Nearly uniformly noncreasy Banach spaces

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Received 3 August 2004
Available online 9 February 2005
Submitted by W.A. Kirk

Abstract

We introduce and study the class of nearly uniformly noncreasy Banach spaces. It is proved that they have the weak fixed point property. A stability result for this property is obtained.

Keywords: Nearly uniformly noncreasy spaces; Weak fixed point property; Banach–Mazur distance

1. Introduction

Uniform convexity and uniform smoothness are basic notions of the geometry of Banach spaces with numerous applications to the fixed point theory. A new geometrical property was introduced in [26]. It can be seen as a combination of uniform convexity and uniform smoothness. Spaces with this property are called uniformly noncreasy. It was shown that they have the fixed point property for nonexpansive mappings. This result was consecutively generalized in [9,22], [5], and [4].

The infinite-dimensional counterparts of uniform convexity and uniform smoothness were studied in [23]. They also have many applications in the metric fixed point theory (see [21]). In this paper we consider a class of spaces which we call nearly uniformly noncreasy. In their definition we follow the idea from [26], but this time we combine the

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concepts of infinite-dimensional uniform convexity and uniform smoothness. This gives us a class of spaces larger than the class of uniformly noncreasy spaces. It contains also the spaces considered in [4,5,9,22], all nearly uniformly convex spaces (see [11]), and all nearly uniformly smooth spaces (see [25]). The name “noncreasy” has its origin in the notion of a crease of the unit sphere which was introduced in [26]. We show that an infinite counterpart of this notion is strongly related to our property. We also find a generalization of the modulus of uniform noncreasiness introduced in [9].

In the last section we use a modification of the ultrapower technique to prove that nearly uniformly noncreasy spaces have the weak fixed point property. As a corollary, we obtain a result concerning stability of this property with respect to the Banach–Mazur distance. We show that our result is essentially stronger than the previous results in this direction. However, it does not improve the stability constants for $l_p$ spaces and in particular for Hilbert spaces.

2. Nearly uniformly noncreasy spaces

In this paper we will consider real Banach spaces. However, the case of complex spaces requires only minor changes. Let $X$ be a Banach space. By $B_X$ and $S_X$ we denote the closed unit ball and the unit sphere of $X$, respectively. Given a weakly convergent sequence $(x_n)$ in $X$, by $w\lim_{n\to\infty} x_n$ we denote its weak limit.

In [26] the following definitions were formulated. Given two functionals $x^*, y^* \in S_{X^*}$ and a scalar $\delta \in [0, 1]$, we put

$$S(x^*, y^*, \delta) = \{x \in B_X: \min\{x^*(x), y^*(x)\} \geq 1 - \delta\}.$$ 

A Banach space $X$ is uniformly noncreasy (UNC for short) provided that for every $\epsilon > 0$ there is $\delta \in (0, 1]$ such that if $x^*, y^* \in S_{X^*}$ and $\|x^* - y^*\| \geq \epsilon$, then

$$\text{diam } S(x^*, y^*, \delta) \leq \epsilon.$$ 

UNC spaces can be also characterized in terms of local moduli of convexity and smoothness. Let $x \in S_X$ and $t \geq 0$. We set

$$\delta(t, x) = \inf_{y \in S_X} \max\{\|x + ty\|, \|x - ty\|\} - 1$$

and

$$\rho(t, x) = \sup_{y \in S_X} \frac{1}{2} (\|x + ty\| + \|x - ty\|) - 1.$$ 

The first formula gives the local version of a modulus defined in [23]. It is strongly related to uniform convexity (see [6]). The second one gives the local version of the well-known modulus of smoothness introduced in [20].

**Theorem 1** [26]. A Banach space $X$ is UNC if and only if for every $\epsilon > 0$ there exists $t > 0$ such that for every $x \in S_X$ it is the case that $\delta(\epsilon, x) \geq t$ or $\rho(t, x) \leq \epsilon t$.

The condition appearing in Theorem 1 is the starting point for the main definition of this paper. Namely, we will replace the moduli $\delta$ and $\rho$ by the functions $d$ and $b$ which were
introduced in [21]. Before formulating their definitions let us recall that a Banach space $X$ has the Schur property if weak convergence of a sequence in $X$ implies norm convergence. Finite-dimensional spaces have this property and the same is true for $l_1$. On the other hand, using Rosenthal’s Theorem (see [28]), one can show that if an infinite-dimensional Banach space $X$ has the Schur property, then $X$ contains an isomorphic copy of $l_1$.

Assume now that a space $X$ lacks the Schur property. Then the family $N_X$ of all weakly null sequences $(x_n)$ in $S_X$ is nonempty. Given $\epsilon \geq 0$ and $x \in X$, we put

$$d(\epsilon, x) = \inf_{(y_m) \in N_X} \limsup_{m \to \infty} \|x + \epsilon y_m\| - \|x\|$$

and

$$b(\epsilon, x) = \sup_{(y_m) \in N_X} \liminf_{m \to \infty} \|x + \epsilon y_m\| - \|x\|.$$ 

To avoid confusion when dealing with different spaces, we will in some cases add the name of a space as a subscript to the name of a modulus.

In case $x \in S_X$, the moduli $d$ and $b$ coincide with those studied in [21]. It was shown that $d$ is strongly related to nearly uniform convexity introduced in [11], which is an infinite-dimensional counterpart of uniform convexity. Namely, a space $X$ is nearly uniformly convex if and only if $X$ is reflexive and $\inf_{x \in S_X} d(\epsilon, x) > 0$ for every $\epsilon > 0$. The dual property is called nearly uniform smoothness (see [25]). A space $X$ is nearly uniformly smooth if and only if $X$ is reflexive and

$$\lim_{\epsilon \to 0^+} \left( \frac{1}{\epsilon} \sup_{x \in S_X} b(\epsilon, x) \right) = 0.$$ 

**Definition.** Let $X$ be a Banach space without the Schur property. We say that $X$ is nearly uniformly noncreasy (NUNC for short) if for every $\epsilon > 0$ there is $t > 0$ such that for every $x \in S_X$ it is the case that $d(\epsilon, x) \geq t$ or $b(t, x) \leq \epsilon t$. Additionally, we treat spaces with the Schur property as being NUNC.

Clearly, the class of NUNC Banach spaces contains all nearly uniformly convex spaces and all nearly uniformly smooth spaces. Later we shall show that it contains also all UNC spaces.

Consider now the space $X = (\mathbb{R} \oplus c_0)_{l_1}$, i.e. the product $\mathbb{R} \times c_0$ endowed with the norm

$$\|(a, u)\| = |a| + \|u\|_{c_0}$$

where $a \in \mathbb{R}$ and $u \in c_0$. This space is not reflexive, so it is not NUC, nor NUS, nor UNC. However, $X$ is NUNC. Indeed, it is easy to see that if $x = (a, u) \in X$, then

$$d_X(\epsilon, x) = b_X(\epsilon, x) = \max\{\|u\|_{c_0}, \epsilon\} - \|u\|_{c_0}$$

for every $\epsilon \geq 0$. Given $\epsilon > 0$, we therefore have $d_X(\epsilon, x) = \epsilon - \|u\|_{c_0} \geq \epsilon/2$ whenever $\|u\|_{c_0} \leq \epsilon/2$ and $b_X(\epsilon/2, x) = 0$ whenever $\|u\|_{c_0} > \epsilon/2$.

Let $(x_n)$ be either a finite or infinite sequence in a Banach space $X$. We put $\text{sep}(x_n) = \inf_{n \neq m} \|x_n - x_m\|$. Next, given a nonempty bounded subset $A$ of $X$, by $\beta(A)$ we denote the separation measure of noncompactness of $A$, i.e. $\beta(A) = \sup\{\text{sep}(x_n)\}$ where the supremum is taken over all infinite sequences $(x_n)$ in $A$ (see [1]). In the definition of UNC
spaces, the sets \( S(x^*, y^*, \delta) \) are used. We shall define their counterparts for NUNC spaces. Let \((x^*_n)\) be a bounded sequence in \( X^* \) and \( \alpha \in \mathbb{R} \). We set

\[
S\left((x^*_n), \alpha\right) = \left\{ x \in B_X : \liminf_{n \to \infty} x^*_n(x) \geq \alpha \right\}.
\]

Let \( X \) be a Banach space without the Schur property and \( \epsilon \in [0, 1] \). We define

\[
\Delta_X(\epsilon) = \inf \{ 1 - x^*(x) \}
\]

where the infimum is taken over all elements \( x \in X \) which are weak limits of sequences \((x_n)\) in \( B_X \) with \( \liminf_{n \to \infty} \|x_n - x\| \geq \epsilon \) and all elements \( x^* \in X^* \) which are weak* limit points of sequences \((x^*_n)\) in \( B_{X^*} \) with \( \liminf_{n \to \infty} \|x^*_n - x^*\| \geq \epsilon \). Existence of such sequences follows form the assumption that \( X \) lacks the Schur property and the Josefson–Nissenzweig Theorem (see [14] or [24]).

Observe that in the definition of \( \Delta_X(\epsilon) \) one can replace the unit balls \( B_X \) and \( B_{X^*} \) by the unit spheres \( S_X \) and \( S_{X^*} \), respectively. Indeed, let \( x \in X \) be a weak limit of a sequence \((x_n)\) in \( B_X \) with \( \liminf_{n \to \infty} \|x_n - x\| \geq \epsilon \) and \( x^* \in X^* \) be a weak* limit point of a sequence \((x^*_n)\) in \( B_{X^*} \) with \( \liminf_{n \to \infty} \|x^*_n - x^*\| \geq \epsilon \). Passing to an appropriate subspace if necessary, we can assume that \( X \) is separable and \((x^*_n)\) converges weakly* to \( x^* \). We can also assume that \( x^*(x), \|x_n\|, \|x^*_n\| \) are positive and the limits \( a = \lim_{n \to \infty} \|x_n\| \) and \( b = \lim_{n \to \infty} \|x^*_n\| \) exist. Then \( a, b > 0 \), the sequence \((x_n/\|x_n\|)\) converges weakly to \( x/a \), and the sequence \((x^*_n/\|x^*_n\|)\) converges weakly* to \( x^*/b \). Moreover, \( \liminf_{n \to \infty} \|x_n/\|x_n\| - x/a\| \geq \epsilon/a \geq \epsilon \), \( \liminf_{n \to \infty} \|x^*_n/\|x^*_n\| - x^*/b\| \geq \epsilon/b \geq \epsilon \), and \( 1 - x^*(x) \geq 1 - x^*(x)/(ab) \).

As an example, let us consider the space \( l_p \) with \( 1 < p < \infty \). Then

\[
\Delta_{l_p}(\epsilon) = 1 - (1 - \epsilon^p)^{1/p}(1 - \epsilon^q)^{1/q}
\]

for every \( \epsilon \in [0, 1] \), where \( 1/p + 1/q = 1 \). Indeed, let \((e_n)\) be the standard basis of \( l_p \).

Considering the sequence of vectors \( x_n = (1 - \epsilon^p)^{1/p}e_1 + \epsilon e_n \) in \( l_p \) and the sequence of vectors \( x^*_n = (1 - \epsilon^q)^{1/q}e_1 + \epsilon e_n \) in \( l_q \), we see that

\[
\Delta_{l_p}(\epsilon) \leq 1 - (1 - \epsilon^p)^{1/p}(1 - \epsilon^q)^{1/q}.
\]

To show the opposite inequality, consider a sequence \((x_n)\) in \( B_{l_p} \) converging weakly to \( x \). Then

\[
\|x\|^p = \liminf_{n \to \infty} \|x_n\|^p - \limsup_{n \to \infty} \|x_n - x\|^p
\]

(see, for instance, [27]). Consequently, if \( \liminf_{n \to \infty} \|x_n - x\| \geq \epsilon \), then \( \|x\|^p \leq 1 - \epsilon^p \).

Similarly, if \((x^*_n)\) in \( B_{l_q} \) converges weakly* to \( x^* \) and \( \liminf_{n \to \infty} \|x^*_n - x^*\| \geq \epsilon \), then \( \|x^*\|^q \leq 1 - \epsilon^q \). Hence

\[
1 - x^*(x) \geq 1 - \|x^*\|^q \|x\| \geq 1 - (1 - \epsilon^p)^{1/p}(1 - \epsilon^q)^{1/q}.
\]

Notice that \( \Delta_{l_2}(\epsilon) = \epsilon^2 \leq \Delta_{l_p}(\epsilon) \) for every \( p \in (1, \infty) \) and every \( \epsilon \in [0, 1] \).

**Theorem 2.** Let \( X \) be a Banach space without the Schur property and \( 0 < \epsilon_1 < \epsilon_2 < \epsilon_3 \leq 1 \). Then each of the following conditions implies the next one.
(i) There exists \( \gamma \in [0, 1) \) such that if \( (x_n^*) \) is a sequence in \( S_{X^*} \) with \( \text{sep}(x_n^*) \geq \epsilon_1 \), then 
\[
\beta(S((x_n^*), \gamma)) < \epsilon_1.
\]
(ii) For every \( \gamma \in [0, 1) \) such that if \( (x_n^*) \) is a sequence in \( S_{X^*} \) with 
\[
\text{sep}(x_n^*) \geq \epsilon_1,
\]
then \( \beta(S((x_n^*), \gamma)) < \epsilon_1 \).

Proof. To show that (i) implies (ii), we assume that \( \Delta_X(\epsilon_2) > 0 \). Then for every \( \gamma \in [0, 1) \) we can find a sequence \( (x_n^*) \) in \( S_{X^*} \) with a weak* limit point \( x^* \) and a sequence \( (x_k) \) in \( S_X \) converging weakly to \( x \) so that \( \liminf_{n \to \infty} \|x_n^* - x^*\| \geq \epsilon_2, \liminf_{n \to \infty} \|x_n - x\| \geq \epsilon_2 \), and \( x^*(x) > \gamma \). We can assume that \( X \) is separable and \( (x_n^*) \) converges weakly* to \( x^* \). Then \( \epsilon_2 \leq \liminf_{n \to \infty} \|x_n - x\| \leq \liminf_{n \to \infty} \liminf_{m \to \infty} \|x_n - x_m\| \) and similarly \( \epsilon_2 \leq \liminf_{n \to \infty} \liminf_{m \to \infty} \|x_n^* - x_m^*\| \). Passing to subsequences, we can therefore assume that \( x^*(x_m) \geq \gamma \) for every \( m \) and if \( n \neq m \), then \( \|x_n^* - x_m^*\| > \epsilon_1 \) and \( \|x_n - x_m\| > \epsilon_1 \). Thus \( \lim_{m \to \infty} x_n^* (x_m) \geq \gamma \) for every \( m \). This shows that the sequence \( (x_n^*) \) is contained in \( S((x_n^*), \gamma) \). Consequently, \( \beta(S((x_n^*), \gamma)) \geq \epsilon_1 \) which gives us the negation of (i).

To prove that (ii) implies (iii) we assume that (iii) does not hold. Then for every \( t \in (0, \epsilon_3/\epsilon_2 - 1) \) we can find \( x \in S_X \) and sequences \( (y_n), (z_n) \in N_X \) for which
\[
\limsup_{n \to \infty} \|x + \epsilon_3 y_n\| < 1 + t, \quad \liminf_{n \to \infty} \|x + t z_n\| > 1 + \epsilon_3 t.
\]
The vectors \( v_n = (x + \epsilon_3 y_n)/(1 + t) \) form a sequence in \( B_X \) converging weakly to \( v = x/(1 + t) \). Moreover, \( \|v_n - v\| = \epsilon_3/(1 + t) > \epsilon_2 \) for every \( n \).

Given \( n \), we choose a functional \( x_n^* \in S_{X^*} \) so that \( x_n^*(x + t z_n) = \|x + t z_n\| \). Let \( x^* \) be a weak* limit point of \( (x_n^*) \). Clearly,
\[
1 + \epsilon_3 t \leq \liminf_{n \to \infty} x_n^*(x + t z_n) \leq 1 + \liminf_{n \to \infty} x_n^*(z_n),
\]
which shows that
\[
\epsilon_3 \leq \liminf_{n \to \infty} x_n^*(z_n) = \liminf_{n \to \infty} (x_n^* - x^*)(z_n) \leq \liminf_{n \to \infty} \|x_n^* - x^*\|.
\]
Moreover,
\[
1 + \epsilon_3 t \leq \liminf_{n \to \infty} x_n^*(x) + t \leq x^*(x) + t,
\]
which shows that
\[
x^*(t) = \frac{1}{1+t} x^*(x) \geq \frac{1 - t(1 - \epsilon_3)}{1 + t}.
\]
Consequently,
\[
\Delta_X(\epsilon_2) \leq 1 - x^*(v) \leq 2t.
\]
Passing to the limit with \( t \) tending to 0, we see that \( \Delta_X(\epsilon_2) = 0 \). \Box

Corollary 3. Let \( X \) be a Banach space without the Schur property. Then each of the following conditions implies the next one.

(i) For every \( \epsilon > 0 \) there is \( \gamma \in [0, 1) \) such that if \( (x_n^*) \) is a sequence in \( S_{X^*} \) with 
\[
\text{sep}(x_n^*) \geq \epsilon,
\]
then \( \beta(S((x_n^*), \gamma)) < \epsilon \).
(ii) $\Delta_X(\epsilon) > 0$ for every $\epsilon \in (0, 1]$.

(iii) The space $X$ is NUNC.

In the sequel we will need the following lemma.

**Lemma 4.** Let $X$ be a Banach space, $x \in X$ and $(y_n)$ be a weakly null sequence in $X$. If $(a_n)$ and $(b_n)$ are sequences of positive numbers such that $\limsup_{n \to \infty} a_n \leq \liminf_{n \to \infty} b_n$, then

$$\limsup_{n \to \infty} \|x + a_n y_n\| \leq \limsup_{n \to \infty} \|x + b_n y_n\|. $$

The above inequality holds also if “$\limsup$” is replaced by “$\liminf$” on both sides.

**Proof.** Clearly, $\liminf_{m \to \infty} \|x + b_m y_m\| \geq \|x\|$, so

$$\|x + a_n y_n\| \leq \frac{a_n}{b_n} \|x + b_n y_n\| + \left| 1 - \frac{a_n}{b_n} \right| \|x\|$$

$$\leq \frac{a_n}{b_n} \|x + b_n y_n\| + \left| 1 - \frac{a_n}{b_n} \right| \liminf_{m \to \infty} \|x + b_m y_m\|$$

for every $n$. Passing to a subsequence, we can assume that limits $\lim_{n \to \infty} \|x + a_n y_n\|$, $a = \lim_{n \to \infty} a_n$, and $b = \lim_{n \to \infty} b_n$ exist. Assuming that $b > 0$, we obtain

$$\lim_{n \to \infty} \|x + a_n y_n\| \leq \frac{a}{b} \limsup_{n \to \infty} \|x + b_n y_n\| + \left(1 - \frac{a}{b}\right) \limsup_{n \to \infty} \|x + b_n y_n\|$$

$$= \limsup_{n \to \infty} \|x + b_n y_n\|. $$

The case when $b = 0$ is trivial. The proof of the inequality with “$\limsup$” replaced by “$\liminf$” is similar. $\square$

Lemma 4 shows in particular that $d(\epsilon, x)$ and $b(\epsilon, x)$ are nondecreasing functions of $\epsilon$ in the interval $[0, +\infty)$.

**Theorem 5.** If an infinite-dimensional Banach space $X$ does not contain an isomorphic copy of $l_1$, then conditions (ii) and (iii) of Corollary 3 are equivalent.

**Proof.** Assume that an infinite-dimensional Banach space $X$ does not contain an isomorphic copy of $l_1$. Then $X$ does not have the Schur property, so in view of Corollary 3 it suffices to prove that (iii) implies (ii). To this end assume that (ii) does not hold. Then there exists $\epsilon \in (0, 1)$ such that $\Delta_X(\epsilon) = 0$. For every $\gamma \in (0, 1)$, we can therefore find a sequence $(x_n^\gamma)$ in $S_X$, with a weak* limit point $x^*$ and a sequence $(x_n)$ in $S_X$ converging weakly to $x$ so that $\|x_n^\gamma - x^*\| > \epsilon/2$ and $\|x_n - x\| > \epsilon/2$ for every $n$, and $x^*(x_n) > 1 - \gamma \epsilon/8$. We put $v = x/\|x\|$ and $v_n = (x_n - x)/\|x_n - x\|$ for every $n$. Then $(v_n) \in N_X$. Moreover,

$$\|v + x_n - x\| \leq 2 - \|x\| < 1 + \frac{\gamma}{8}.$$
for every $n$. Using Lemma 4, we therefore see that
\[
\limsup_{n \to \infty} \left\| v + \frac{\epsilon}{8} v_n \right\| \leq \limsup_{n \to \infty} \left\| v + x_n - x \right\| \leq 1 + \frac{\gamma}{8}.
\]
It follows that $d(\epsilon/8, v) \leq \gamma$.

Given $n$, we now choose $w_n \in S_X$ so that $(x_n^* - x^*)(w_n) > \epsilon/2$. We can assume that the space $X$ is separable and $x^*$ is the weak* limit of the sequence $(x_n^*)$. Using Rosenthal’s Theorem (see [28]), we can also assume that the vectors $y_n = w_{2n} - w_{2n-1}$ tend weakly to 0 and $\lim_{n \to \infty} (x_{2n}^* - x^*)(w_{2n-1}) = 0$. Then
\[
\liminf_{n \to \infty} \left\| v + \frac{\gamma}{2} y_n \right\| \geq x^*(v) + \frac{\gamma}{2} \liminf_{n \to \infty} x_{2n}^*(y_n) > 1 - \frac{\gamma \epsilon}{8} + \frac{\gamma}{2} \liminf_{n \to \infty} (x_{2n}^* - x^*)(y_n)
\]
\[\geq 1 + \frac{\epsilon}{8} \gamma.
\]
This in particular shows that $\liminf_{n \to \infty} \left\| y_n \right\| > 0$. We can therefore assume that $\left\| y_n \right\| \neq 0$ and set $z_n = y_n/\left\| y_n \right\|$ for every $n$. Then $(z_n) \in S_X$ and by Lemma 4,
\[
\liminf_{n \to \infty} \left\| v + z_n \right\| \geq \liminf_{n \to \infty} \left\| v + \frac{\gamma}{2} y_n \right\| \geq 1 + \frac{\epsilon}{8} \gamma,
\]
which shows that $b(\gamma, v) \geq \gamma \epsilon/8$. We therefore see that $X$ is not NUNC. \qed

The assumption that $X$ does not contain an isomorphic copy of $l_1$ is essential in Theorem 5. Indeed, let $X = (l_1 \oplus c_0)_{l_1}$. In much the same way as in the case of the space $(\mathbb{R} \oplus c_0)_{l_0}$, one can show that $X$ is NUNC. We put $e_n = (0, \ldots, 0, 1, 0, \ldots)$ where 1 occupies the $n$th place, $n = 1, 2, \ldots$. The space $X^*$ can be identified with $(l_\infty \oplus l_1)_{\ell_\infty}$, and we consider the sequences of elements $x_n = (0, e_1 + e_n) \in S_X$ and $x_n^* = (e_n, e_1) \in S_{X^*}$

Clearly, $(x_n)$ converges weakly to $x = (0, e_1)$ and $(x_n^*)$ converges weakly* to $x^* = (0, e_1)$. Moreover, $\liminf_{n \to \infty} \left\| x_n - x \right\| = 1 = \liminf_{n \to \infty} \left\| x_n^* - x^* \right\|$ and $x^*(x) = 1$. This shows that $\Delta_X(1) = 0$, so $X$ does not satisfy condition (ii).

A modification of this example shows that (ii) does not imply (i). Namely, let $Y = (l_1 \oplus l_2)_{l_1}$. Clearly, $Y$ does not have the Schur property. Applying formula (1), one can easily show that if $(x_n)$ is a sequence in $BY$ such that $(x_n)$ converges weakly to $x$ and $\liminf_{n \to \infty} \left\| x_n - x \right\| \geq \epsilon$ where $\epsilon \in [0, 1]$, then $\left\| x \right\| \leq (1 - \epsilon^2)^{1/2}$. This gives us the estimate $\Delta_Y(\epsilon) \leq 1 - (1 - \epsilon^2)^{1/2}$. It is also easy to obtain the opposite inequality, which finally leads to the formula $\Delta_Y(\epsilon) = 1 - (1 - \epsilon^2)^{1/2}$. Consequently, the space $Y$ satisfies condition (ii). On the other hand, considering the sequence of functionals $x_n^* = (\sum_{k=1}^n e_k, 0)$ in $S_Y$ and the sequence of elements $x_n = (e_n, 0)$ in $S((x_n^*), 1)$, we see that $Y$ does not satisfy condition (i). However, the following result holds.

**Theorem 6.** If $X$ is an infinite-dimensional reflexive Banach space, then conditions (i), (ii), and (iii) of Corollary 3 are equivalent.

**Proof.** In view of Corollary 3 and Theorem 5, it suffices to prove that (ii) implies (i). For this purpose, we assume that condition (i) is not satisfied. Then there exists $\epsilon > 0$ such that for every $\gamma \in [0, 1)$ we can find sequences $(x_n^*)$ in $S_{X^*}$ and $(x_n)$ in $S((x_n^*), \gamma)$ such...
that \( \text{sep}(x^*_n) \geq \epsilon \) and \( \text{sep}(x_n) \geq \epsilon \). Passing to subsequences, we can assume that the sequence \((x^*_n)\) converges weakly to some \( x^* \), the sequence \((x_n)\) converges weakly to some \( x \), \( \|x^*_n - x^*\| > \epsilon / 2 \), and \( \|x_n - x\| > \epsilon / 2 \) for all \( n \). Clearly,
\[
x^*(x) = \lim_{n \to \infty} x^*(x_n) = \lim_{n \to \infty} \lim_{m \to \infty} x^*_m(x_n) \geq \gamma.
\]
This shows that \( \Delta_X(\epsilon / 2) \leq 1 - \gamma \). Passing to the limit with \( \gamma \) tending to 1, we see that \( \Delta_X(\epsilon / 2) = 0 \).

**Corollary 7.** If a Banach space \( X \) is UNC, then \( X \) is NUNC.

**Proof.** UNC spaces are reflexive, so it is enough to show that if \( X \) is UNC, then \( X \) satisfies condition (i) of Corollary 3. Let \( X \) be an UNC space and let \( \epsilon > 0 \). There is \( \delta > 0 \) such that if \( x^*, y^* \in S_X^* \) and \( \|x^* - y^*\| \geq \epsilon \), then
\[
\text{diam}(S^{x^*, y^*, \delta}) \leq \epsilon.
\]
Given a sequence \((x^*_n)\) in \( S_X^* \) with \( \text{sep}(x^*_n) \geq \epsilon \) and \( x_1, x_2 \in S((x^*_n), 1 - \delta / 2) \), we find \( m \in \mathbb{N} \) such that \( x^*_m(x_i) > 1 - \delta \) for all \( n \geq m \) and \( i = 1, 2 \). Then \( \|x^*_m - x^*_{m+1}\| \geq \epsilon \) and \( x_1, x_2 \in S((x^*_m, x^*_{m+1}, \delta)) \). Hence \( \|x_1 - x_2\| \leq \text{diam}(S(x^*_m, x^*_{m+1}, \delta)) \leq \epsilon \). This shows that \( \text{diam}(S(1, 1 - \delta / 2)) \leq \epsilon \) and consequently, \( \beta(S((x^*_n), 1 - \delta / 2)) \leq \epsilon \). We therefore see that \( X \) is NUNC.

It is clear that if a Banach space \( X \) is reflexive, then the moduli \( \Delta_X \) and \( \Delta_{X^*} \) are equal. This gives us the second corollary of Theorem 6.

**Corollary 8.** A reflexive space \( X \) is NUNC if and only if \( X^* \) is NUNC.

### 3. Fixed point theorems

In this section we shall give some applications of the moduli \( d \) and \( b \) to the fixed point theory. It turns out that the results become stronger if \( b \) is replaced by an equivalent modulus. Given a Banach space \( X \), by \( \mathcal{M}_X \) we denote the set of all weakly null sequences \((y_m)\) in \( B_X \) such that
\[
\limsup_{n \to \infty} \limsup_{m \to \infty} \|y_n - y_m\| \leq 1.
\]
Let \( x \in X \) and \( \epsilon \geq 0 \). We put
\[
b_1(\epsilon, x) = \sup_{(y_m) \in \mathcal{M}_X} \liminf_{m \to \infty} \|x + \epsilon y_m\| - \|x\|.
\]
Observe that “\( \liminf \)” can be replaced by “\( \limsup \)” in the definition of \( b_1(\epsilon, x) \). It follows that \( b_1(\epsilon, x) \) is a convex function of \( \epsilon \in [0, +\infty) \). Moreover, \( b_1(0, x) = 0 \), so \( b_1(\epsilon, x) / \epsilon \) is nondecreasing in the interval \( (0, +\infty) \).

Using Lemma 4, one can show that \( b_1(\epsilon, x) \leq b(\epsilon, x) \leq b_1(2\epsilon, x) \) for every \( x \in X \) and every \( \epsilon \geq 0 \). The modulus \( b \) can be therefore replaced by \( b_1 \) in the definition of NUNC.
spaces. To avoid confusion, we will add the name of a space as a subscript to the name \( b_1 \) if necessary.

In the proof of our fixed point theorem, we will use the method developed in [7,13] (see also [16]). Before passing to the theorem, we briefly recall the notation and preliminary results. Let \( C \) be a nonempty bounded closed convex subset of a Banach space \( X \) and \( T : C \to C \) be a nonexpansive mapping, i.e.

\[
\| Tx - Ty \| \leq \| x - y \|
\]

for all \( x, y \in C \). Then there is a sequence \( (x_n) \) in \( C \) such that \( \lim_{n \to \infty} \| x_n - Tx_n \| = 0 \). Such sequence is called an approximate fixed point sequence. Assume additionally that \( C \) is weakly compact. Using the Zorn lemma, one can show that \( C \) contains a subset \( K \) which is minimal in the family of all nonempty closed convex subsets of \( C \) invariant for \( T \). Such set \( K \) is briefly called a minimal invariant set for \( T \). Basic properties of approximate fixed point sequences in minimal invariant sets were independently given in [10,15].

**Goebel–Karlovitz Lemma.** Let \( K \) be a minimal invariant set for a nonexpansive mapping \( T \) and \( (x_n) \) be an approximate fixed point sequence in \( K \). Then

\[
\lim_{n \to \infty} \| x - x_n \| = \text{diam } K \quad \text{for every } x \in K.
\]

Given a Banach space \( X \), by \( \tilde{X} \) we denote the quotient space \( l_\infty(X)/c_0(X) \). Let \( (x_n) \in l_\infty(X) \). We put \( [(x_n)] = (x_n) + c_0(X) \). It is easy to see that the quotient norm is given by the formula

\[
\| [(x_n)] \| = \limsup_{n \to \infty} \| x_n \|.
\]

We identify an element \( x \in X \) with \( [(x,x,x,\ldots)] \). Next, if \( K \) is a nonempty subset of \( X \), we put

\[
\tilde{K} = \{ [(x_n)] \in \tilde{X} : x_n \in K \}
\]

and given a mapping \( T : K \to K \), we define \( \tilde{T} : \tilde{K} \to \tilde{K} \) by the formula

\[
\tilde{T}[(x_n)] = [(Tx_n)].
\]

If \( T \) is nonexpansive, then \( \tilde{T} \) has the same property and cosets of approximate fixed point sequences are fixed points of \( \tilde{T} \). The following result was essentially proved in [17] (see also [16]).

**Lin’s Lemma.** Let \( K \) be a minimal invariant set for a nonexpansive mapping \( T \). If \( (\xi_n) \) is an approximate fixed point sequence for \( \tilde{T} \) in \( \tilde{K} \), then

\[
\lim_{n \to \infty} \| \xi_n - x \| = \text{diam } K \quad \text{for every } x \in K.
\]

Let us recall that a Banach space \( X \) is said to have the weak fixed point property if every nonexpansive mapping \( T : C \to C \), where \( C \subset X \) is convex and weakly compact, has a fixed point. Now we can pass to our main fixed point result.
Theorem 9. Let \( X \) be a Banach space without the Schur property. If there exists \( \epsilon \in (0, 1) \) such that for every \( x \in S_X \) it is the case that \( b_1(1, x) < 1 - \epsilon \) or \( d(1, x) > \epsilon \), then \( X \) has the weak fixed point property.

Proof. Assume that \( X \) lacks the weak fixed point property. Then there is a convex weakly compact set \( K \subset X \) with diameter 1 which is minimal invariant for a nonexpansive mapping \( T \). We can assume that \( K \) contains a weakly null approximate fixed point sequence \((x_n)\) for \( T \). We put
\[
W = \left\{ [(w_n)] \in \bar{K} : ||[(w_n)] - [(x_n)]|| \leq \frac{1}{2}, \limsup_{n \to \infty} \limsup_{m \to \infty} ||w_n - w_m|| \leq \frac{1}{2} \right\}.
\]

Goebel–Karlovitz Lemma shows that \([(x_n)/2] \in W\). The set \( W \) is therefore nonempty, closed, convex, and invariant for \( \bar{T} \). In view of Lin’s Lemma, we see that for every \( \epsilon \in (0, 1) \) there is \([(w_n)] \in W\) such that \(||[(w_n)]|| > 1 - \epsilon/4\). We choose a subsequence \((w_n_k)\) so that
\[
||[(w_n_k)]|| = \lim_k \Vert w_{n_k} - x_{n_k} \Vert > 1 - \epsilon/4.
\]

We set \( u = v/||v|| \) and \( u_{k} = 2(w_{n_k} - v) \) for every \( k \). Then \( u \in S_X \) and
\[
\lim k \to \infty \limsup k \to \infty ||u_k - u|| = 2 \lim k \to \infty \limsup k \to \infty ||w_{n_k} - w_{n_l}|| < 1.
\]

Moreover,
\[
\liminf_{k \to \infty} ||u_k + u_k|| \geq \liminf_{k \to \infty} \Vert 2v + u_k \Vert - \left\| 2v - \frac{v}{||v||} \right\| = 2 \liminf_{k \to \infty} ||w_{n_k}|| + 2 ||v|| - 1 > 2 - \epsilon.
\]

It follows that \( b_1(1, u) > 1 - \epsilon\).

Consider now the sequence of vectors \( y_k = 2(w_{n_k} - v - x_{n_k}) \). It converges weakly to 0.
From Goebel–Karlovitz Lemma and (2), we obtain
\[
\liminf_{k \to \infty} ||y_k|| \geq 2 \left( \lim k \to \infty \Vert x_{n_k} \Vert - \limsup k \to \infty ||w_{n_k} - v|| \right) > 1.
\]
Moreover,
\[
\limsup_{k \to \infty} \|u + y_k\| \leq \limsup_{k \to \infty} \|2v + y_k\| + \left\| \frac{v}{\|v\|} - 2v \right\| < 2 \limsup_{k \to \infty} \|w_{nk} - x_{nk}\| + \frac{\epsilon}{2} \\
\leq 1 + \frac{\epsilon}{2}.
\]
Using Lemma 4 we therefore see that
\[
\limsup_{k \to \infty} \left\| \frac{y_k}{\|y_k\|} \right\| \leq \limsup_{k \to \infty} \|u + y_k\| < 1 + \frac{\epsilon}{2}.
\]
Consequently, \(d(1, u) < \epsilon\).

We shall establish some corollaries of Theorem 9. For this purpose, we need the following technical lemma.

**Lemma 10.** Let \(X\) be a Banach space without the Schur property and let \(x \in S_X\). If \(b_1(t, x) < (1 - \epsilon)t\) for some \(\epsilon \in (0, 1)\) and \(t > 0\), then \(b_1(1, x) < 1 - \min\{1, t\} \epsilon/2\).

**Proof.** If \(t \geq 1\), then \(1 - \epsilon > b_1(t, x)/t \geq b_1(1, x)\). Assume now that \(t \in (0, 1)\). Then we find a sequence \((y_n) \in M_X\) so that
\[
b_1(1, x) - \frac{\epsilon t}{2} < \limsup_{n \to \infty} \|x + y_n\| - 1.
\]
But
\[
\limsup_{n \to \infty} \|x + y_n\| - 1 \leq \limsup_{n \to \infty} \|x + ty_n\| + (1 - t) \limsup_{n \to \infty} \|y_n\| - 1 \\
\leq \limsup_{n \to \infty} \|x + ty_n\| - t \leq b_1(t, x) + 1 - t.
\]
Applying our assumption, we therefore see that \(b_1(1, x) < 1 - \epsilon t/2\).

**Corollary 11.** If a Banach space \(X\) is NUNC, then \(X\) has the weak fixed point property.

**Proof.** Let \(X\) be a NUNC Banach space. If \(X\) has the Schur property, then every weakly compact subset of \(X\) is actually compact in norm. Therefore \(X\) has the weak fixed point property even for continuous mappings (see [1, p. 11]).

Assume now that \(X\) does not have the Schur property. We shall show that \(X\) satisfies the assumption of Theorem 9. Since \(X\) is NUNC, there is \(t \in (0, 1)\) such that for every \(x \in S_X\) we have \(d(1/2, x) \geq t\) or \(b(t, x) \leq t/2\). In the first case, we obtain \(d(1, x) \geq t > t/8\). In the second case, we have \(b_1(t, x) \leq b(t, x) < 3t/4\), which by Lemma 10 gives us the estimate \(b_1(1, x) < 1 - t/8\).

Theorem 2 and Lemma 10 give us other conditions which guarantee the weak fixed point property.

**Corollary 12.** Let \(X\) be a Banach space without the Schur property. Each of the following conditions is sufficient for the weak fixed point property.
(i) There exists \( \gamma \in (0, 1) \) such that if \( (x_n^*) \subset Sx^* \) and \( \text{sep}(x_n^*) \geq \gamma \), then
\[
\beta(S((x_n^*), \gamma)) < \gamma.
\]
(ii) \( \lim_{\epsilon \to 1^-} \Delta x(\epsilon) > 0 \).

Corollary 11 generalizes the fixed point theorem for UNC spaces which was proved in [26]. Consecutive generalizations of that theorem were also obtained in [4,5,9]. We recall here the result from [4]. Given \( k \in \mathbb{N} \) and a nonempty bounded subset \( A \) of a Banach space \( X \), we put
\[
\beta_k(A) = \sup \{ \text{sep}(x_i)^k + 1 : x_1, \ldots, x_{k+1} \in A \}.
\]
Next, we set
\[
S(x_1^*, \ldots, x_{l+1}^*, \delta) = \left\{ x \in Bx^* : \min_{1 \leq i \leq k} x_i^*(x) \geq 1 - \delta \right\}
\]
where \( x_1^*, \ldots, x_{l+1}^* \in Sx^* \) and \( \delta \in [0, 1] \). Let now \( r \in (0, 1], k, l \in \mathbb{N} \). A Banach space \( X \) is said to be \( (r, k, l) \)-somewhat uniformly noncreasy if there exist \( \epsilon \in (0, r) \) and \( \delta \in (0, 1) \) such that if \( x_1^*, \ldots, x_{l+1}^* \in Sx^* \) and \( \text{sep}(x_i^*)_{i=1}^{l+1} \geq \epsilon \), then
\[
\beta_k(S(x_1^*, \ldots, x_{l+1}^*, \delta)) \leq \epsilon.
\]
In [4] it was proved that if \( X \) is \( (1, k, l) \)-somewhat uniformly noncreasy for some \( k, l \in \mathbb{N} \), then \( X \) is superreflexive and it has the weak fixed point property. Slight modification of the proof of Corollary 7 shows that \( (1, k, l) \)-somewhat uniformly noncreasy spaces satisfy condition (i) in Corollary 12, so our result is more general than those in [4,5,9].

Let \( X, Y \) be isomorphic Banach spaces. The Banach–Mazur distance for these spaces is defined by the formula
\[
d(X, Y) = \inf \{ \| T \| \| T^{-1} \| : T \text{ isomorphism} \}
\]
where the infimum is taken over all isomorphisms \( T : X \to Y \). In [2] a coefficient \( M(X) \) was introduced and it was proved that if \( d(X, Y) < M(X) \), then \( Y \) has the weak fixed point property. In our notation
\[
M(X) = \sup_{s > 0} \left( \frac{1 + s}{\sup_{\| x \| \leq 1} \{ b_1(X(1, x) + \| x \|) \}} \right).
\]
We shall show a stronger result in this direction. For this purpose, we need the following lemma.

**Lemma 13.** Assume that \( X \) is a Banach space without the Schur property. Let \( \| \cdot \| \) be the initial norm in \( X \) and let a norm \( | \cdot | \) in \( X \) satisfy the condition \( \| x \| \leq | x | \leq \sigma \| x \| \) for every \( x \in X \). Setting \( Y = (X, | \cdot |) \), we have
\[
d_Y(\epsilon, x) + | x | \geq d_X \left( \frac{\epsilon}{\sigma}, x \right) + \| x \| \quad \text{and} \quad (3)
\]
\[
b_{1,Y}(\epsilon, x) + | x | \leq \sigma \{ b_{1,X}(\epsilon, x) + \| x \| \} \quad \text{(4)}
\]
for every \( x \in X \) and every \( \epsilon \geq 0 \).
Proof. We fix $\epsilon \geq 0$ and $x \in X$. If $(y_n) \in \mathcal{N}_T$, then $\|\sigma y_n\| \geq 1$ for all $n \in \mathbb{N}$ and the sequence $(\sigma y_n)$ converges weakly to zero. Using Lemma 4, we see that

$$d_Y(\epsilon, x) + |x| = \inf_{(y_n) \in \mathcal{N}_T} \limsup_{n \to \infty} |x + \epsilon y_n| \geq \inf_{(y_n) \in \mathcal{N}_T} \limsup_{n \to \infty} \left\| x + \frac{\epsilon \|\sigma y_n\|}{\sigma} \sigma y_n \right\| \geq d_X\left(\frac{\epsilon}{\sigma}, x\right) + \|x\|.$$ 

In order to prove the second inequality, observe that $M_Y \subset M_X$, so

$$b_{1,Y}(\epsilon, x) + |x| = \sup_{(y_n) \in \mathcal{M}_Y} \liminf_{n \to \infty} |x + \epsilon y_n| \leq \sup_{(y_n) \in \mathcal{M}_Y} \liminf_{n \to \infty} \|x + \epsilon y_n\| \leq \sup_{(y_n) \in \mathcal{M}_X} \liminf_{n \to \infty} \|x + \epsilon y_n\| = \sigma \left( b_{1,X}(\epsilon, x) + \|x\| \right).$$

Let $X$ be a Banach space without the Schur property. Given $t \geq 0$ and $x \in X$, we put

$$d_X^{-1}(t, x) = \max\{\epsilon \geq 0 : d_X(\epsilon, x) \leq t\}.$$ 

Next, we set

$$M_1(X) = \sup_{0 < \epsilon < 1} \sup_{t > 0} \inf_{x \in B_X} \left\{ \frac{1}{d_X^{-1}(1 - \|x\| + \epsilon, x)}, \sup_{s \geq t} \left( \frac{(1 - \epsilon)s + 1}{b_{1,X}(s, x) + \|x\|} \right) \right\}.$$ 

Observe that the supremum over all $0 < \epsilon < 1$ can be replaced by the limit with $\epsilon \to 0^+$. Therefore

$$M_1(X) \geq \lim_{\epsilon \to 0^+} \inf_{t > 0} \sup_{x \in B_X} \left\{ \frac{1}{d_X^{-1}(1 - \|x\| + \epsilon, x)}, \sup_{s \geq t} \left( \frac{s + 1}{b_{1,X}(s, x) + \|x\|} \right) \right\}$$

$$= \lim_{\epsilon \to 0^+} \inf_{t > 0} \sup_{x \in B_X} \left\{ \frac{1}{d_X^{-1}(1 - \|x\| + \epsilon, x)}, \sup_{s \geq t} \left( \frac{s + 1}{b_{1,X}(s, x) + \|x\|} \right) \right\}.$$ 

The opposite inequality is obvious, so we get the formula

$$M_1(X) = \lim_{\epsilon \to 0^+} \inf_{t > 0} \sup_{x \in B_X} \left\{ \frac{1}{d_X^{-1}(1 - \|x\| + \epsilon, x)}, \sup_{s \geq t} \left( \frac{s + 1}{b_{1,X}(s, x) + \|x\|} \right) \right\}.$$ 

Observe also that

$$M_1(X) \geq \sup_{t > 0} \left( \frac{t + 1}{\sup_{x \in B_X} \left( b_{1,X}(t, x) + \|x\| \right)} \right)$$

$$= \sup_{t > 0} \left( \frac{1 + \frac{1}{t}}{\sup_{x \in B_X} \left( b_{1,X}(1, \frac{1}{t} x) + \|\frac{1}{t} x\| \right)} \right)$$

$$= M(X).$$

Theorem 14. Assume that $X$ is a Banach space without the Schur property. Let $\| \cdot \|$ be the initial norm in $X$ and let a norm $| \cdot |$ in $X$ satisfy the condition $\|x\| \leq |x| \leq \sigma \|x\|$ for every $x \in X$. We set $Y = (X, | \cdot |)$. If $\sigma < M_1(X)$, then $Y$ has the weak fixed point property.
Proof. Assume that the space \( Y \) fails the weak fixed point property. Then by Theorem 9, for every \( \epsilon \in (0,1) \) and every \( t > 0 \) there exists \( x \in SY \) such that \( b_{1,Y}(1, x) \geq 1 - \epsilon \min[1, t]/4 \) and \( d_Y(1, x) \leq \epsilon \min[1, t]/4 \). Lemma 10 shows that \( b_{1,Y}(t, x)/t \geq 1 - \epsilon/2 > 1 - \epsilon \). Moreover, \( d_Y(1, x) < \epsilon \). Using Lemma 13, we therefore see that

\[
\frac{1}{\sigma} \leq d_X \left( \frac{1}{\sigma}, x \right) + \| x \| - 1 \leq d_Y(1, x) < \epsilon,
\]

for every \( s \geq t \). Hence \( 1/\sigma \leq d_X^{-1}(1 - \| x \| + \epsilon, x) \) and

\[
\sigma \geq \sup_{s \geq t} \left( \frac{1 - \epsilon}{s} \right) \left( 1 - \| x \| + \epsilon, x \right).
\]

Consequently, \( \sigma \geq M_1(X) \).

Corollary 15. Let \( X \) be a Banach space without the Schur property. If \( Y \) is a Banach space such that

\[
d(X, Y) < M_1(X),
\]

then \( Y \) has the weak fixed point property.

Since \( M_1(X) \geq M(X) \), Corollary 15 extends the fixed point theorem proved in [2]. That theorem was also strengthened in [13]. We shall show that Corollary 15 extends also the last result. For this purpose, we need to recall some more terminology. Given \( c \geq 0 \), we set

\[
r_X(c) = \inf \left\{ \liminf_{n \to \infty} \| x + x_n \| - 1 \right\}
\]

where the infimum is taken over all \( x \in X \) with \( \| x \| \geq c \) and all weakly null sequences \((x_n)\) in \( X \) with \( \liminf_{n \to \infty} \| x_n \| \geq 1 \). The function \( r_X \) is called the Opial modulus of \( X \). In [19] it was shown that \( r_X \) is continuous on \([0, +\infty)\). We put

\[
C_X(B) = \sup\{c \geq 0: r_X(c) \leq B - 1\}
\]

where \( B \geq 1 \). In [2] the following coefficient was introduced:

\[
R(a, X) = \sup_{n \to \infty} \left\{ \liminf_{n \to \infty} \| x + x_n \| \right\}
\]

where the supremum is taken over all \( x \in X \) with \( \| x \| \leq a \) and all sequences \((x_n) \in M_X \).

Observe that

\[
R(a, X) = \sup_{n \to \infty} \left\{ \liminf_{n \to \infty} \| ax + x_n \|: x \in B_X, (x_n) \in M_X \right\}.
\]

Moreover, for fixed \( x \in B_X \) and \((x_n) \in M_X \), \( \liminf_{n \to \infty} \| ax + x_n \| \) is a lipschitzian function of \( a \in [0, +\infty) \) with the Lipschitz constant 1. Consequently, \( R(a, X) \) has the same property.
Assume that $X$ is a Banach space without the Schur property. Let $\| \cdot \|$ be the initial norm in $X$ and let a norm $| \cdot |$ satisfy the condition $\| x \| \leq | x | \leq \sigma \| x \|$ for every $x \in X$. We set $Y = (X, | \cdot |)$. In [13] (see also [8]) it was proved that if

$$
\sigma < \sup \left\{ \frac{1 + a}{R(\alpha \in C_X(\sigma), X)} : a \geq 0 \right\},
$$

then $Y$ has the weak fixed point property.

**Proposition 16.** If a number $\sigma \geq 1$ satisfies condition (5), then $\sigma < M_1(X)$.

**Proof.** Assume that $\sigma \geq M_1(X)$. Then for every $t > 0$ and every $n \in \mathbb{N}$ there exists $x_n \in B_X$ such that

$$
\sigma \left(1 + \frac{1}{n}\right) \geq \max \left\{ \frac{1}{d_X^{-1}(1 - \| x_n \| + \frac{1}{n}, x_n)}, \frac{(1 - \frac{1}{n})t + 1}{b_{1,X}(t, x_n) + \| x_n \|} \right\}.
$$

(6)

In particular,

$$
\frac{1}{\sigma(1 + \frac{1}{n})} \leq d_X^{-1} \left(1 - \| x_n \| + \frac{1}{n}, x_n\right),
$$

which shows that

$$
d_X \left(\frac{1}{\sigma(1 + \frac{1}{n})}, x_n\right) \leq 1 - \| x_n \| + \frac{1}{n}.
$$

We can therefore find a sequence $(y_m) \in N_X$ such that

$$
\limsup_{m \to \infty} \left\| x_n + \frac{1}{\sigma(1 + \frac{1}{n})} y_m \right\| < 1 + \frac{2}{n}.
$$

Consequently,

$$
r_X \left(\sigma \left(1 + \frac{1}{n}\right) \| x_n \| \right) \leq \limsup_{m \to \infty} \left\| \sigma \left(1 + \frac{1}{n}\right) x_n + y_m \right\| - 1
$$

$$
< \sigma \left(1 + \frac{1}{n}\right) \left(1 + \frac{2}{n}\right) - 1.
$$

(7)

Passing to a subsequence, we can assume that the limit $\alpha = \lim_{n \to \infty} \| x_n \|$ exists. Then from (7) it follows that $r_X(\sigma \alpha) \leq \sigma - 1$ which shows that

$$
\sigma \alpha \leq C_X(\sigma).
$$

(8)

From (6) we also see that

$$
\sigma \left(1 + \frac{1}{n}\right) \left(b_{1,X}(t, x_n) + \| x_n \| \right) \geq \left(1 - \frac{1}{n}\right)t + 1.
$$

We can therefore choose a sequence $(z_m) \in M_X$ such that

$$
\sigma \left(1 + \frac{1}{n}\right) \liminf_{m \to \infty} \| x_n + t z_m \| > \left(1 - \frac{2}{n}\right)t + 1.
$$
Hence
\[
\left(1 - \frac{2}{n}\right) + \frac{1}{t} < \sigma \left(1 - \frac{1}{n}\right) \liminf_{m \to \infty} \left\lVert \frac{1}{t} x_n + z_m \right\lVert \leq \sigma \left(1 + \frac{1}{n}\right) R \left(\frac{1}{t} \left\lVert x_n \right\lVert, X\right),
\]
and passing to the limit with \(n\) tending to infinity, we obtain
\[
1 + \frac{1}{t} \leq \sigma R \left(\frac{1}{t} \alpha, X\right).
\]
In view of (8) this gives us the inequality
\[
1 + \frac{1}{t} \leq \sigma R \left(\frac{1}{t} \frac{C_X(\sigma)}{\sigma}, X\right).
\]
Since \(t > 0\) is arbitrary, this shows that \(\sigma\) does not satisfy condition (5). \(\Box\)

For some spaces \(X\) the condition \(\sigma < M_1(X)\) is equivalent to (5). We shall prove that this is the case when \(X = l_p\) with \(1 < p < \infty\). In view of Proposition 16, it suffices to show that if \(\sigma < M_1(X)\), then \(\sigma\) satisfies (5). First recall that
\[
R(a, l_p) = \left(a^p + \frac{1}{2}\right)^{1/p}
\]
for every \(a \geq 0\) (see [2]) and \(r_{l_p}(c) = (c^p + 1)^{1/p} - 1\) for all \(c \geq 0\) (see [19]), which shows that
\[
C_{l_p}(\sigma) = (\sigma^p - 1)^{1/p}
\]
for every \(\sigma \geq 1\) (see [13]).

Let now \(x \in l_p\) and \(\epsilon \geq 0\). Using (1), one can easily show that \(d_{l_p}(\epsilon, x) = (\|x\|^p + \epsilon^p)^{1/p} - \|x\|\). Consequently,
\[
d_{l_p}^{-1}(t, x) = \left((\|x\|^p + t)^{p} - \|x\|^p\right)^{1/p}
\]
for every \(t \geq 0\). Moreover,
\[
b_{l_p}(\epsilon, x) = \left(\|x\|^p + \frac{\epsilon^p}{2}\right)^{1/p} - \|x\|.
\]
Hence
\[
M_1(l_p) = \lim_{\epsilon \to 0^+} \sup_{t > 0} \inf_{x \in B_{l_p}} \max \left\{ \frac{1}{((1 + \epsilon)^p - \|x\|^p)^{1/p}}, \sup_{s \geq t} \left(\frac{s + 1}{(\|x\|^p + \frac{s^p}{2})^{1/p}}\right)\right\}.
\]
Assume that \(1 \leq \sigma < M_1(l_p)\). Then
\[
\sigma < \inf_{\alpha \in (0, 1)} \max \left\{ \frac{1}{(1 - \alpha p)^{1/p}}, \sup_{s > 0} \left(\frac{s + 1}{(\alpha p + \frac{s^p}{2})^{1/p}}\right)\right\}.
\]
The infimum is attained at some \(\xi \in (0, 1)\) for which
\[
\frac{1}{(1 - \xi p)^{1/p}} = \sup_{s > 0} \left(\frac{s + 1}{(\xi p + \frac{s^p}{2})^{1/p}}\right).
\]
Hence \( \sigma < 1/(1 - \xi^p)^{1/p} \), which gives us the inequality

\[
\frac{\sigma^p - 1}{\sigma^p} < \xi^p.
\]

Consequently,

\[
\sigma < \sup_{s > 0} \left( \frac{s + 1}{(\xi^p + \frac{s^p}{2})^{1/p}} \right) \leq \sup_{s > 0} \left( \frac{s + 1}{(\frac{s_p - 1}{2} + \frac{s^p}{2})^{1/p}} \right) = \sup_{a > 0} \left( \frac{1 + a}{\sqrt[2p]{a^p - 1 + \frac{1}{2}}} \right),
\]

and the last expression is just the one which appears on the right-hand side of (5).

In case \( p = 2 \), condition (5) is equivalent to \( \sigma < \sqrt{2} + \sqrt{2} \) (see [13]). We therefore see that \( M_1(l_2) = \sqrt{2} + \sqrt{2} \). It is worth noting that for \( X = l_2 \) fixed point theorems stronger than Corollary 15 can be found in the literature (see [18,22]).

In contrast, let us consider the space \( X = (\mathbb{R} \oplus c_0)_{l_1} \). We shall show that condition (5) fails for every \( \sigma \geq 1 \)

\[
M_1(X) = \frac{1 + \sqrt{5}}{2}.
\]

If \( x = (\alpha, u) \in X \) where \( \alpha \in \mathbb{R}, u \in c_0 \), then

\[
d_X(t, x) = b_{1,X}(t, x) = \max\{\|u\|_{l_0}, t\} - \|u\|_{c_0}
\]

for every \( t \geq 0 \). Hence \( d_X^{-1}(t, x) = t + \|u\|_{c_0} \). It is also easy to obtain \( r_X(c) = \max\{0, c - 1\} \) for every \( c \geq 0 \) and, consequently, \( C_X(\sigma) = \sigma \) for every \( \sigma \geq 1 \). Thus the coefficient on the right-hand side of (5) equals \( M(X) \). In order to compute its value consider the point \( x = (1, 0) \in X \). The second formula in (10) shows that \( b_{1,X}(1, sx) = 1 \) for every \( s \geq 1 \). It follows that \( M\{X\} \leq 1 \). Since the opposite inequality holds for every space, \( M(X) = 1 \). We therefore see that condition (5) does not hold for any \( \sigma \geq 1 \).

To establish formula (9), we set \( \alpha_0 = (3 - \sqrt{5})/2 \) and \( \eta_0 = (\sqrt{5} - 1)/2 \). If \( 0 < t < \eta_0 \), then

\[
\sup_{s \geq \eta_0} \left( \frac{s + 1}{\max\{\eta_0, s\} + \alpha_0} \right) = \eta_0 + 1.
\]

Considering the point \( x = (\alpha_0, \eta_0 e_1) \in B_X \), where \( e_1 \) is the first vector of the standard basis of \( c_0 \), we therefore obtain

\[
M_1(X) \leq \lim_{t \to 0^+} \max \left\{ \frac{1}{1 - \|x\|_{l_0}}, \sup_{\eta_0 > \eta} \left( \frac{s + 1}{\max\{\eta_0, s\} + \alpha_0} \right) \right\} = \eta_0 + 1 = \frac{1 + \sqrt{5}}{2}.
\]

To get the opposite estimate, we put \( t_0 = 1/10 \) and take \( \alpha, \eta \geq 0 \) such that \( \alpha + \eta \leq 1 \). Then

\[
\sup_{s \geq \eta_0} \left( \frac{s + 1}{\max\{\eta, s\} + \alpha} \right) \geq \min \left\{ \frac{t_0 + 1}{t_0 + \alpha}, \frac{2 - \alpha}{\alpha} \right\}
\]

and if \( \alpha \in [0, 9/10] \), then \( 2 - \alpha \leq (t_0 + 1)/(t_0 + \alpha) \). For each \( \epsilon \in (0, 1/2) \), the equation \( 1/(1 - \alpha + \epsilon) = 2 - \alpha \) has the unique solution \( \alpha_\epsilon \) in the interval \((0, 9/10)\). Moreover, if \( \alpha \in (\alpha_\epsilon, 1] \), then \( 1/(1 - \alpha + \epsilon) > 2 - \alpha \). Hence
\[ M_1(X) \geq \lim_{\epsilon \to 0^+} \inf_{\alpha, \eta \geq 0} \max \left\{ \frac{1}{1 - \alpha + \epsilon}, \sup_{s \geq \eta_0} \left( \frac{s + 1}{\max\{\eta, s\} + \alpha} \right) \right\} \]

\[ \geq \lim_{\epsilon \to 0^+} \frac{1}{1 - \alpha_0 + \epsilon} = \frac{1}{1 - \alpha_0} = \frac{1 + \sqrt{5}}{2}. \]

It is worth mentioning that for the notion of NUNC spaces and Corollary 11 can be given a more general form. Namely, let \( \tau \) be a Hausdorff vector topology in a Banach space \( X \) such that there exists a sequence in \( SX \) which converges to zero with respect to \( \tau \). Then one can consider the property NUNC(\( \tau \)) whose definition is obtained from the definition of NUNC by replacing the weak convergence of sequences in the formulae for \( b \) and \( d \) by the convergence with respect to \( \tau \).

Let now \( (x_n) \) be a sequence in \( X \) converging to zero with respect to \( \tau \). The function

\[ \Gamma(x) = \limsup_{n \to \infty} \|x_n - x\| \]

where \( x \in X \), is called a \( \tau \)-null type associated to \( (x_n) \). Using results from [3] (see also [12]), one can easily modify the proof of Theorem 9 to obtain the following generalization of Corollary 11.

**Theorem 17.** Let \( X \) be a Banach space and \( \tau \) be a Hausdorff vector topology in \( X \) such that every \( \tau \)-sequentially compact set in \( X \) is \( \tau \)-compact and every \( \tau \)-null type is \( \tau \)-sequentially lower semicontinuous. If \( X \) is NUNC(\( \tau \)) and a nonempty set \( C \subset X \) is bounded, convex, and \( \tau \)-sequentially compact, then each nonexpansive mapping \( T : C \to C \) has a fixed point.

**Acknowledgment**

The authors thank the reviewer for suggestions which greatly improved the paper.

**References**


