Gödel, Escher, Bach" by Douglas R. Hofstadter, pp. 135–137 (Basic Books, New York, 1979). Let $G(n)$ be defined by

$$G(0)=0, \tag{1}$$

$$G(n)=n-G(G(n-1)) \qquad (n \geqslant 1). \tag{2}$$

Can we find a closed form for $G(n)$?

It is easily proved that $0 \leqslant G(n) \leqslant n$ and that G is monotonic non-decreasing. In fact, $G(n+1)-G(n)=0$ or 1. A short run on my baby computer suggested that $G(n)$ increases rather regularly. If we assume that $G(n) \approx an$ for a constant **a**, then (2) (for large **n**) shows that $a=1-a^2$, so $a=(\sqrt{5}-1)/2$.

A longer run, calculating $\Delta(n)=G(n)-an$, seemed to indicated that $\max \Delta(n)-\min \Delta(n)<1$. A little further experimentation suggested that $-1<\Delta(n)-a<0$, so we were led to conjecture that

$$G(n)=[a(n+1)] \qquad (n \geqslant 0) \tag{3}$$

($[\ ]$ = greater integer).

When we showed this to Harley Flanders, he ran it up on his grown-up computer to $n=15{,}000$. It coughed up four cases where (3) was violated. But calculating **a** to a few more decimal places remedied those exceptions. Thus convinced of the truth of (3), we both went home that night and proved it, rather easily.

The natural generalization of (2),

$$G(n)=n-G_k(n-1), \tag{4}$$

where G_k is the kth iterate of G, proved to be a slight disappointment.

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With a = the root of $a^k + a = 1$ lying between 0 and 1, let $D(n) = a(n+1) - G(n)$. Then for $k = 3$, max $D(n)$ − min $D(n) > 1$, so no formula of the form

$$G(n) = [an + C],$$

with C constant, can be true. Nevertheless, the values of $D(n)$ stay remarkably small, and a very plausible conjecture is that $D(n)$ is bounded.

The following generalization does work, however.

THEOREM. *Let r be a positive integer, and let G be defined by $G(0) = 0$ and*

$$G(n) = n - \left[\frac{1}{r} G(G(n-1))\right] \qquad (n \geqslant 1). \tag{5}$$

Let a be the positive root of $a^2/r + a - 1 = 0$. Then

$$G(n) = [a(n+1)] \qquad (n \geqslant 0). \tag{6}$$

Proof. Let $F(n) = [a(n+1)]$, and define

$$S(n) = F(n) + \left[\frac{1}{r} F(F(n-1))\right] \qquad (n \geqslant 1).$$

Since $F(0) = 0$, it will be sufficient to prove that $S(n) = n$, for then F will satisfy (5).

Let $an = J + \theta$, where J is an integer and $0 < \theta < 1$. Then $[an] = J$ and

$$F(n) = [an + a] = J + [\theta + a],$$

$$\left[\frac{1}{r} F(F(n-1))\right] = \left[\frac{1}{r}[aJ + a]\right] = \left[\frac{a}{r}(J+1)\right];$$

we have used the fact that if **r** is a positive integer, **x** real, then

$$\left[\frac{1}{r}[x]\right] = \left[\frac{x}{r}\right].$$

Now

$$\begin{aligned}\frac{a}{r}(J+1) &= \frac{a}{r} + \frac{a}{r}(an - \theta)\\ &= \frac{a}{r}(1-\theta) + (1-a)\,n\\ &= \frac{a}{r}(1-\theta) + n - J - \theta.\end{aligned}$$

Hence $S(n)=n+[a+\theta]+[T]$, where

$$T=\frac{a}{r}(1-\theta)-\theta.$$

Clearly $T<a/r<1$ and $T>-\theta>-1$, so $[T]=0$ if $T\geqslant 0$ and $[T]=-1$ if $T<0$. But

$$\begin{aligned} T&=\frac{a}{r}-\theta\left(1+\frac{a}{r}\right)\\ &=\left(1+\frac{a}{r}\right)\left(\frac{a}{a+r}-\theta\right)\\ &=\left(1+\frac{a}{r}\right)(1-(a+\theta)). \end{aligned}$$

Thus, if $[a+\theta]=0$, $a+\theta<1$, $T>0$ and $[T]=0$. If $[a+\theta]=1$, $a+\theta>1$, $T<0$ and $[T]=-1$. In both cases, $[a+\theta]+[T]=0$, so $S(n)=n$. This completes the proof.

If we replace $1/r$ in (5) by an arbitrary $b<1$, the result (6) (with appropriate **a**) fails in general. Here, again, the approximation is remarkably close, and we conjecture that $G(n)-an$ is bounded.