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Constructing Irreducible Representations of Finitely Presented Algebras

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We describe an algorithmic test, using the "standard polynomial identity" (and elementary computational commutative algebra), for determining whether or not a finitely presented associative algebra has an irreducible n-dimensional representation. When n-dimensional irreducible representations do exist, our proposed procedure can (in principle) produce explicit constructions.

1. Introduction

Our aim in this paper is to suggest a general algorithmic approach to the finite-dimensional irreducible representations of finitely presented algebras, combining well-known methods from both noncommutative ring theory and computational commutative algebra. (There have been numerous previous studies, from an algorithmic perspective, on matrix representations of finitely presented groups and algebras; see, e.g. Labonté (1990), Linton (1991), Plesken and Souvignier (1997) for analyses that—as in our study below—do not place additional technical conditions on the groups or algebras involved.)

1.1. To briefly describe the content of this paper, assume that k is a computable field, and that $\overline{k}$ denotes the algebraic closure of k. Suppose further that n is a fixed positive integer, that $M_n(\overline{k})$ is the algebra of $n\times n$ matrices over $\overline{k}$, and that R is a finitely presented k-algebra. We will always use the expression *n-dimensional representation of R* to mean a k-algebra homomorphism $\rho\colon R \to M_n(\overline{k})$, and we will say that ρ is *irreducible* when $M_n(\overline{k})$ is $\overline{k}$-linearly spanned by $\rho(R)$ (cf. e.g. Artin (1969, Section 9)). Note that ρ is irreducible if and only if $\rho\otimes 1\colon R\otimes_k \overline{k} \to M_n(\overline{k})$ is surjective, if and only if $\rho\otimes 1$ is irreducible in the more common use of the term.

1.2. Calculating over k, the procedure described in this paper always (in principle):

(a) decides whether irreducible representations $R \to M_n(\overline{k})$ exist,
(b) explicitly constructs an irreducible representation $R \to M_n(\overline{k})$ if at least one exists (assuming that $k[x]$ is equipped with a factoring algorithm).

1.3. The finite-dimensional irreducible representations of finitely generated noncommutative algebras were parametrized, up to equivalence (i.e. up to isomorphisms among the corresponding modules), in the famous work of Artin (1969), Formanek (1972), Procesi

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(1974), Razmyslov (1973), and others (see, for example, Formanek, 1991; McConnell and Robson, 1987; or Rowen, 1980). The algorithm we describe, however, does not distinguish among equivalence classes of irreducible representations; it depends only on the Amitsur–Levitzky theorem (e.g. McConnell and Robson, 1987, 13.3.3) and the recent work of Pappacena (1997). In Letzter (2001) we present a procedure that counts the number (possibly infinite) of equivalence classes of irreducible representations, in characteristic zero.

1.4. Examples are discussed in Section 4. All of the computational commutative algebra used in this paper is elementary, and the necessary background can be found in Cox *et al.* (1997), for example.

2. Representations of Finitely Presented Algebras

While most of the material in this section is known, we provide a complete treatment, for the reader's convenience.

2.1. (i) Retaining the notation of (1.1), let $\mathbb{M}$ denote the affine space $(M_n(\overline{k}))^s$ of s-tuples of $n\times n$ matrices over $\overline{k}$.

(ii) Let

$$B = k[x_{ij}(\ell) : 1 \leq i, j \leq n; 1 \leq \ell \leq s]$$

be the commutative algebra of polynomial functions, with coefficients in k, on $\mathbb{M}$.

(iii) For $1 \leq \ell \leq s$, let $\mathbf{x}_\ell$ denote the generic matrix

$$(x_{ij}(\ell)) \in M_n(B).$$

(iv) Let K be a commutative ring, and let $K\{X_1, \ldots, X_m\}$ be the free associative K-algebra in the noncommuting variables $X_1, \ldots, X_m$. Given a K-module M, we will also regard $K\{X_1, \ldots, X_m\}$ as an algebra of noncommutative polynomial functions from M^m to M.

(v) Choose $f_1, \ldots, f_t \in k\{X_1, \ldots, X_s\}$, and set

$$R = k\{X_1, \ldots, X_s\}/\langle f_1, \ldots, f_t\rangle.$$

We will let $X_1, \ldots, X_s$ also denote their images in R.

Our aim now is to algorithmically determine whether irreducible representations $R \to M_n(\overline{k})$ exist, and to construct one if they do.

2.2. (i) Each representation $\rho\colon R \to M_n(\overline{k})$ is determined exactly by the point

$$(\rho(X_1), \ldots, \rho(X_s)) \in \mathbb{M}.$$

In particular, the set of representations $R \to M_n(\overline{k})$ can be identified with

$$\{(\Gamma_1, \ldots, \Gamma_s) \in \mathbb{M} \mid f_1(\Gamma_1, \ldots, \Gamma_s) = \cdots = f_t(\Gamma_1, \ldots, \Gamma_s) = 0\},$$

which is equal to the closed subvariety $V(\mathrm{Rel}(B))$ of $\mathbb{M}$, where $\mathrm{Rel}(B)$ is the ideal of B generated by the entries of the matrices

$$f_1(\mathbf{x}_1, \ldots, \mathbf{x}_s), \ldots, f_t(\mathbf{x}_1, \ldots, \mathbf{x}_s)$$

in $M_n(B)$.

(ii) Let P denote the set of s-tuples $(\Gamma_1, \ldots, \Gamma_s) \in \mathbb{M}$ for which the $\overline{k}$-algebra generated by the $\Gamma_1, \ldots, \Gamma_s$ is not equal to $M_n(\overline{k})$. Since P is equal to the set of s-tuples of simultaneously block-upper-triangularizable matrices (for non-$n \times n$ blocks), we see that P is a closed subvariety of $\mathbb{M}$.

(iii) Suppose that P is defined by the equations $g_1 = \cdots = g_q = 0$ in B. By definition, there exists an irreducible representation $R \to M_n(\overline{k})$ if and only if $V(\mathrm{Rel}(B)) \not\subseteq P$. Therefore, there exists an irreducible representation $R \to M_n(\overline{k})$ if and only if at least one g_i is not contained in the radical of $\mathrm{Rel}(B)$. Consequently, the radical membership algorithm can be used to determine whether or not there exists an irreducible n-dimensional representation of R.

It remains, then, to specify a set of defining equations for P.

2.3. Let K be a field, and let A be a K-subalgebra of $M_n(K)$. Suppose further that A is generated, as a K-algebra, by the set G. Let $p = n^2$. It is easy to see that A is K-linearly spanned by

$$\{a_1 \cdots a_i \mid a_1, \ldots, a_i \in G,\ 0 \le i < p\},$$

where the product corresponding to $i = 0$ is the identity matrix. It follows from Pappacena (1997) that the preceding conclusion remains true if we instead use

$$p = n\sqrt{2n^2/(n-1) + 1/4} + n/2 - 2.$$

(Moreover, by the Cayley–Hamilton theorem, we can always replace a^n, for $a \in A$, by a K-linear combination of $1, a, a^2, \ldots, a^{n-1}$.)

2.4. We now turn to polynomial identities. Our brief treatment here is distilled from McConnell and Robson (1987, Chapter 13) (cf. Formanek, 1991, Rowen, 1980). Let A be a k-algebra, and let $g \in \mathbb{Z}\{Y_1, \ldots, Y_m\}$.

(i) If X is a subset of A then the set $\{g(a_1, \ldots, a_m) \mid a_1, \ldots, a_m \in X\}$ will be designated $g(X)$.

(ii) The *mth standard identity* is

$$s_m \;=\; \sum_{\sigma \in S_m} (\operatorname{sgn}\sigma) Y_{\sigma(1)} \cdots Y_{\sigma(m)} \;\in\; \mathbb{Z}\{Y_1, \ldots, Y_m\}.$$

Observe that $s_m \colon A^m \to A$ is $Z(A)$-multilinear and alternating, where $Z(A)$ denotes the center of A.

2.5. (i) Let K be a commutative ring. The Amitsur–Levitzky theorem (see, for example, McConnell and Robson, 1987, 13.3.3) ensures that

$$s_{2m'}(M_m(K)) = 0$$

for all $m' \ge m$. Moreover, $s_{2m'}(M_m(K)) \ne 0$ for all $m' < m$ (e.g. McConnell and Robson, 1987, 13.3.2).

(ii) Let K be a field, and suppose that A is a proper K-subalgebra of $M_n(K)$. Let J denote the Jacobson radical of A. The semisimple algebra A/J will embed (as a non-unital subring) into a direct sum of copies of $M_m(K)$, for some $m < n$. It therefore follows from (i) that $s_{2(n-1)}(A/J) = 0$, and so $s_{2(n-1)}(A) \subseteq J$.

2.6. Let K be a field, and let A be a K-subalgebra of $M_n(K)$. Let J denote the Jacobson radical of A, and set

$$L = A \cdot s_{2(n-1)}(A),$$

a left ideal of A.

(i) Suppose that A is a proper subalgebra of $M_n(K)$. Then L is a left ideal contained within J, by (2.5), and hence L is a nilpotent left ideal of A. In particular, every matrix in L has trace zero.

(ii) If $A = M_n(K)$ then L is a left ideal of $M_n(K)$, and at least one matrix in L has nonzero trace.

(iii) Let $T = \{b_1, \ldots, b_N\}$ be a K-linear spanning set for A. Set

$$V = \{b_{m_0} \cdot s_{2(n-1)}(b_{m_1}, \ldots, b_{m_{2(n-1)}}) \;\mid\; 1 \le m_0 \le N, \quad 1 \le m_1 < \cdots < m_{2(n-1)} \le N\},$$

and note that V is a linear generating set for L. (Recall that $s_{2(n-1)}$ is multilinear and alternating.)

(iv) We conclude that A is a proper subalgebra of $M_n(K)$ if and only if $\{\text{trace}(v) \mid v \in V\} = \{0\}$.

(v) Suppose that A is generated, as a K-algebra, by $G \subseteq M_n(K)$. Choosing p as in (2.3), we may take

$$T = \{a_1 \cdots a_i \;\mid\; a_1, \ldots, a_i \in G,\ 0 \le i < p\}.$$

2.7. We now prove that the proposed algorithm satisfies the claims made in (1.1). A summary of the procedure, and comments on it, will be presented in the next section.

(i) Let p be as in (2.3), and set

$$S = \{\mathbf{x}_{\ell_1} \cdots \mathbf{x}_{\ell_i} \;\mid\; 0 \le i < p\} \subseteq M_n(B).$$

(ii) Write $S = \{M_1, \ldots, M_N\}$, and set $U =$

$$\{M_{m_0} \cdot s_{2(n-1)}(M_{m_1}, \ldots, M_{m_{2(n-1)}}) \;\mid\; 1 \le m_0 \le N, \quad 1 \le m_1 < \cdots < m_{2(n-1)} \le N\}.$$

(iii) Recall $P \subseteq \mathbb{M}$ from (2.2ii). It follows from (2.6) that

$$\{\text{trace}(u) = 0 \mid u \in U\}$$

is a set of defining equations, in B, for P.

(iv) Following (2.2), there exists an irreducible representation $R \to M_n(\overline{k})$ if and only if $\text{trace}(U) = \{\text{trace}(u) \mid u \in U\}$ is not contained in the radical of $\text{Rel}(B)$, and we may therefore use the radical membership test to determine whether or not R has an irreducible n-dimensional representation.

(v) Suppose that $y \in \text{trace}(U) \subseteq B$ is not contained in the radical of $\text{Rel}(B)$. Further suppose that $k[x]$ is equipped with a factoring algorithm. Elimination methods can now be applied to find a homomorphism $\varphi\colon B \to \overline{k}$ such that $y \notin \ker\varphi$ and such that $\text{Rel}(B) \subseteq \ker\varphi$. The assignment

$$X_\ell \longmapsto (\varphi(x_{ij}(\ell))) \in M_n(\overline{k}),$$

for $1 \le \ell \le s$, will then produce an irreducible n-dimensional representation of R.

(vi) Other sets of polynomials can be used to define P. For example, we can rewrite the matrices in S as $n^2 \times 1$ column matrices, and then concatenate all possible combinations of n^2 of them, to form $n^2 \times n^2$-matrices over B. Letting D denote the set of determinants of these matrices, we see that $P = V(D)$. (My thanks to Zinovy Reichstein for this observation.) The variety P can also be defined using the well-known central polynomials described, for example, in Formanek (1972) and Razmyslov (1973).

(vii) Suppose that $s = n = 2$. Note that $\Gamma_1, \Gamma_2 \in M_2(\overline{k})$ generate $M_2(\overline{k})$ as a $\overline{k}$-algebra if and only if Γ_1 and Γ_2 are not simultaneously upper triangularizable. By considering the possible Jordan canonical forms of Γ_1, it is not hard to verify that Γ_1 and Γ_2 generate $M_2(\overline{k})$ if and only if $\det(\Gamma_1\Gamma_2 - \Gamma_2\Gamma_1) \neq 0$. Therefore, in this case, R has an irreducible two-dimensional representation if and only if $\det(\mathbf{x}_1\mathbf{x}_2 - \mathbf{x}_2\mathbf{x}_1)$ is not contained in the radical of $\mathrm{Rel}(B)$. The reader is referred to Boularas and Bouzar (1996) and Friedland (1983) for a complete discussion of similarity classes of $2{\times}2$ matrices.

3. The Procedure

3.1. We now outline a procedure, based on the preceding section, that satisfies (1.1). A proof that the process works follows from (2.7).

1. Input

(i) n is a positive integer.
(ii) k is a computable field, and $\overline{k}$ is the algebraic closure of k.
(iii) $R = k\{X_1, \ldots, X_s\}/\langle f_1, \ldots, f_t\rangle$.

2. Notation

(i) B is the polynomial ring in commuting variables $x_{ij}(\ell)$, for $1 \leq i, j \leq n$ and $1 \leq \ell \leq s$.
(ii) $M_n(B)$ is the k-algebra of $n{\times}n$ matrices over B, and $\mathbf{x}_\ell$ denotes the ℓth generic matrix $(x_{ij}(\ell)) \in M_n(B)$.
(iii) $\mathrm{Rel}(B)$ denotes the ideal of B generated by the entries of $f_1(\mathbf{x}_1, \ldots, \mathbf{x}_s)$, $\ldots$, $f_t(\mathbf{x}_1, \ldots, x_s)$.
(iv) $s_{2(n-1)} = \sum_{\sigma \in S_{2(n-1)}} (\operatorname{sgn}\sigma) Y_{\sigma(1)} \cdots Y_{\sigma(2(n-1))} \in \mathbb{Z}\{Y_1, \ldots, Y_{2(n-1)}\}$, the $2(n-1)$th standard polynomial.

3. Decision

(i) For $p = 4$ when $n = 2$, and for (e.g.) $p = n\sqrt{2n^2/(n-1) + 1/4} + n/2 - 2$ otherwise (see (2.3)), set

$$S = \{\mathbf{x}_{\ell_1} \cdot \cdots \cdot \mathbf{x}_{\ell_m} \mid m < p\}.$$

(By the Cayley–Hamilton theorem, we may—for example—exclude from the preceding set those terms containing $\mathbf{x}_\ell^n$, for $1 \leq \ell \leq s$.) Choose an ordering for S, say $S = \{M_1, \ldots, M_N\}$.
(ii) Set $U =$

$$\{M_{m_0} \cdot s_{2(n-1)}(M_{m_1}, \ldots, M_{m_{2(n-1)}}) \mid 1 \leq m_0 \leq N, \quad 1 \leq m_1 < \cdots < m_{2(n-1)} \leq N\}.$$

(Recall that $s_{2(n-1)}$ is alternating.)
(iii) Applying the radical membership algorithm, determine whether any elements in $\mathrm{trace}(U)$ are contained in the radical of $\mathrm{Rel}(B)$. (Not every element of U needs its trace evaluated, since $\mathrm{trace}(YZ) = \mathrm{trace}(ZY)$). In addition, for $y \in \mathrm{trace}(U)$, it may be easier to test whether the image of y in $B/\mathrm{Rel}(B)$ is contained in the nilradical of $B/\mathrm{Rel}(B)$; working modulo $\mathrm{Rel}(B)$, the generic matrix arithmetic can often be significantly simplified.) If every element in $\mathrm{trace}(U)$ is contained in the radical of $\mathrm{Rel}(B)$ then there exist no irreducible representations $R \to M_n(\overline{k})$; see (2.7iv). If at least one element in $\mathrm{trace}(U)$ is not contained in the radical of $\mathrm{Rel}(B)$, then there exist irreducible representations $R \to M_n(\overline{k})$, and we may proceed to step 4.

4. Construction

If $k[x]$ is equipped with a factoring algorithm, and if $y \in \mathrm{trace}(U)$ is not contained in the radical of $\mathrm{Rel}(B)$:

(i) Apply elimination methods to solve the sn^2+1 commutative polynomial equations, in $B[z]$, obtained by setting $yz-1$ and the entries of $f_1(\mathbf{x}_1,\dots,\mathbf{x}_s),\dots,f_t(\mathbf{x}_1,\dots,\mathbf{x}_s)$ equal to zero. In this solution, say, $x_{ij}(\ell)=\lambda_{ij}(\ell)\in\overline{k}$, for $1\le i,j\le n$ and $1\le \ell\le s$.

(ii) The representation

$$\begin{array}{ccc} R & \longrightarrow & M_n(\overline{k}) \\ \\ X_\ell & \longmapsto & (\lambda_{ij}(\ell)) \end{array}$$

is irreducible, by (2.7v).

3.2. Remarks

(i) It is sensible, in practice, to first look for irreducible representations $\rho\colon R\to M_n(k)$ under simplifying assumptions. For example, one can initially suppose that one (or more) of the $\rho(X_\ell)$ are diagonal, or that a subset of the images of the $\rho(X_\ell)$ are triangular; see example (4.2). (Of course, for any commuting subset of the generators $X_\ell\in R$, there is no loss of generality in assuming that the images are all upper triangular.)

(ii) Roughly speaking, the cost of employing this procedure depends on the degrees of the polynomials involved in applications of the radical membership algorithm. Note, in general, that the polynomials in $\mathrm{trace}(U)$ may have degree p^{2n-1}. Another consideration will be the number of polynomials in $\mathrm{trace}(U)$ to which the radical membership algorithm, modulo $\mathrm{Rel}(B)$, is actually applied. This quantity appears difficult—in general—to precisely estimate and can vary greatly for different choices of $f_1,\dots,f_t$; see example (4.3). Observe, if the number of elements of S used in step 3 is equal to q, that the number of terms $M_{m_0}\cdot s_{2(n-1)}(M_{m_1},\dots,M_{m_{2(n-1)}})$ is $q\binom{q}{2(n-1)}$.

(iii) Recalling (2.7vi), one can use D instead of $\mathrm{trace}(U)$ in steps 3 and 4. In general, the polynomials in D can have degree p^{n^2}, and if q is the number of elements from S used in this approach then there will be $\binom{q}{n^2}$ polynomials to which the radical membership algorithm must be applied.

(iv) By (2.7vii), when $s=n=2$, we can replace $\mathrm{trace}(U)$ with the single polynomial $\det(\mathbf{x}_1\mathbf{x}_2-\mathbf{x}_2\mathbf{x}_1)$.

4. Examples

Retain the notation of the previous sections.

4.1. We begin with two-dimensional representations.

(i) Set

$$\mathbf{x}_1=\mathbf{x}=\begin{bmatrix} x_{11} & x_{12}\\ x_{21} & x_{22}\end{bmatrix},\qquad \mathbf{x}_2=\mathbf{y}=\begin{bmatrix} y_{11} & y_{12}\\ y_{21} & y_{22}\end{bmatrix},\qquad \mathbf{x}_3=\mathbf{z}=\begin{bmatrix} z_{11} & z_{12}\\ z_{21} & z_{22}\end{bmatrix}.$$

and

$$B=\mathbb{Q}[x_{11},x_{12},x_{21},x_{22},y_{11},y_{12},y_{21},y_{22},z_{11},z_{12},z_{21},z_{22}].$$

(ii) The value of p, as defined in (3.1.3i), is 4, and

$$S=\{abc \mid a,b,c\in\{1,\mathbf{x},\mathbf{y},\mathbf{z}\}\}.$$

(iii) Here, $s_{2(n-1)} = s_2(a,b) = ab - ba$ is the commutator. As in (3.1.3ii), order S, and let

$$U = \{a(a'a'' - a''a') \mid a, a', a'' \in S, \quad a' < a''\}.$$

(iv) Now set

$$R = \mathbb{Q}\{X, Y, Z\}/\langle (XY - YX)^2, (XZ - ZX)^2, (YZ - ZY)^2 \rangle.$$

Using Macaulay2, we found that

$$T = \operatorname{trace}(\mathbf{x}(\mathbf{yz} - \mathbf{zy}))$$

is not contained in the radical of Rel(B); see (3.1.3iii). Therefore, R has an irreducible two-dimensional representation. For example,

$$X \mapsto \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}, \qquad Y \mapsto \begin{bmatrix} 2 & 2 \\ 0 & -2 \end{bmatrix}, \qquad Z \mapsto \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix},$$

defines an irreducible two-dimensional representation in which $T \mapsto 16$. By (3.2iv), the subalgebras of R generated by any two of the generators X, Y, Z have no irreducible two-dimensional representations.

4.2. Continue to let R be as in (4.1); we now consider the case when $n = 3$. Let $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ denote the 3×3 generic matrices respectively corresponding to X, Y, and Z.

To make the calculations more manageable, one can first check to see if R has an irreducible three-dimensional representation in which X is diagonal, Y is upper triangular, and Z is lower triangular. With this simplification, using Macaulay2, we found that

$$T = \operatorname{trace}(\mathbf{x} \cdot s_4(\mathbf{y}, \mathbf{z}, \mathbf{xy}, \mathbf{xz}))$$

is not contained in the radical of Rel(B), and so R must have a three-dimensional irreducible representation. For instance,

$$X \mapsto \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \qquad Y \mapsto \begin{bmatrix} 2 & -1 & 2 \\ 0 & -2 & 8 \\ 0 & 0 & 2 \end{bmatrix}, \qquad Z \mapsto \begin{bmatrix} 2 & 0 & 0 \\ 4 & 2 & 0 \\ 2 & 2 & -1 \end{bmatrix}$$

produces a three-dimensional irreducible representation in which $T \mapsto 8192$.

4.3. Set $n = 3$ and $R =$

$$\mathbb{Q}\{a, b, X, Y\} \Big/ \!\!\Big/ \left\langle \begin{matrix} X^2 - a, Y^2 - b, \\ uv - vu \text{ for } u \in \{a, b\} \text{ and } v \in \{a, b, X, Y\} \end{matrix} \right\rangle.$$

Let $\mathbf{x}$ and $\mathbf{y}$ be the 3×3 generic matrices corresponding, respectively, to X and Y.

Following (3.1.3i), we can take $8 < p < 9$. In view of the defining relations for R, we may now set $S = \{M_1, \ldots, M_{17}\} =$

$$\left\{ \begin{matrix} 1, \mathbf{x}, \mathbf{y}, \mathbf{xy}, \mathbf{yx}, \mathbf{xyx}, \mathbf{yxy}, \mathbf{xyxy}, \mathbf{yxyx}, \mathbf{xyxyx}, \mathbf{yxyxy}, \mathbf{xyxyxy}, \\ \mathbf{yxyxyx}, \mathbf{xyxyxyx}, \mathbf{yxyxyxy}, \mathbf{yxyxyxyx}, \mathbf{xyxyxyxy} \end{matrix} \right\},$$

as in (3.1.3i). Following (3.1.3ii),

$$U = \{M_{m_0} \cdot s_4(M_{m_1}, M_{m_2}, M_{m_3}, M_{m_4}) \mid 1 \le m_0 \le 17,\ 1 \le m_1 < m_2 < m_3 < m_4 \le 17\}.$$

Using Macaulay2, we checked directly that every member of trace(U) is contained in the radical of Rel(B). Therefore, by (3.1.3iii), there exist no three-dimensional irreducible representations of R.

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References

Artin, M. (1969). On Azumaya algebras and finite-dimensional representations of rings. *J. Algebra*, **11**, 532–563.

Boularas, D., Bouzar, Z. (1996). Concomitants et p-uplets de matrices $2{\times}2$. *Linear Multilinear Algebr.*, **41**, 161–173.

Cox, D., Little, J., O'Shea, D. (1997). *Ideals, Varieties, and Algorithms*, 2nd edn, New York, Springer-Verlag.

Formanek, E. (1972). Central polynomials for matrix rings. *J. Algebra*, **23**, 129–132.

Formanek, E. (1991). The Polynomial Identites and Invariants of $n{\times}n$ Matrices, *Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics* **78**. Rhode Island, American Mathematical Society.

Friedland, S. (1983). Simultaneous similarity of matrices. *Adv. Math.*, **50**, 189–265.

Labonté, G. (1990). An algorithm for the construction of matrix representations for finitely presented noncommutative algebras. *J. Symbolic Computation*, **9**, 27–38.

Letzter, E. S. (2001). Counting equivalence classes of irreducible representations, in preparation.

Linton, S. A. (1991). Constructing matrix representations of finitely presented groups, Computational group theory, Part 2. *J. Symbolic Computation*, **12**, 427–438.

McConnell, J. C., Robson, J. C. (1987). *Noncommutative Noetherian Rings.* Chichester, John Wiley and Sons.

Pappacena, C. J. (1997). An upper bound for the length of a finite-dimensional algebra. *J. Algebra*, **197**, 535–545.

Plesken, W., Souvignier, B. (1997). Analysing finitely presented groups by constructing representations. *J. Symbolic Computation*, **24**, 335–349.

Procesi, C. (1974). Finite-dimensional representations of algebras. *Israel J. Math.*, **19**, 169–182.

Razmyslov, Ju. P. (1973). A certain problem of Kaplansky. *Izv. Akad. Nauk SSSR Ser. Mat.*, **37**, 483–501.

Rowen, L. H. (1980). *Polynomial Identities in Ring Theory.* New York, Academic Press.

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