Journal of Algebra 365 (2012) 29–41

Contents lists available at SciVerse ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

The Abhyankar–Jung Theorem

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ARTICLE INFO

Article history:
Received 11 April 2011
Available online 24 May 2012
Communicated by Steven Dale Cutkosky

MSC:
primary 13F25
secondary 13J15, 26E10

Keywords:
Quasi-ordinary polynomials
Newton polyhedron

ABSTRACT

We show that every quasi-ordinary Weierstrass polynomial $P(Z) = Z^d + a_1(X)Z^{d-1} + \cdots + a_d(X) \in \mathbb{K}[[X]][Z]$, $X = (X_1, \ldots, X_n)$, over an algebraically closed field of characteristic zero $\mathbb{K}$, such that $a_1 = 0$, is ν-quasi-ordinary. That means that if the discriminant $\Delta_P \in \mathbb{K}[[X]]$ is equal to a monomial times a unit then the ideal $(a_i^{d!/i}(X))_{i=2,\ldots,d}$ is monomial and generated by one of $a_i^{d!/i}(X)$. We use this result to give a constructive proof of the Abhyankar–Jung Theorem that works for any Henselian local subring of $\mathbb{K}[[X]]$ and the function germs of quasi-analytic families.

1. Introduction

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero and let

$$P(Z) = Z^d + a_1(X_1, \ldots, X_n)Z^{d-1} + \cdots + a_d(X_1, \ldots, X_n) \in \mathbb{K}[[X]][Z] \quad (1)$$

be a unitary polynomial with coefficients formal power series in $X = (X_1, \ldots, X_n)$. Such a polynomial P is called *quasi-ordinary* if its discriminant $\Delta_P(X)$ equals $X_1^{\alpha_1} \cdots X_n^{\alpha_n} U(X)$, with $\alpha_i \in \mathbb{N}$ and $U(0) \neq 0$. We call $P(Z)$ *a Weierstrass polynomial* if $a_i(0) = 0$ for all $i = 1, \ldots, d$.

We show the following result.

Theorem 1.1. *Let $\mathbb{K}$ be an algebraically closed field of characteristic zero and let $P \in \mathbb{K}[[X]][Z]$ be a quasi-ordinary Weierstrass polynomial such that $a_1 = 0$. Then the ideal $(a_i^{d!/i}(X))_{i=2,\ldots,d}$ is monomial and generated by one of $a_i^{d!/i}(X)$.*

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0021-8693/$ – see front matter
http://dx.doi.org/10.1016/j.jalgebra.2012.05.003

The latter condition is equivalent to P being ν-quasi-ordinary in the sense of Hironaka [H1,Lu], and satisfying $a_1 = 0$. Being ν-quasi-ordinary is a condition on the Newton polyhedron of P that we recall in Section 3 below. Thus Theorem 1.1 can be rephrased as follows.

Theorem 1.2. *(See [Lu], Theorem 1.) If P is a quasi-ordinary Weierstrass polynomial with $a_1 = 0$ then P is ν-quasi-ordinary.*

As noticed in [K-V], Luengo's proof of Theorem 1.2 is not complete. We complete the proof of Luengo and thus we complete his proof of the Abhyankar–Jung Theorem.

Theorem 1.3 *(Abhyankar–Jung Theorem). Let $\mathbb{K}$ be an algebraically closed field of characteristic zero and let $P \in \mathbb{K}[[X]][Z]$ be a quasi-ordinary Weierstrass polynomial such that the discriminant of P satisfies $\Delta_P(X) = X_1^{\alpha_1} \cdots X_r^{\alpha_r} U(X)$, where $U(0) \neq 0$, and $r \leqslant n$. Then there is $q \in \mathbb{N} \setminus \{0\}$ such that $P(Z)$ has its roots in $\mathbb{K}[[X_1^{\frac{1}{q}}, \ldots, X_r^{\frac{1}{q}}, X_{r+1}, \ldots, X_n]]$.*

Theorem 1.3 has first been proven by Jung in 1908 for $n = 2$ and $\mathbb{K} = \mathbb{C}$ in order to give a local uniformisation of singular complex analytic surfaces [J]. His method has been then used by Walker [W] and Zariski [Z] to give proofs of resolution of singularities of surfaces, see [PP] for a detailed account of the Jung's method of resolution of singularities of complex surfaces. The first complete proof of Theorem 1.3 appeared in [Ab]. As shown in [Lu], Theorem 1.3 follows fairly easily from his Theorem 1 (our Theorem 1.2). Since then there were other proofs of Theorem 1.3 based on Theorem 1 of [Lu], see e.g. [Zu].

Theorem 1.1 is proven in Section 3. In Section 5 we show how Theorem 1.1 gives a procedure to compute the roots of P, similar to the Newton algorithm for $n = 1$ (as done in [B-M2]), and thus implies the Abhyankar–Jung Theorem. Unlike the one in [Lu], our procedure does not use the Weierstrass Preparation Theorem, but only the Implicit Function Theorem. Thanks to this we are able to extend the Abhyankar–Jung Theorem to Henselian subrings of $\mathbb{K}[[X]]$, and quasi-analytic families of function germs answering thus a question posed in [R]. A similar proof of this latter result was given in [N1] assuming Theorem 1 of [Lu]. In [N2] is also given a proof of the Abhyankar–Jung Theorem for excellent Henselian subrings of $\mathbb{K}[[X]]$ using model theoretic methods and Artin Approximation.

It is not difficult to see that Theorem 1.1 and the Abhyankar–Jung Theorem are equivalent, one implies easily the other. As we mentioned above Theorem 1.1 gives the Abhyankar–Jung Theorem. We show in Section 4 how Theorem 1.1 can be proven using the Abhyankar–Jung Theorem, see also [Z]. We also give in Section 4 an alternative proof of Theorem 1.1 that uses the complex analytic version of the Abhyankar–Jung Theorem and the Artin Approximation Theorem.

Finally, in Section 6, we extend Abhyankar–Jung Theorem to the toric case following our alternative proof of Theorem 4 and using a complex analytic version of the Abhyankar–Jung Theorem in the toric case proven by P. González Pérez [G-P1].

Remark 1.4. Neither in Theorem 1.1 nor in the Abhyankar–Jung Theorem the assumption that P is Weierstrass is necessary. Moreover, in Theorem 1.1 the assumption that $\mathbb{K}$ is algebraically closed is not necessary. If $\mathbb{K}$ is not algebraically closed then the roots of P may have coefficients in a finite extension of $\mathbb{K}$, see Proposition 5.1 below.

Notation. The set of natural numbers including zero is denoted by $\mathbb{N}$. We denote $\mathbb{Q}_{\geqslant 0} = \{x \in \mathbb{Q};\ x \geqslant 0\}$ and $\mathbb{Q}_+ = \{x \in \mathbb{Q};\ x > 0\}$. Similarly, by $\mathbb{R}_{\geqslant 0}$ we denote the set $\{x \in \mathbb{R};\ x \geqslant 0\}$.

2. Preliminary results

The following proposition is well known, see for instance [S] or [N2]. We present its proof for the reader's convenience.

Proposition 2.1. *The Abhyankar–Jung Theorem holds for quasi-ordinary polynomials with complex analytic coefficients, $P \in \mathbb{C}\{X\}[Z]$.*

Proof. Fix a polydisc $U = \prod_{i=1}^{n} D_\varepsilon = \{X \in \mathbb{C}^n;\ |X_i| < \varepsilon,\ i = 1, \ldots, n\}$ such that the coefficients $a_i(X)$ of P are analytic on a neighbourhood of $\overline{U}$. By assumption, the projection of $\{(X, Z) \in U \times \mathbb{C};\ P(X, Z) = 0\}$ onto U is a finite branched covering. Its restriction over $U^* = \{X \in U;\ X_i \neq 0, i = 1, \ldots, r\}$ is a finite covering of degree d. Thus there is a substitution of powers

$$X(Y) = \left(Y_1^q, \ldots, Y_r^q, Y_{r+1}, \ldots, Y_n\right) : U_1 \to U,$$

where $U_1 = \prod_{i=1}^{r} D_{\varepsilon^{1/q}} \times \prod_{i=r+1}^{n} D_\varepsilon$, such that the induced covering over $U_1^* = \prod_{i=1}^{r} D^*_{\varepsilon^{1/q}} \times \prod_{i=r+1}^{n} D_\varepsilon$ is trivial. That is to say on U_1^*, $P(X(Y), Z)$ factors

$$P\big(X(Y), Z\big) = \prod \big(Z - f_i(Y)\big),$$

with f_i complex analytic and bounded on U_1^*.[1]

Hence, by Riemann Removable Singularity Theorem, see e.g. [GR], Theorem 3, p. 19, each f_i extends to an analytic function on U_1. □

Remark 2.2. If the coefficients of $P(Z)$ are global analytic functions, and $\Delta_P(X) = X^\alpha u(X)$ globally, where $u(X)$ is nowhere vanishing in $\mathbb{C}$, then we can choose $U = \mathbb{C}^n$ in the former proof. Thus, using the notations of this proof, we see that after a substitution of powers $X_i = Y_i^q$, for $1 \leqslant i \leqslant r$, we may assume that the roots of $P(Z)$ are global analytic.

Given a polynomial $P(X_1, \ldots, X_n, Z) \in \mathbb{K}[X_1, \ldots, X_n, Z]$, where $\mathbb{K}$ is a field of characteristic zero, denote by $\mathbb{K}_1$ the field generated by the coefficients of P. Since $\mathbb{K}_1$ is finitely generated over $\mathbb{Q}$ there exists a field embedding $\mathbb{K}_1 \hookrightarrow \mathbb{C}$. This allows us to extend some results from complex polynomials to the polynomials over $\mathbb{K}$. This is a special case of the Lefschetz principle. We shall need later two such results.

Proposition 2.3. *Let*

$$P(Z) = Z^d + a_1(X)Z^{d-1} + \cdots + a_d(X) \in \mathbb{K}[X][Z] \tag{2}$$

be quasi-ordinary (as a polynomial with coefficients in $\mathbb{K}[[X]]$). Then $(P_{|X_n=0})_{red}$ is quasi-ordinary. Moreover, the discriminant of $(P_{|X_n=0})_{red}$ divides the discriminant of P.

Proof. Denote $Q(X', Z) = P(X_1, X_2, \ldots, X_{n-1}, 0, Z)$, where $X' = (X_1, X_2, \ldots, X_{n-1})$. Let $Q = \prod Q_i^{m_i}$ be the factorisation into irreducible factors. Then $(P_{|X_n=0})_{red} = \prod Q_i$. We may assume that P and each of the Q_i's are defined over a subfield of $\mathbb{C}$. Thus, by Proposition 2.1, the roots of P are complex analytic after a substitution of powers, we write them as $Z_1(X), \ldots, Z_d(X) \in \mathbb{C}\{X_1^{1/q}, \ldots, X_n^{1/q}\}$ for some $q \in \mathbb{N}$. Since $\Delta_P(X) = \prod_{i \neq j}(Z_i(X) - Z_j(X))$,

$$Z_{i,j}(X) = Z_i(X) - Z_j(X) = X^{\beta_{ij}} u_{ij}(X),$$

[1] Fix $u_0 \in U^*$. The fundamental group $\pi_1(U^*, u_0)$ is equal to $\mathbb{Z}^r$. To each connected finite covering $h : \tilde{U}^* \to U^*$ and each $\tilde{u}_0 \in h^{-1}(u_0)$ corresponds a subgroup $h_*(\pi_1(\tilde{U}^*, \tilde{u}_0)) \subset \pi_1(U^*, u_0)$ of finite index. If $(q\mathbb{Z})^r \subset h_*(\pi_1(\tilde{U}^*, \tilde{u}_0))$, then the covering corresponding to $(q\mathbb{Z})^r \subset \mathbb{Z}^r$, that is the substitution of powers $X(Y) : U_1 \to U$, factors through h. That is there exists an analytic map $\tilde{U}^* \to \{(X, Z) \in U^* \times \mathbb{C};\ P(X, Z) = 0\}$, of the form $Y \to (X(Y), Z(Y))$. This $Z(Y)$ is one of the functions f_i. If we apply this argument to each connected component $\tilde{U}^*$ of $\{(X, Z) \in U^* \times \mathbb{C};\ P(X, Z) = 0\}$ and to each point of the fiber over u_0 we obtain d distinct analytic functions f_i.

where $\beta_{ij} \in \mathbb{Q}_{\geqslant 0}$, $u_{ij} \in \mathbb{C}\{X_1^{1/q}, \ldots, X_n^{1/q}\}$, $u_{ij}(0) \neq 0$. Taking $X_n = 0$ we see that the differences of the roots of $Q_{red} = \prod Q_i$ are the restrictions $Z_{ij}|_{X_n=0}$ and hence their product is a monomial times a unit, that is Q_{red} is quasi-ordinary. □

Proposition 2.4. *(See [Lu], Proposition 1.) Let $P \in \mathbb{K}[X][Z]$ be a polynomial of the form* (2) *such that $a_1 = 0$ and the discriminant $\Delta_P(X) = c_0 X^\alpha$, $c_0 \in \mathbb{K} \setminus 0$, $\alpha \neq 0$. Then, for each $i = 2, 3, \ldots, d$, $a_i(X) = c_i X^{i\alpha/d(d-1)}$, $c_i \in \mathbb{K}$.*

Proof. Consider first the case $n = 1$, $\mathbb{K} = \mathbb{C}$. By Proposition 2.1 and Remark 2.2, after a substitution of powers $X = Y^q$, there are analytic functions $f_i(Y)$, $i = 1, \ldots, d$, such that

$$P(Y^q, Z) = \prod_i \bigl(Z - f_i(Y)\bigr).$$

As a root of a polynomial each f_i satisfies

$$\bigl|f_i(Y)\bigr| \leqslant C\bigl(1 + |Y|^N\bigr)$$

for $C, N \in \mathbb{R}$, see e.g. [BR], 1.2.1. Hence, by Liouville's Theorem, cf. [Ti], Section 2.52, p. 85, f_i is a polynomial. By assumption, $\Delta_P(Y^q) = c_0 Y^{q\alpha}$, and hence each difference $f_i - f_j$ is a monomial. For i, j, k distinct we have $(f_i - f_j) + (f_j - f_k) + (f_k - f_i) = 0$, and therefore all these monomials should have the same exponent $(f_i - f_j)(Y) = c_{i,j} Y^\beta$, where $\beta = \frac{q}{d(d-1)}\alpha$. Finally, since $a_1 = 0$, each f_i is a monomial:

$$f_i = \sum_j \frac{f_i - f_j}{d}.$$

In the general case we consider $P \in \mathbb{K}[X_1, \ldots, X_n][Z]$ as a polynomial in X_n, Z with coefficients in $\mathbb{K}' = \mathbb{K}(X_1, \ldots, X_{n-1})$ and $\mathbb{K}' \hookrightarrow \mathbb{C}$. Therefore, for every i, a_i equals $X_n^{i\alpha_n/d(d-1)}$ times a constant of the algebraic closure of $\mathbb{K}'$. Since a_i is a polynomial in (X', X_n) it must be equal to $X_n^{i\alpha_n/d(d-1)}$ times a polynomial in X'. Applying this argument to each variable X_j, $j = 1, \ldots, n$, we see that a_i is the product of all $X_j^{i\alpha_j/d(d-1)}$ and a constant of $\mathbb{K}$. This ends the proof. □

3. 1st proof of Theorem 1.1

Given $P \in \mathbb{K}[[X_1, \ldots, X_n, Z]]$. Write

$$P(X, Z) = \sum_{(i_1, \ldots, i_{n+1})} P_{i_1, \ldots, i_{n+1}} X^{i_1} \cdots X^{i_n} Z^{i_{n+1}}.$$

Let $H(P) = \{(i_1, \ldots, i_{n+1}) \in \mathbb{N}^{n+1};\ P_{i_1, \ldots, i_{n+1}} \neq 0\}$. The Newton polyhedron of P is the convex hull in $\mathbb{R}^{n+1}$ of $\bigcup_{a \in H(P)} (a + \mathbb{R}_{\geqslant 0}^{n+1})$, and we will denote it by $\mathrm{NP}(P)$.

A Weierstrass polynomial (1) is called *ν-quasi-ordinary* if there is a point R_1 of the Newton polyhedron $\mathrm{NP}(P)$, $R_1 \neq R_0 = (0, \ldots, 0, d)$, such that if R_1' denotes the projection of R_1 onto $\mathbb{R}^n \times 0$ from R_0, and $S = |R_0, R_1'|$ is the segment joining R_0 and R_1', then

(1) $\mathrm{NP}(P) \subset |S| = \bigcup_{s \in S} (s + \mathbb{R}_{\geqslant 0}^{n+1})$.
(2) $P_S = \sum_{(i_1, \ldots, i_{n+1}) \in S} P_{i_1, \ldots, i_{n+1}} X_1^{i_1} \cdots X_n^{i_n} Z^{i_{n+1}}$ is not a power of a linear form in Z.

The second condition is satisfied automatically if $a_1 = 0$.

Lemma 3.1. *Let $P(Z) \in \mathbb{K}[[X]][Z]$ be a Weierstrass polynomial* (1) *such that $a_1 = 0$. The following conditions are equivalent*:

(1) *P is ν-quasi-ordinary.*
(2) $\mathrm{NP}(P)$ *has only one compact edge containing R_0.*
(3) *the ideal $(a_i^{d!/i}(X))_{i=2,\dots,d} \subset \mathbb{K}[[X]]$ is monomial and generated by one of $a_i^{d!/i}(X)$.*

Remark 3.2. The ideal $(a_i^{d!/i}(X))_{i=2,\dots,d} \subset \mathbb{K}[[X]]$ is exactly the idealistic exponent introduced by Hironaka (see [H2]).

Proof of Lemma 3.1. (3) holds if and only if there is $\gamma \in \mathbb{N}^n$ and $i_0 \in \{2,\dots,d\}$ such that $a_{i_0}(X) = X^\gamma U_{i_0}(X)$, $U_{i_0}(0) \neq 0$, and for all $j \in \{2,\dots,d\}$, $X^{j\gamma}$ divides $a_j^{i_0}$. Thus we may take $R_1 = (\gamma, d - i_0)$ and conversely by this formula R_1 defines i_0 and γ. Thus (1) is equivalent to (3).

Let us denote by π_0 the projection from R_0 onto $\mathbb{R}^n \times 0$. Then both (1) and (2) are equivalent to $\pi_0(\mathrm{NP}(P))$ being of the form $p + \mathbb{R}^n_{\geqslant 0}$ for some $p \in \mathbb{R}^n$. □

Let $P(Z) \in \mathbb{K}[[X]][Z]$ be a quasi-ordinary polynomial of degree d with $a_1 = 0$. Let us assume that $P(Z)$ is not ν-quasi-ordinary. Then, as shown in the proof of Theorem 1 of [Lu], p. 403, there exists $\beta = (\beta_1,\dots,\beta_{n+1}) \in (\mathbb{N} \setminus \{0\})^{n+1}$ such that

(1) $L(u) := \beta_1 u_1 + \dots + \beta_{n+1} u_{n+1} - d\beta_{n+1} = 0$ is the equation of a hyperplane H of $\mathbb{N}^{n+1}$ containing $(0,\dots,0,d)$,
(2) $H \cap \mathrm{NP}(P)$ is a compact face of $\mathrm{NP}(P)$ of dimension $\geqslant 2$,
(3) $L(\mathrm{NP}(P)) \geqslant 0$.

The existence of such β can be also shown as follows. Each $\beta \in \mathbb{R}^{n+1}_{\geqslant 0}$ defines a face Γ_β of $\mathrm{NP}(P)$ by

$$\Gamma_\beta = \Big\{ v \in \mathrm{NP}(P);\ \langle \beta, v\rangle = \min_{u \in \mathrm{NP}(P)} \langle \beta, u\rangle \Big\}.$$

Each face of $\mathrm{NP}(P)$ can be obtained this way. Moreover, since the vertices of $\mathrm{NP}(P)$ have integer coefficients, each face can be defined by $\beta \in \mathbb{Q}^{n+1}_{\geqslant 0}$ and even $\beta \in \mathbb{N}^{n+1}$ by multiplying it by an integer. If one of the coordinates of β is zero then Γ_β is not compact. Thus it suffices to take as β a vector in $(\mathbb{N} \setminus \{0\})^{n+1}$ defining a compact face containing R_0 and of dimension $\geqslant 2$.

Let

$$P_H(X, Z) := \sum_{i_1,\dots,i_{n+1} \in H} P_{i_1,\dots,i_{n+1}} X_1^{i_1} \dots X_n^{i_n} Z^{i_{n+1}},$$

and define $\tilde{P}_H$ as P_H reduced. If $\mathrm{NP}(P_H) = H \cap \mathrm{NP}(P)$ is not included in a segment, neither is $\mathrm{NP}(\tilde{P}_H)$. Thus, by Proposition 2.4, there is $c \in (\mathbb{K}^*)^n$ such that $\Delta_{\tilde{P}_H}(c) = 0$. We show that this contradicts the assumption that P is quasi-ordinary.

Let

$$\begin{aligned} Q(\tilde{X}_1,\dots,\tilde{X}_n, T, Z) &= T^{-d\beta_{n+1}} P\big((c_1 + \tilde{X}_1)T^{\beta_1},\dots,(c_n + \tilde{X}_n)T^{\beta_n}, ZT^{\beta_{n+1}}\big) \\ &= P_H(c + \tilde{X}, Z) + \sum_{m=1}^{\infty} P_m(\tilde{X}_1,\dots,\tilde{X}_n, Z)T^m. \end{aligned}$$

Write $(c+X)T^\beta$ for $((c_1+\tilde{X}_1)T^{\beta_1},\dots,(c_n+\tilde{X}_n)T^{\beta_n})$. If $\Delta_P(X) = X^\alpha U(X)$ then the discriminant of Q is given by

$$\Delta_Q(\tilde{X},T)=T^{-d(d-1)}\Delta_P\big((c+\tilde{X})T^\beta\big)=T^M(c+\tilde{X})^\alpha U\big((c+\tilde{X})T^\beta\big),$$

where $M=\sum_i \alpha_i\beta_i-d(d-1)$. Let $Q_k(\tilde{X},T,Z)=P_H(c+\tilde{X},Z)+\sum_{m=1}^{k-1}P_mT^m$. Then $Q(\tilde{X},T,Z)-Q_k(\tilde{X},T,Z)\in(T)^k$ and hence $\Delta_Q(\tilde{X},T)-\Delta_{Q_k}(\tilde{X},T)\in(T)^{(d-1)k}$. That means that for k sufficiently large, $k(d-1)>M$, $\Delta_{Q_k}(\tilde{X},T)$ equals $T^MU_1(\tilde{X},T)$ in $\mathbb{K}[[\tilde{X},T]]$, where $U_1(0)\neq 0$. Here we use the fact that all $c_i\neq 0$ and hence $(c+\tilde{X})^\alpha$ is invertible.

Since $\beta_i>0$ for $i=1,\dots,n$, all P_m are polynomials and hence Q_k is a polynomial. By Proposition 2.3, the discriminant of $(P_H(c+\tilde{X},Z))_{red}=\tilde{P}_H(c+\tilde{X},Z)$ divides Δ_{Q_k}, and therefore has to be nonzero at $\tilde{X}=0$. This contradicts the fact that $\Delta_{\tilde{P}_H}(c)=0$. This ends the proof of Theorem 1.1. □

Corollary 3.3. *If $\Delta_P(X)=X_1^{\alpha_1}\cdots X_r^{\alpha_r}U(X)$ with $r\leqslant n$, then there is $\gamma\in\mathbb{N}^r\times 0$ and $i_0\in\{2,\dots,d\}$ such that $a_{i_0}(X)=X^\gamma U_{i_0}(X)$, $U_{i_0}(0)\neq 0$, and $X^{j\gamma}$ divides $a_j^{i_0}$ for all $j\in\{2,\dots,d\}$.*

Proof. By Theorem 1.1 there is such $\gamma\in\mathbb{N}^n$. If $\Delta_P(X)$ is not divisible by X_k then there is at least one coefficient a_i that is not divisible by X_k. □

4. 2nd proof of Theorem 1.1

First we show that Theorem 1.3 implies Theorem 1.1. This proposition is well known, see [Z] for instance.

Proposition 4.1. *Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. Let $P(Z)\in\mathbb{K}[[X]][Z]$ be a quasi-ordinary Weierstrass polynomial with $a_1=0$. If there is $q\in\mathbb{N}\setminus\{0\}$ such that $P(Z)$ has its roots in $\mathbb{K}[[X^{\frac{1}{q}},\dots,X_n^{\frac{1}{q}}]]$ then P is ν-quasi-ordinary.*

Proof. Let $P(Z)\in\mathbb{K}[[X_1,\dots,X_n]][Z]$ be a quasi-ordinary polynomial such that its roots $Z_1(X),\dots,Z_d(X)\in\mathbb{K}[[X^{1/q}]]$ for some $q\in\mathbb{N}\setminus\{0\}$. In what follows we assume for simplicity $q=1$, substituting the powers if necessary. For $i\neq j$,

$$Z_{i,j}(X)=Z_i(X)-Z_j(X)=X^{\beta_{ij}}u_{ij}(X),\quad u_{ij}(0)\neq 0.$$

For each i fixed, the series $Z_{i,j}$, $j\neq i$, and their differences are normal crossings (that is monomial times a unit). By the lemma below, the set $\{\beta_{i,j},\ j=1,\dots,d\}$ is totally ordered.

Lemma 4.2. *(See [Z,B-M1], Lemma 4.7.) Let $\alpha,\beta,\gamma\in\mathbb{N}^n$ and let $a(X),b(X),c(X)$ be invertible elements of $\mathbb{K}[[X]]$. If*

$$a(X)X^\alpha-b(X)X^\beta=c(X)X^\gamma,$$

then either $\alpha_i\leqslant\beta_i$ for all $i=1,\dots,n$ or $\beta_i\leqslant\alpha_i$ for all $i=1,\dots,n$.

Proof of Lemma 4.2. For $f=\sum_{\alpha\in\mathbb{N}^n}f_\alpha X^\alpha\in\mathbb{K}[[X]]$, let $\operatorname{Supp}(f):=\{\alpha\in\mathbb{N}^n\ /\ f_\alpha\neq 0\}$ be the support of f. We always have $\operatorname{Supp}(c(X)X^\gamma)\subset\gamma+\mathbb{N}^n$ and, since $c(X)$ is invertible, $\gamma\in\operatorname{Supp}(c(X)X^\gamma)$. Since $a(X)X^\alpha-b(X)X^b=c(X)X^\gamma$, then

$$\operatorname{Supp}\big(c(X)X^\gamma\big)\subset\operatorname{Supp}\big(a(X)X^\alpha\big)\cup\operatorname{Supp}\big(b(X)X^\beta\big)\subset\big(\alpha+\mathbb{N}^n\big)\cup\big(\beta+\mathbb{N}^n\big).$$

Thus either $\gamma\in\alpha+\mathbb{N}^n$ or $\gamma\in\beta+\mathbb{N}^n$. If $\gamma\in\alpha+\mathbb{N}^n$, then X^α divides X^γ, hence $b(X)X^\beta$ is divisible by X^α which means that $\alpha\leqslant\beta$ component-wise. □

Denote $\beta_i = \min_{j\neq i} \beta_{i,j}$.

Lemma 4.3. *We have $\beta_1 = \beta_2 = \cdots = \beta_d$. Denote this common exponent by β. Then each a_i is divisible by $(X^\beta)^i$.*

Proof. For i, j, k distinct we have $\beta_{i,j} \geqslant \min\{\beta_{i,k}, \beta_{j,k}\}$ (with the equality if $\beta_{i,k} \neq \beta_{j,k}$). Therefore $\beta_{i,j} \geqslant \beta_k$ and hence $\beta_i \geqslant \beta_k$. This shows $\beta_1 = \beta_2 = \cdots = \beta_d$. Because $a_1 = 0$,

$$Z_i = Z_i - \frac{1}{d}\sum_{k=1}^{d} Z_k = \sum_{k=1}^{d} \frac{Z_i - Z_k}{d}$$

is divisible by X^β. □

To complete the proof we show that there is i_0 such that $a_{i_0}/X^{i_0\beta}$ does not vanish at the origin. By Lemma 4.3 we may write

$$Z_i(X) = X^\beta \tilde{Z}_i(X).$$

Then i_0 is the number of i such that $\tilde{Z}_i(0) \neq 0$, and then $\gamma = i_0\beta$.

Remark 4.4. The set $\{\beta_{i,j}\}$ determines many properties of the hypersurface germ defined by the quasi-ordinary polynomial P (see for instance [G,Li]).

Remark 4.5. It is possible to define a change of coordinates of the form $Z' = Z + a(X)$ in such a way that the Newton polyhedron of the quasi-ordinary polynomial $P(Z' - a(X))$ has only one compact face of dimension $\leqslant 1$ (see [G-P2]).

In order to prove Theorem 1.1 we use Proposition 2.1. Hence by Proposition 4.1, Theorem 1.1 is true for any $P(Z) \in \mathbb{C}\{X\}[Z]$.

Let $\mathbb{K}$ be any algebraically closed field of characteristic zero. Let $P(Z) \in \mathbb{K}[[X]][Z]$, $P(Z) = Z^d + a_1(X)Z^{d-1} + \cdots + a_d(X)$. Then the coefficients of the a_i's are in a field extension of $\mathbb{Q}$ generated by countably many elements and denoted by $\mathbb{K}_1$. Since $\operatorname{trdeg}_{\mathbb{Q}} \mathbb{C}$ is not countable and since $\mathbb{C}$ is algebraically closed, there is an embedding $\mathbb{K}_1 \hookrightarrow \mathbb{C}$. Since the conditions of being quasi-ordinary and ν-quasi-ordinary does not depend on the embedding $\mathbb{K}_1 \hookrightarrow \mathbb{C}$, we may assume that $P(Z) \in \mathbb{C}[[X_1, \dots, X_n]][Z]$.

Then let us assume that $P(Z) \in \mathbb{C}[[X_1, \dots, X_n]][Z]$ such that $a_1 = 0$ and $\Delta_P(X) = X^\alpha u(X)$ with $u(0) \neq 0$. Let us remark that $\Delta(X) = R(a_2(X), \dots, a_n(X))$ for some polynomial $R(A_2, \dots, A_d) \in \mathbb{Q}[A_2, \dots, A_d]$. Let us denote by $Q \in \mathbb{Q}[X_1, \dots, X_n][A_2, \dots, A_d, U]$ the following polynomial:

$$Q(A_2, \dots, A_d, U) := \Delta(A_2, \dots, A_d) - X^\alpha U.$$

Then $Q(a_2(X), \dots, a_d(X), u(X)) = 0$. By the Artin Approximation Theorem (cf. [Ar1], Theorem 1.2), for every integer $j \in \mathbb{N}$, there exist $a_{2,j}(X), \dots, a_{n,j}(X)$, $u_j(X) \in \mathbb{C}\{X_1, \dots, X_n\}$ such that

$$Q\big(a_{2,j}(X), \dots, a_{n,j}(X), u_j(X)\big) = 0,$$

$a_k(X) - a_{k,j}(X) \in (X)^j$ and $u(X) - u_j(X) \in (X)^j$. Let us denote

$$P_j(Z) := Z^d + a_{2,j}(X)Z^{d-2} + \cdots + a_{d,j}(X) \in \mathbb{C}\{X_1, \dots, X_n\}[Z].$$

Then $P_j(Z)$ is quasi-ordinary for $j \geqslant 1$ and $\mathrm{NP}(P) \subset \mathrm{NP}(P_j) + \mathbb{N}^n_{\geqslant j}$ where

$$\mathbb{N}^n_{\geqslant j} := \{k \in \mathbb{N}^n \mathbin{/} k_1 + \cdots + k_n \geqslant j\}.$$

If $P(Z)$ were not ν-quasi-ordinary then $\mathrm{NP}(P)$ would have a compact face of dimension at least 2 and containing the point $(0, \ldots, 0, d)$. For $j > j_0 := \max|\gamma|$ where γ runs through the vertices of $\mathrm{NP}(P)$, we see that this compact face is also a face of $\mathrm{NP}(P_j)$ and this contradicts the fact that $P_j(Z)$ is ν-quasi-ordinary. Thus Theorem 1.1 is proven.

In fact, by using the Strong Artin Approximation Theorem, we can prove the following result about the continuity of the Newton polyhedra of $P(Z)$ with respect to its discriminant.

Proposition 4.6. *For any $d \in \mathbb{N}$ and any $\alpha \in \mathbb{N}^n$, there exists a function $\beta : \mathbb{N} \to \mathbb{N}$ satisfying the following property: for any $k \in \mathbb{N}$ and any Weierstrass polynomial $P(Z) = Z^d + a_1 Z^{d-1} + \cdots + a_d \in \mathbb{K}[[X_1, \ldots, X_n]][Z]$ of degree d such that $a_1 = 0$ and its discriminant $\Delta_P = X^\alpha U(X)$ mod. $(X)^{\beta(k)}$ there exists a compact edge S containing $R_0 := (0, \ldots, 0, d)$ such that one has* $\mathrm{NP}(P) \subset |S| + \mathbb{N}^n_{\geqslant k}$.

Proof. Let $d \in \mathbb{N}$ and $\alpha \in \mathbb{N}^n$. Let us denote by $Q \in \mathbb{Q}[[X_1, \ldots, X_n]][A_2, \ldots, A_d, U]$ the polynomial $Q(A_2, \ldots, A_d, U) := \Delta(A_2, \ldots, A_d) - X^\alpha U$ where $\Delta(A)$ is the discriminant of the polynomial $Z^d + A_2 Z^{d-2} + \cdots + A_d$.

By the Strong Artin Approximation Theorem (cf. [Ar2], Theorem 6.1), there exists a function $\beta : \mathbb{N} \to \mathbb{N}$ such that for any $k \in \mathbb{N}$ and any $a_2, \ldots, a_d, u \in \mathbb{K}[[X]]$ with $Q(a_2, \ldots, a_d, u) \in (X)^{\beta(k)}$ there exist $\bar{a}_2, \ldots, \bar{a}_d, \bar{u} \in \mathbb{K}[[X]]$ such that $Q(\bar{a}_2, \ldots, \bar{a}_d, \bar{u}) = 0$ and $\bar{a}_i - a_i$, $\bar{u} - u \in (X)^k$ for all i.

Let $P(Z) = Z^d + a_2 Z^{d-2} + \cdots + a_d$ such that $\Delta(P) = X^\alpha U(X)$ mod. $(X)^{\beta(k)}$. Then there exists a polynomial $\overline{P}(Z)$ such that its discriminant $\overline{\Delta}(\overline{P}) = X^\alpha \overline{U}(X)$ with $\overline{U}(0) \neq 0$ and $P(Z) - \overline{P}(Z) \in (X)^k$. By Theorem 1.1 $NP(\overline{P}) \subset |S|$ hence $NP(P) \subset |S| + \mathbb{N}^n_{\geqslant k}$. □

5. Applications

5.1. Proof of the Abhyankar–Jung Theorem

Let

$$P(Z) = Z^d + a_1(X) Z^{d-1} + \cdots + a_d(X),$$

$a_i \in \mathbb{K}[[X]]$, and let $\mathbb{K}$ be an algebraically closed field of characteristic zero. We suppose that the discriminant of P is of the form $X^\alpha U(X)$, $U(0) \neq 0$. It is not necessary to suppose that all $a_i(0) = 0$ (of course Theorem 1.1 holds if one of the a_i's does not vanish at the origin). The procedure consists of a number of steps simplifying the polynomial and finally factorising it to two polynomials of smaller degree. Theorem 1.1 is used in Step 2.

Step 1 *(Tschirnhausen transformation).* Replace Z by $Z - \frac{a_1(X)}{d}$. The coefficients $a = (a_1, a_2, \ldots, a_d)$ are replaced by $(0, \tilde{a}_2, \ldots, \tilde{a}_d)$ so we can assume $a_1 = 0$.

Step 2. Write $Z = X^\beta \tilde{Z}$, and divide each a_i by $X^{i\beta}$, where $\beta = \gamma / i_0$ for γ and i_0 given by Corollary 3.3. If the coordinates of β are not integers this step involves a substitution of powers. Then

$$P(Z) = P(X^\beta \tilde{Z}) = X^{d\beta}(\tilde{Z}^d + \tilde{a}_1(X)\tilde{Z}^{d-1} + \cdots + \tilde{a}_d(X)), \tag{3}$$

where $\tilde{a}_i = a_i / X^{i\beta}$. We replace $P(Z)$ by $\tilde{P}(Z) = Z^d + \tilde{a}_1(X) Z^{d-1} + \cdots + \tilde{a}_d(X)$.

Step 3. Now $\tilde{a}_{i_0} \neq 0$ and since $\tilde{a}_1 = 0$ the polynomial $Q(Z) = \tilde{P}(Z)|_{X=0} \in \mathbb{K}[Z]$ has at least two distinct roots in $\mathbb{K}$ and can be factored $Q(Z) = Q_1(Z)Q_2(Z)$, $d_i := \deg Q_i < d$, $i = 1, 2$, where $Q_1(Z)$ and $Q_2(Z)$ are two polynomials of $\mathbb{K}[Z]$ without common root.

Step 4. By the Implicit Function Theorem there is a factorisation $\tilde{P}(Z) = \tilde{P}_1(Z)\tilde{P}_2(Z)$ with $\tilde{P}_i(Z)_{|X=0} = Q_i(Z)$ for $i = 1, 2$. More precisely, let

$$q(Z) = Z^d + a_1 Z^{d-1} + \cdots + a_d,$$
$$q_1(Z) = Z^{d_1} + b_1 Z^{d_1-1} + \cdots + b_{d_1}, \qquad q_2(Z) = Z^{d_2} + c_1 Z^{d_2-1} + \cdots + c_{d_2}$$

where $a = (a_1, \dots, a_d) \in \mathbb{K}^d$, $b = (b_1, \dots, b_{d_1}) \in \mathbb{K}^{d_1}$, $c = (c_1, \dots, c_{d_2}) \in \mathbb{K}^{d_2}$. The product of polynomials $q = q_1 q_2$ defines a map $a = \Phi(b, c)$, $\Phi : \mathbb{K}^d \to \mathbb{K}^d$, that is polynomial in b and c. The Jacobian determinant of Φ equals the resultant of q_1 and q_2. Denote by b_0, c_0 the coefficient vectors of Q_1 and Q_2 and consider $\tilde{\Phi} : \mathbb{K}[[X]]^d \to \mathbb{K}[[X]]^d$ given by $\tilde{\Phi}(b, c) = \Phi(b + b_0, c + c_0) - \tilde{a}(0)$. Then the Jacobian determinant of $\tilde{\Phi}$ is invertible and hence, by the Implicit Function Theorem for formal power series, the inverse of $\tilde{\Phi}$ is a well-defined power series. Define $(b(X), c(X)) \in \mathbb{K}[[X]]^d$ as $\tilde{\Psi}^{-1}(\tilde{a}(X) - \tilde{a}(0)) + (b_0, c_0)$. Then $\tilde{P}(Z) = \tilde{P}_1(Z)\tilde{P}_2(Z)$ where $\tilde{P}_1(Z) = Z^{d_1} + b_1(X)Z^{d_1-1} + \cdots + b_{d_1}(X)$ and $\tilde{P}_2(Z) = Z^{d_2} + c_1(X)Z^{d_2-1} + \cdots + c_{d_2}(X)$.

We may describe the outcome of Steps 2–4 by the following. Denote the new polynomial obtained in Step 2 by $\tilde{P}(\tilde{Z})$, where $\tilde{Z} = Z/X^\beta$,

$$\tilde{P}(\tilde{Z}) = \tilde{Z}^d + \tilde{a}_1(X)\tilde{Z}^{d-1} + \cdots + \tilde{a}_d(X).$$

Then by Step 4 we may factor $\tilde{P} = \tilde{P}_1\tilde{P}_2$, $d_1 = \deg \tilde{P}_1 < \deg P$, $d_2 = \deg \tilde{P}_1 < \deg P$, and

$$P(Z) = X^{d\beta}\tilde{P}(\tilde{Z}) = X^{d_1\beta}\tilde{P}_1(\tilde{Z})X^{d_2\beta}\tilde{P}_2(\tilde{Z}) = P_1(Z)P_2(Z). \tag{4}$$

The discriminant of P is equal to the product of the discriminants of P_1 and P_2 and the resultant of P_1 and P_2. Hence Δ_{P_1}, and similarly Δ_{P_2}, is equal to a monomial times a unit. Thus we continue the procedure for $P_1(Z)$ and $P_2(Z)$ until we reduce to polynomials of degree one. This ends the proof. □

Note that if $\mathbb{K}$ is a field of characteristic zero not necessarily algebraically closed then in Step 3 we may need a finite field extension. Thus we obtain the following result, see [Lu] the last page proposition.

Proposition 5.1. *Let $P \in \mathbb{K}[[X_1, \dots, X_n]][Z]$ be a quasi-ordinary Weierstrass polynomial with coefficients in a field of characteristic zero (not necessarily algebraically closed). Then there is a finite extension $\mathbb{K}' \supset \mathbb{K}$ such that the roots of $P(Z)$ are in $\mathbb{K}'[[X^{\frac{1}{q}}]]$ for some $q \geqslant 1$.*

Remark 5.2. It is not true in general that the roots of ν-quasi-ordinary Weierstrass polynomials are Puiseux series in several variables. In the latter algorithm, its is not difficult to check that in (4), P_1 and P_2 satisfy property (1) of the definition of ν-quasi-ordinary polynomials but not property (2). For example let

$$P(Z) := Z^4 - 2X_1X_2(1 - X_1 - X_2)Z^2 + (X_1X_2)^2(1 + X_1 + X_2)^2.$$

This polynomial is ν-quasi-ordinary and factors as

$$P(Z) = \left(Z^2 + 2(X_1X_2)^{\frac{1}{2}}Z + X_1X_2(1 + X_1 + X_2)\right)\left(Z^2 - 2(X_1X_2)^{\frac{1}{2}}Z + X_1X_2(1 + X_1 + X_2)\right)$$

in $\mathbb{K}[[X^{\frac{1}{2}}]][Z]$. One has

$$Z^2+2(X_1X_2)^{\frac{1}{2}}Z+X_1X_2(1+X_1+X_2)=\left(Z+(X_1X_2)^{\frac{1}{2}}\right)^2+X_1X_2(X_1+X_2).$$

This shows that $P(Z)$ is irreducible in $\mathbb{K}[[X]][Z]$ and that none of its roots is a Puiseux series in X_1, X_2 (all roots are branched along $X_1+X_2=0$).

5.2. Abhyankar–Jung Theorem for Henselian subrings of $K[[X]]$

Consider Henselian subrings of $\mathbb{K}[[X]]$ which do not necessarily have the Weierstrass division property.

Definition 5.3. We will consider $\mathbb{K}\{\{X_1,\dots,X_n\}\}$ a subring of $\mathbb{K}[[X_1,\dots,X_n]]$ such that:

(i) $\mathbb{K}\{\{X_1,\dots,X_n\}\}$ contains $\mathbb{K}[X_1,\dots,X_n]$.
(ii) $\mathbb{K}\{\{X_1,\dots,X_n\}\}$ is a Henselian local ring with maximal ideal generated by $X_1,\dots,X_n$.
(iii) $\mathbb{K}\{\{X_1,\dots,X_n\}\}\cap(X_i)\mathbb{K}[[X_1,\dots,X_n]]=(X_i)\mathbb{K}\{\{X\}\}$.
(iv) If $f\in\mathbb{K}\{\{X\}\}$ then $f(X_1^{e_1},\dots,X_n^{e_n})\in\mathbb{K}\{\{X\}\}$ for any $e_i\in\mathbb{N}\setminus\{0\}$.

Example 5.4. The rings of algebraic or formal power series over a field satisfy Definition 5.3. If $\mathbb{K}$ is a valued field, then the ring of convergent power series over $\mathbb{K}$ satisfies also this definition. The ring of germs of quasi-analytic functions over $\mathbb{R}$ also satisfies this definition (even if there is no Weierstrass Division Theorem in this case, see [C] or [ES]). We come back to this example in the next subsection.

Since the Implicit Function Theorem holds for such rings (they are Henselian) we obtain by the procedure of Section 5.1 the following result.

Theorem 5.5. *Let $\mathbb{K}\{\{X_1,\dots,X_n\}\}$ be a subring of $\mathbb{K}[[X_1,\dots,X_n]]$ like in Definition 5.3. Moreover let us assume that $\mathbb{K}$ is an algebraically closed field of characteristic zero. Let $P(Z)\in\mathbb{K}\{\{X\}\}$ be a quasi-ordinary Weierstrass polynomial such that its discriminant,*

$$\Delta(X)=X_1^{\alpha_1}\cdots X_r^{\alpha_r}U(X),$$

$r\leqslant n$, where α_i are positive integers and $U(0)\neq 0$. Then there exists an integer $q\in\mathbb{N}\setminus\{0\}$ such that the roots of $P(Z)$ are in $\mathbb{K}\{\{X_1^{\frac{1}{q}},\dots,X_r^{\frac{1}{q}},X_{r+1},\dots,X_n\}\}$.

5.3. Quasi-analytic functions

Denote by $\mathcal{E}_n$ the algebra of complex valued C^∞ germs of n real variables: $f:(\mathbb{R}^n,0)\to\mathbb{C}$. We call a subalgebra $\mathcal{C}_n\subset\mathcal{E}_n$ *quasi-analytic* if the Taylor series morphism $\mathcal{C}_n\to\mathbb{C}[[X_1,\dots,X_n]]$ is injective. If this is the case we identify $\mathcal{C}_n$ with its image in $\mathbb{C}[[X_1,\dots,X_n]]$. Usually one considers families of algebras $\mathcal{C}_n$ defined for all $n\in\mathbb{N}$ and satisfying some additional properties, such as stability by differentiation, taking implicit functions, composition, etc., see [T].

If $\mathcal{C}_n$ is Henselian in the sense of Definition 5.3, that is practically always the case, then we may apply Theorem 5.5. Since the arguments of quasi-analytic functions are real, the substitution of powers $X_i=Y_i^{\gamma_i}$ is not surjective if one of γ_i's is even. Thus for the sake of applications, cf. [R], it is natural to consider the power substitutions with signs $X_i=\varepsilon_iY_i^{\gamma_i}$, $\varepsilon_i=\pm1$. Thus Theorem 5.5 implies the following.

Theorem 5.6 *(Abhyankar–Jung Theorem for quasi-analytic germs). Let $\mathcal{C}_n$ be a quasi-analytic algebra satisfying Definition 5.3. Let the discriminant $\Delta_P(X)$ of*

$$P(Z) = Z^d + a_1(X_1, \ldots, X_n)Z^{d-1} + \cdots + a_d(X_1, \ldots, X_n) \in \mathcal{C}_n[Z]$$

satisfy $\Delta_P(X) = X_1^{\alpha_1} \cdots X_r^{\alpha_r} U(X)$, *where* $U(0) \neq 0$, *and* $r \leqslant n$. *Then there is* $\gamma \in (\mathbb{N} \setminus \{0\})^r$ *such that for every combination of signs* $\varepsilon \in \{-1, 1\}^r$, *the polynomial*

$$Z^d + a_1\big(\varepsilon_1 Y_1^{\gamma_1}, \ldots, \varepsilon_r Y_r^{\gamma_r}, Y_{r+1}, \ldots, Y_n\big)Z^{d-1} + \cdots + a_d\big(\varepsilon_1 Y_1^{\gamma_1}, \ldots, \varepsilon_r Y_r^{\gamma_r}, Y_{r+1}, \ldots, Y_n\big)$$

has d *distinct roots in* $\mathcal{C}_n$.

6. Toric case

We thank Pedro González Pérez who pointed out that our proof of Abhyankar–Jung Theorem may be generalised to the toric case. This is the aim of this section.

Let $\sigma \subset \mathbb{R}^n$ be a rational strictly convex polyhedral cone of dimension d. Let

$$\sigma^\vee := \big\{v \in \big(\mathbb{R}^n\big)^* \ / \ \langle v, u\rangle \geqslant 0, \ \forall u \in \sigma\big\}$$

be the dual cone of σ. Let $V_\sigma := \operatorname{Spec}(\mathbb{K}[X^v \,/\, v \in \sigma^\vee \cap \mathbb{Z}^d])$ the associated affine toric variety. The ideal $\mathfrak{m}_\sigma$ generated by the X^v, when v runs through $\sigma^\vee \cap (\mathbb{Z}^d)^*$, is a maximal ideal defining a closed point of V_σ denoted by 0. In fact $\mathbb{K}[X^v \,/\, v \in \sigma^\vee \cap \mathbb{Z}^d] \simeq \mathbb{K}[Y]/I$ where $Y = (Y_1, \ldots, Y_m)$ for some integer m and I is a binomial ideal. In this case $\mathfrak{m}_\sigma \simeq (Y)$.

When $\mathbb{K} = \mathbb{C}$ we define $\mathcal{O}_{V_\sigma,0} := \mathbb{C}\{X^v\}_{v \in \sigma^\vee \cap \mathbb{Z}^d} \simeq \mathbb{C}\{Y\}/I$ the ring of germs of analytic functions at $(V_\sigma, 0)$.

Let $P(Z) = Z^d + a_1 Z^{d-1} + \cdots + a_d \in \mathbb{K}[[X^v]]_{v \in \sigma^\vee \cap \mathbb{Z}^d}[Z]$ be a toric polynomial. The polynomial $P(Z)$ is called *quasi-ordinary* if its discriminant equals $X^\alpha U(X)$, $\alpha \in \sigma^\vee$ and $U(X)$ being a unit of $\mathbb{K}[[X^v]]_{v \in \sigma^\vee \cap \mathbb{Z}^d}$. We call $P(Z)$ *a Weierstrass polynomial* if $a_i \in \mathfrak{m}_\sigma$ for $i = 1, \ldots, d$. In [G-P1], P. González Pérez proved the following theorem:

Theorem 6.1. *(See [G-P1].) Let* $P(Z) \in \mathbb{C}\{X^v\}_{v \in \sigma^\vee \cap \mathbb{Z}^d}[Z]$ *be a toric quasi-ordinary polynomial. Then there exists* $q \in \mathbb{N}$ *such that* $P(Z)$ *has its roots in* $\mathbb{C}\{X^v\}_{v \in \sigma^\vee \cap \frac{1}{q}\mathbb{Z}^d}$.

Here we will prove a generalisation of this result over any algebraically closed field $\mathbb{K}$ of characteristic zero.

Theorem 6.2 *(Toric Abhyankar–Jung Theorem). Let* $\mathbb{K}\{\{X_1, \ldots, X_n\}\}$ *be a subring of* $\mathbb{K}[[X_1, \ldots, X_n]]$ *like in Definition 5.3. Moreover let us assume that* $\mathbb{K}$ *is an algebraically closed field of characteristic zero. Let* $P \in \mathbb{K}\{\{X^v\}\}_{v \in \sigma^\vee \cap \mathbb{Z}^d}[Z]$ *be a toric quasi-ordinary Weierstrass polynomial. Then there is* $q \in \mathbb{N} \setminus \{0\}$ *such that* $P(Z)$ *has its roots in* $\mathbb{K}\{\{X^v\}\}_{v \in \sigma^\vee \cap \frac{1}{q}\mathbb{Z}^d}$.

First we define ν-quasi-ordinary polynomials in the toric case. Given $P \in \mathbb{K}[[X^v, Z]]_{v \in \sigma^\vee \cap \mathbb{Z}^d}$. Write

$$P(X, Z) = \sum_{(i_1, \ldots, i_{n+1})} P_{i_1, \ldots, i_{n+1}} X^{i_1} \cdots X^{i_n} Z^{i_{n+1}}.$$

Let $H(P) = \{(i_1, \ldots, i_{n+1}) \in (\sigma^\vee \cap \mathbb{Z}^d) \times \mathbb{N};\ P_{i_1, \ldots, i_{n+1}} \neq 0\}$. The Newton polyhedron of P is the convex hull in $\mathbb{R}^{n+1}$ of $\bigcup_{a \in H(P)} (a + (\sigma^\vee \cap \mathbb{Z}^d) \times \mathbb{R}_{\geqslant 0})$, and we will denote it by $\mathrm{NP}(P)$.

A Weierstrass polynomial as before is called *ν-quasi-ordinary* if there is a point R_1 of the Newton polyhedron $\mathrm{NP}(P)$, $R_1 \neq R_0 = (0, \ldots, 0, d)$, such that if R_1' denotes the projection of R_1 onto $\mathbb{R}^n \times 0$ from R_0, and $S = |R_0, R_1'|$ is the segment joining R_0 and R_1', then

(1) $\mathrm{NP}(P) \subset |S| = \bigcup_{s\in S}(s + (\sigma^\vee \cap \mathbb{Z}^d) \times \mathbb{R}_{\geqslant 0})$.
(2) $P_S = \sum_{(i_1,\dots,i_{n+1})\in S} P_{i_1,\dots,i_{n+1}} X_1^{i_1}\cdots X_n^{i_n} Z^{i_{n+1}}$ is not a power of a linear form in Z.

The second condition is satisfied automatically if $a_1 = 0$. The proof of the following lemma is the same as the proof of Lemma 3.1.

Lemma 6.3. *Let $P(Z) \in \mathbb{K}[[X^\nu]]_{\nu\in\sigma^\vee\cap\mathbb{Z}^d}[Z]$ be a Weierstrass polynomial* (1) *such that $a_1 = 0$. The following conditions are equivalent*:

(1) *P is ν-quasi-ordinary.*
(2) *$\mathrm{NP}(P)$ has only one compact edge containing R_0.*
(3) *The ideal $(a_i^{d!/i}(X))_{i=2,\dots,d} \subset \mathbb{K}[[X^\nu]]_{\nu\in\sigma^\vee\cap\mathbb{Z}^d}$ is monomial and principal.*

In order to prove Theorem 6.2 we first show the toric version of Theorem 1.1.

Theorem 6.4. *Let $\mathbb{K}$ be an algebraically closed field of characteristic zero and let $P \in \mathbb{K}[[X^\nu]]_{\nu\in\sigma^\vee\cap\mathbb{Z}^d}[Z]$ be a toric quasi-ordinary Weierstrass polynomial such that $a_1 = 0$. Then the ideal $(a_i^{d!/i})_{i=2,\dots,d} \subset \mathbb{K}[[X^\nu]]_{\nu\in\sigma^\vee\cap\mathbb{Z}^d}$ is principal and generated by a monomial.*

As before, this theorem may be reformulated as the following:

Theorem 6.5. *If P is a toric quasi-ordinary Weierstrass polynomial with $a_1 = 0$ then P is ν-quasi-ordinary.*

Proposition 6.6. *Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. Let $P(Z) \in \mathbb{K}[[X^\nu]]_{\nu\in\sigma^\vee\cap\mathbb{Z}^d}[Z]$ be a quasi-ordinary Weierstrass polynomial with $a_1 = 0$. If there is $q \in \mathbb{N}\setminus\{0\}$ such that $P(Z)$ has its roots in $\mathbb{K}[[X^\nu]]_{\nu\in\sigma^\vee\cap\frac{1}{q}\mathbb{Z}^d}$ then P is ν-quasi-ordinary.*

Proof of Proposition 6.6. The proof of Proposition 6.6 is exactly the same as the proof of Proposition 4.1. We only need the following lemma:

Lemma 6.7. *Let $\alpha, \beta, \gamma \in \sigma^\vee \cap \mathbb{Z}^d$ and let $a(X), b(X), c(X)$ be invertible elements of $\mathbb{K}[[X^\nu]]_{\nu\in\sigma^\vee\cap\mathbb{Z}^d}$. If*

$$a(X)X^\alpha - b(X)X^\beta = c(X)X^\gamma,$$

then either $\alpha \in \beta + \sigma^\vee$ or $\beta \in \alpha + \sigma^\vee$.

The proof of Lemma 6.7 is exactly the same as the proof of Lemma 4.2, we only need to replace $\mathbb{N}^n$ by $\sigma^\vee$. □

Proof of Theorem 6.5. The proof of Theorem 6.5 is similar to the second proof of Theorem 1.1, Section 4. We replace Proposition 4.1 by Proposition 6.6, $\mathbb{K}[[X]]$ by $\mathbb{K}[[X^\nu]]_{\nu\in\sigma^\vee\cap\mathbb{Z}^d}$ and $\mathbb{C}\{X\}$ by $\mathbb{C}\{X^\nu\}_{\nu\in\sigma^\vee\cap\mathbb{Z}^d} \simeq \mathbb{C}\{Y\}/I$. Then we use the fact that the ring $\mathbb{C}\{Y\}/I$ satisfies the Artin Approximation Theorem (cf. [Ar1], Theorem 1.3). □

Proof of Theorem 6.2. We prove Theorem 6.2 exactly as we proved Theorem 1.3. Step 1, Step 2 and Step 3 are exactly the same (we just replace $P(Z)|_{X=0}$ by the image of $P(Z)$ in $\mathbb{K}\{\{X^\nu\}\}_{\nu\in\sigma^\vee\cap\mathbb{Z}^d}[Z]/\mathfrak{m}_\sigma$). For Step 4, we replace the Implicit Function Theorem by Hensel's Lemma (cf. [EGA], 18.5.13) since $\mathbb{K}\{\{X^\nu\}\}_{\nu\in\sigma^\vee\cap\mathbb{Z}^d} \simeq \mathbb{K}\{\{Y\}\}/I$ is a local Henselian ring. □

Acknowledgment

The authors would like to thank P. González Pérez for his valuable suggestions, in particular concerning the toric case.

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