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THE COEFFICIENTS OF DIFFERENTIATED EXPANSIONS AND DERIVATIVES OF ULTRASPHERICAL POLYNOMIALS

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Abstract—A formula expressing the ultraspherical coefficients of the general order derivative of an infinitely differentiable function in terms of its original ultraspherical coefficients is stated in a more compact form and proved in a simpler way than the formula suggested by Karageorghis and Phillips in their recent report [5].
Formulas expressing explicitly the derivatives of ultraspherical polynomials of any degree and for any order in terms of the ultraspherical polynomials are given. The special cases of Chebyshev polynomials of the first and second kinds and of Legendre polynomials are considered.
An application of how to use ultraspherical polynomials for solving ordinary and partial differential equations is described.

1. INTRODUCTION

Classical orthogonal polynomials are used extensively for the numerical solution of differential equations in spectral and pseudospectral methods, see for instance, Gottlieb and Orszag [3], Voigt *et al.* [8], and Doha [2]. If these polynomials are used as basis functions, then the rate of decay of the expansion coefficients is determined by the smoothness properties of the function being expanded. It is well-known that if the solution of the differential equation is infinitely differentiable then the nth expansion coefficient will decrease faster than any finite power of $(1/n)$, cf. [3].

For spectral and pseudo-spectral methods, explicit expressions for the expansion coefficients of the derivatives in terms of the expansion coefficients of the solution are needed. Also explicit expressions for the derivatives of the basis function in terms of the basis functions themselves are required.

A formula expressing the Chebyshev coefficients of the general order derivative of an infinitely differentiable function in terms of its Chebyshev coefficients is given by Karageorghis [4], and a corresponding formula for the Legendre coefficients is obtained by Phillips [6].

A formula for the coefficients of an expansion of ultraspherical polynomials which has been differentiated an arbitrary number of times to those in the original expansion is proved by Karageorghis and Phillips [5]. Their formula is somewhat complicated and its derivation is too lengthy, also the particular cases considered from it are not direct. No formula expressing explicitly the derivatives of the ultraspherical polynomials in terms of ultraspherical polynomials are known yet.

In the present paper we rederive the formula given in [5] in a simpler way and write it in a more compact form, and we obtain explicit expressions for the derivatives of the ultraspherical polynomials of any degree and for any order in terms of the ultraspherical polynomials.

In Section 2, some properties of the ultraspherical polynomials are given. In Section 3, we rederive the simpler and more compact formula corresponding to that given in [5]. Formulas expressing directly the derivatives of ultraspherical polynomials in terms of ultraspherical polynomials are given in Section 4; results for the Chebyshev polynomials of the first and second kinds and for the Legendre polynomials are obtained. In Section 5, we describe how the ultraspherical polynomials are used to solve differential equations of higher orders.

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2. SOME PROPERTIES OF ULTRASPHERICAL POLYNOMIALS

The ultraspherical (Gegenbauer) polynomials associated with the real parameter ($\alpha > -\frac{1}{2}$) are a sequence of polynomials $C_n^{(\alpha)}(x) \quad (n = 0, 1, 2, \dots)$, each respectively of degree n, satisfying the orthogonality relation

$$\int_{-1}^{+1} (1-x^2)^{\alpha-\frac{1}{2}} C_m^{(\alpha)}(x) C_n^{(\alpha)}(x)\,dx = 0 \qquad (m \neq n)$$

For the present purposes it is convenient to standardize the ultraspherical polynomials so that

$$C_n^{(\alpha)}(1) = 1 \qquad (n = 0, 1, 2, \dots) \tag{1}$$

This is not the usual standardization, but has the desirable properties that $C_n^{(0)}(x)$ is identical with the Chebyshev polynomials of the first kind $T_n(x)$, $C_n^{(\frac{1}{2})}(x)$ is the Legendre polynomials $P_n(x)$, and $C_n^{(1)}(x)$ is equal to $(1/(n+1))U_n(x)$, where $U_n(x)$ is the Chebyshev polynomials of the second kind. In this form the polynomials may be generated by Rodrigue's formula

$$C_n^{(\alpha)}(x) = \left(-\frac{1}{2}\right)^n \frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(n+\alpha+\frac{1}{2})}(1-x^2)^{\frac{1}{2}-\alpha} D_x^n\left((1-x^2)^{n+\alpha-\frac{1}{2}}\right) \tag{2}$$

The following two recurrence relations are of fundamental importance in developing the present work. These are

$$(n+2\alpha)C_{n+1}^{(\alpha)}(x) = 2(n+\alpha)xC_n^{(\alpha)}(x) - nC_{n-1}^{(\alpha)}(x) \tag{3}$$

$$2(n+\alpha)C_n^{(\alpha)}(x) = \frac{n+2\alpha}{n+1}D_x C_{n+1}^{(\alpha)}(x) - \frac{n}{n+2\alpha-1}D_x C_{n-1}^{(\alpha)}(x) \tag{4}$$

Note that the recurrence formula (3) may be used to generate the ultraspherical polynomials starting from $C_0^{(\alpha)}(x) = 1$ and $C_1^{(\alpha)}(x) = x$, cf., Szegö [7].

Suppose now we are given a function $f(x)$ which is infinitely differentiable in the closed interval $[-1, 1]$, then we can write

$$f(x) = \sum_{n=0}^{\infty} a_n C_n^{(\alpha)}(x) \tag{5}$$

and for the qth derivative of $f(x)$,

$$f^{(q)}(x) = \sum_{n=0}^{\infty} a_n^{(q)} C_n^{(\alpha)}(x) \tag{6}$$

then

$$f^{(q+1)}(x) = \sum_{n=0}^{\infty} a_n^{(q+1)} C_n^{(\alpha)}(x)$$

and making use of (4) gives

$$f^{(q+1)}(x) = \sum_{n=1}^{\infty} \left[\frac{n+2\alpha-1}{2n(n+\alpha-1)}a_{n-1}^{(q+1)} - \frac{n+1}{2(n+\alpha+1)(n+2\alpha)}a_{n+1}^{(q+1)}\right] D_x C_n^{(\alpha)}(x)$$

On differentiating (6), we find

$$f^{(q+1)}(x) = \sum_{n=0}^{\infty} a_n^{(q)} D_x C_n^{(\alpha)}(x)$$

from which, and equating the coefficients, we have

$$\frac{(n+2\alpha-1)}{2n(n+\alpha-1)}a_{n-1}^{(q+1)} - \frac{(n+1)}{2(n+\alpha+1)(n+2\alpha)}a_{n+1}^{(q+1)} = a_n^{(q)} \qquad n \geq 1 \tag{7}$$

For computing purposes, this equation is not easy to use, since the coefficients on the left hand side are functions of n. To simplify the computing, we define a related set of coefficients $b_n^{(q)}$ by writing

$$a_n^{(q)} = \frac{(n+\alpha)\Gamma(n+2\alpha)}{n!}b_n^{(q)} \qquad n \geq 0, \quad q = 0,1,2,\ldots \tag{8}$$

where $\Gamma(\cdot)$ is the Gamma function. Equation (7) then takes the simpler form

$$b_{n-1}^{(q)} - b_{n+1}^{(q)} = 2(n+\alpha)b_n^{(q-1)} \qquad n \geq 1, \quad q = 1,2,\ldots \tag{9}$$

This difference equation may be solved to give

$$b_n^{(q)} = 2\sum_{i=1}^{\infty}(n+2i+\alpha-1)b_{n+2i-1}^{(q-1)} \tag{10}$$

3. RELATIONS BETWEEN THE COEFFICIENTS $a_n^{(q)}$ AND a_n

The main result of this Section is to prove the following theorem for the coefficients $b_n^{(q)}$:

$$b_n^{(q)} = \frac{2^q}{(q-1)!}\sum_{j=1}^{\infty}\frac{(j+q-2)!\Gamma(n+j+q+\alpha-1)}{(j-1)!\Gamma(n+j+\alpha)}(n+2j+q+\alpha-2)b_{n+2j+q-2} \tag{11}$$

The following lemma is needed to proceed with the proof of the theorem (11).

3.1 Lemma 1:

$$\sum_{i=1}^{p}(n+2i+\alpha-1)\frac{(p-i+q-1)!\Gamma(n+i+p+q+\alpha-1)}{(p-i)!\Gamma(n+i+p+\alpha)} = \frac{1}{q}\frac{(p+q-1)!\Gamma(n+p+q+\alpha)}{(p-1)!\Gamma(n+p+\alpha)} \qquad \forall n, \forall q \geq 1 \tag{12}$$

PROOF. For $p = 1$ the left hand side of (12) equals the right hand side of (12) which is equal to $(q-1)!\Gamma(n+q+\alpha+1)/\Gamma(n+\alpha+1)$. If we apply induction on p, assuming that (12) holds, we have to show that

$$\sum_{i=1}^{p+1}(n+2i+\alpha-1)\frac{(p+q-i)!\Gamma(n+p+q+i+\alpha)}{(p-i+1)!\Gamma(n+p+i+\alpha+1)} = \frac{1}{q}\frac{(p+q)!\Gamma(n+p+q+\alpha+1)}{p!\Gamma(n+p+\alpha+1)} \tag{13}$$

From (12) by taking $n+2$ instead of n and $m = i+1$, we get

$$\sum_{m=2}^{p+1}(n+2m+\alpha-1)\frac{(p+q-m)!\Gamma(n+p+q+m+\alpha)}{(p-m+1)!\Gamma(n+p+m+\alpha+1)} = \frac{1}{q}\frac{(p+q-1)!\Gamma(n+p+q+\alpha+2)}{(p-1)!\Gamma(n+p+\alpha-2)} \tag{14}$$

The left hand side of (13) becomes (with application of (14))

$$\begin{aligned}
&(n+\alpha+1)\frac{(p+q-1)!\Gamma(n+p+q+\alpha+1)}{p!\Gamma(n+p+\alpha+2)} + \sum_{i=2}^{p+1}(n+2i+\alpha-1)\frac{(p+q-i)!\Gamma(n+p+q+i+\alpha)}{(p-i+1)!\Gamma(n+p+i+\alpha+1)} \\
&= (n+\alpha+1)\frac{(p+q-1)!\Gamma(n+p+q+\alpha+1)}{p!\Gamma(n+p+\alpha+2)} + \frac{1}{q}\frac{(p+q-1)!\Gamma(n+p+q+\alpha+2)}{(p-1)!\Gamma(n+p+\alpha+2)} \\
&= \frac{1}{q}\frac{(p+q)!\Gamma(n+p+q+\alpha+1)}{p!\Gamma(n+p+\alpha+1)}
\end{aligned}$$

which completes the induction and proves the lemma.

PROOF OF THEOREM. For $q = 1$, the application of (10) with $q = 1$ yields the required formula. Proceeding by induction, assuming that the theorem is valid for q, we want to prove that

$$b_n^{(q+1)} = \frac{2^{q+1}}{q!} \sum_{j=1}^{\infty} \frac{(j+q-1)!}{(j-1)!} \frac{\Gamma(n+j+q+\alpha)}{\Gamma(n+j+\alpha)} (n+2j+q+\alpha-1) b_{n+2j+q-1} \tag{15}$$

From (10) (by taking $q+1$ instead of q) and assuming the validity of the theorem for q,

$$b_n^{(q+1)} = \frac{2^{q+1}}{(q-1)!} \sum_{i=1}^{\infty} (n+2i+\alpha-1)$$
$$\times \left\{ \sum_{j=1}^{\infty} \frac{(j+q-2)!}{(j-1)!} \frac{\Gamma(n+2i+j+q+\alpha-2)}{\Gamma(n+2i+j+\alpha-1)} (n+2i+2j+q+\alpha-3) b_{n+2i+2j+q-3} \right\} \tag{16}$$

Let $i+j-1 = p$, then (16) may be written in the form

$$b_n^{(q+1)} = \frac{2^{q+1}}{(q-1)!} \sum_{p=1}^{\infty} \Bigg[\sum_{\substack{i,j=1 \\ i+j=p+1}}^{p} (n+2i+\alpha-1) \frac{(j+q-2)!}{(j-1)!} \frac{\Gamma(n+2i+j+q+\alpha-2)}{\Gamma(n+2i+j+\alpha-1)}$$
$$\times (n+2p+q+\alpha-1) b_{n+2p+q-1} \Bigg]$$

which also may be written as

$$b_n^{(q+1)} = \frac{2^{q+1}}{(q-1)!} \sum_{p=1}^{\infty} \Bigg[\sum_{i=1}^{p} (n+2i+\alpha-1) \frac{(p-i+q-1)!}{(p-i)!} \frac{\Gamma(n+p+i+q+\alpha-1)}{\Gamma(n+p+i+\alpha-1)}$$
$$\times (n+2p+q+\alpha-1) \Bigg] b_{n+2p+q-1}$$

Application of lemma 1 given in (12) to the second series yields equation (15) and the proof of the theorem is complete.

Now, substitution of (11) into (8) gives the relation between the coefficients of a general order derivative of an expansion in ultraspherical polynomials in terms of the coefficients of the original expansion as

$$a_n^{(q)} = \frac{2^q (n+\alpha)\Gamma(n+2\alpha)}{(q-1)!n!} \sum_{j=1}^{\infty} \frac{(j+q-2)!}{(j-1)!} \frac{\Gamma(n+j+q+\alpha-1)(n+2j+q-2)!}{\Gamma(n+j+\alpha)\Gamma(n+2j+q+2\alpha-2)} a_{n+2j+q-2}$$
$$\text{for all } n \geq 0, q \geq 1 \tag{17}$$

Formula (17) is more simple and is written in a more compact form than the formula given in [5].

In particular, the special cases for the Chebyshev polynomials of the first and second kinds may be obtained directly by taking $\alpha = 0, 1$ respectively, and for the Legendre polynomials by taking $\alpha = \frac{1}{2}$. These are given as corollaries to the previous theorem.

COROLLARY 1. If $f(x)$ is an infinitely differentiable function on $[-1, 1]$, then the Chebyshev coefficients $a_n^{(q)}$ of the qth derivative of $f(x)$ are related to the Chebyshev coefficients, a_n, of $f(x)$ by

$$c_n a_n^{(q)} = \frac{2^q}{(q-1)!} \sum_{j=1}^{\infty} \frac{(j+q-2)!(n+j+q-2)!}{(j-1)!(n+j-1)!} (n+2j+q-2) a_{n+2j+q-2}$$
$$\text{for all } n \geq 0, q \geq 1 \tag{18}$$

where $c_0 = 2$, $c_n = 1$ for all $n \geq 1$, $c_n = 0$ for $n < 0$.

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$$b_n^{(q+1)} = \frac{2^{q+1}}{q!} \sum_{j=1}^{\infty} \frac{(j+q-1)!}{(j-1)!} \frac{\Gamma(n+j+q+\alpha)}{\Gamma(n+j+\alpha)} (n+2j+q+\alpha-1) b_{n+2j+q-1} \tag{15}$$

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$$\times \left\{ \sum_{j=1}^{\infty} \frac{(j+q-2)!}{(j-1)!} \frac{\Gamma(n+2i+j+q+\alpha-2)}{\Gamma(n+2i+j+\alpha-1)} (n+2i+2j+q+\alpha-3) b_{n+2i+2j+q-3} \right\} \tag{16}$$

Let $i + j - 1 = p$, then (16) may be written in the form

$$b_n^{(q+1)} = \frac{2^{q+1}}{(q-1)!} \sum_{p=1}^{\infty} \left[\sum_{\substack{i,j=1 \\ i+j=p+1}}^{p} (n+2i+\alpha-1) \frac{(j+q-2)!}{(j-1)!} \frac{\Gamma(n+2i+j+q+\alpha-2)}{\Gamma(n+2i+j+\alpha-1)} \right.$$
$$\left. \times (n+2p+q+\alpha-1) b_{n+2p+q-1} \right]$$

which also may be written as

$$b_n^{(q+1)} = \frac{2^{q+1}}{(q-1)!} \sum_{p=1}^{\infty} \left[\sum_{i=1}^{p} (n+2i+\alpha-1) \frac{(p-i+q-1)!}{(p-i)!} \frac{\Gamma(n+p+i+q+\alpha-1)}{\Gamma(n+p+i+\alpha-1)} \right.$$
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In particular, the special cases for the Chebyshev polynomials of the first and second kinds may be obtained directly by taking $\alpha = 0, 1$ respectively, and for the Legendre polynomials by taking $\alpha = \frac{1}{2}$. These are given as corollaries to the previous theorem.

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$$\text{for all } n \geq 0, q \geq 1 \tag{18}$$

where $c_0 = 2$, $c_n = 1$ for all $n \geq 1$, $c_n = 0$ for $n < 0$.

$$D_x^q C_{q+2j+1}^{(\alpha)}(x) = \frac{2^q (q+2j+1)!}{(q-1)!\Gamma(q+2j+2\alpha+1)} \sum_{i=0}^{j} \frac{(2i+\alpha+1)\Gamma(2i+2\alpha+1)}{(2i+1)!(j-i)!} \times \frac{\Gamma(q+j+i+\alpha+1)}{\Gamma(j+i+\alpha+2)} (q+j-i-1)! C_{2i+1}^{(\alpha)}(x) \quad (25)$$

The particular expressions for the derivatives of the Chebyshev polynomials of the first and second kinds, and for the Legendre polynomials may be obtained as specials cases from formulas (24) and (25). we give these as corrollaries as follows:

Corollary 4. Set $\alpha = 0$ in (24) and (25), we get the derivatives of the Chebyshev polynomials of the first kind in the forms:

$$D_x^q T_{q+2j}(x) = \frac{2^q (q+2j)}{(q-1)!} {\sum_{i=0}^{j}}' \frac{(q+j-i-1)!(q+j+i-1)!}{(j-i)!(j+i)!} T_{2i}(x) \quad (26)$$

$$D_x^q T_{q+2j+1}(x) = \frac{2^q (q+2j+1)}{(q-1)!} \sum_{i=0}^{j} \frac{(q+j-i-1)!(q+j+i)!}{(j-i)!(j+i+1)!} T_{2i+1}(x) \quad (27)$$

($\sum'$ means that the first term is taking with factor $\frac{1}{2}$).

Corollary 5. The corresponding formulas for Chebyshev polynomials of the second kind are given by:

$$D_x^q U_{q+2j}(x) = \frac{2^q}{(q-1)!} \sum_{i=0}^{j} (2i+1) \frac{(q+j-i-1)!(q+j+i)!}{(j-i)!(j+i+1)!} U_{2i}(x) \quad (28)$$

$$D_x^q U_{q+2j+1}(x) = \frac{2^{q+1}}{(q-1)!} \sum_{i=0}^{j} (i+1) \frac{(q+j-i-1)!(q+j+i+1)!}{(j-i)!(j+i+2)!} U_{2i+1}(x) \quad (29)$$

These are obtained from (24) and (25) by setting $\alpha = 1$, and noting that $C_n^{(1)}(x) = U_n(x)/(n+1)$.

Corollary 6. The derivatives of Legendre polynomials are given by:

$$\begin{aligned} D_x^q P_{q+2j}(x) &= \frac{2^{q-1}}{(q-1)!} \sum_{i=0}^{j} (4i+1) \frac{(q+j-i-1)!\Gamma(q+j+i+\frac{1}{2})}{(j-i)!\Gamma(j+i+3/2)} P_{2i}(x) \\ &= \frac{1}{2^{q-2}(q-1)!} \sum_{i=0}^{j} (4i+1) \frac{(q+j-i-1)!(2q+2j+2i-1)!}{(j-i)!(2j+2i+2)!} \frac{(j+i+1)!}{(q+j+i-1)!} P_{2i}(x) \quad (30) \end{aligned}$$

$$\begin{aligned} D_x^q P_{q+2j+1}(x) &= \frac{2^{q-1}}{(q-1)!} \sum_{i=0}^{j} (4i+3) \frac{(q+j-i-1)!\Gamma(q+j+i+3/2)}{(j-i)!\Gamma(j+i+5/2)} P_{2i+1}(x) \\ &= \frac{1}{2^{q-2}(q-1)!} \sum_{i=0}^{j} (4i+3) \frac{(q+j-i-1)!(2q+2j+2i+1)!}{(j-i)!(2j+2i+4)!} \frac{(j+i+2)!}{(q+j+i)!} P_{2i+1}(x) \quad (31) \end{aligned}$$

Note here that the first equalities of (30) and (31) are obtained simply by setting $\alpha = \frac{1}{2}$ in (24) and (25) respectively. Making use of (21) with $n = i+1$, $n = i+2$ give the second equalities directly.

5. USE OF ULTRASPHERICAL POLYNOMIALS TO SOLVE DIFFERENTIAL EQUATIONS

Consider the linear ordinary differential equation of order n of the form

$$\sum_{i=0}^{n} f_i(x) D_x^i y(x) = g(x) \tag{32}$$

where $f_i(x)$ and $g(x)$ are functions of x only. Suppose the equation to be solved in the interval $[-1, 1]$ subject to n linear boundary conditions, and assume we approximate $y(x)$ by a truncated expansion of ultraspherical polynomials

$$y(x) = \sum_{j=0}^{N} a_j C_j^{(\alpha)}(x) \tag{33}$$

where N is the degree of approximation, $a_0, a_1, \dots, a_N$ are unknown coefficients to be determined. Substituting (33) into (32) yields

$$\sum_{i=0}^{n} \left\{ f_i(x) \sum_{j=0}^{N} a_j D_x^i C_j^{(\alpha)}(x) \right\} = g(x) \tag{34}$$

which may be written in the form

$$\sum_{j=0}^{N} \left\{ a_j \sum_{i=0}^{n} f_i(x) D_x^i C_j^{(\alpha)}(x) \right\} = g(x) \tag{35}$$

The boundary conditions associated with (32) give rise to n equations connecting the coefficients a_j, and the remaining equations may be obtained in two ways:

(i) we may equate the coefficients of the various $C_j^{(\alpha)}(x)$ after expanding the two sides of (35) in ultraspherical series.

(ii) we may collocate at $m = N - n$ selected points in $(-1, 1)$.

The system of equations obtained from the collocation is of the form

$$\sum_{j=0}^{N} \left\{ a_j \sum_{i=0}^{n} f_i(x_k) D_x^i C_j^{(\alpha)}(x_k) \right\} = g(x_k) \qquad k = 1, 2, \dots, m \tag{36}$$

where x_k are the collocation points, which are usually chosen at the zeros of $C_m^{(\alpha)}(x)$, (See for instance, Abramowitz and Stegun [1]). Since the derivatives $D_x^i C_j^{(\alpha)}(x)$ are now expressible explicitly in terms of $C_j^{(\alpha)}(x)$, then the problem of computing them is solved by using the formulas (24) and (25). Therefore, the resulting linear system obtained from (34) and the n linear boundary conditions can easily be solved using standard direct solvers.

The method just described is easily extended to higher dimensions. Consider, for example, the second order partial differential equation

$$A_1(x,y)u_{xx} + A_2(x,y)u_{xy} + A_3(x,y)u_{yy} + A_4(x,y)u_x + A_5(x,y)u_y + A_6(x,y)u = f(x,y) \tag{37}$$

where the coefficients $A_1, A_2, \dots, A_6$ and f are functions of x and y only. Suppose the solution of the equation is required in the square S $(-1 \le x, y \le 1)$, subject to general linear boundary conditions of the form

$$B_1(x,y)u_x + B_2(x,y)u_y + B_3(x,y)u = g(x,y) \tag{38}$$

on the sides of the square S.

Suppose the function $u(x,y)$ can be approximated by the double finite ultraspherical series

$$u(x,y) = \sum_{m=0}^{M} \sum_{n=0}^{N} a_{mn} C_m^{(\alpha)}(x) C_n^{(\alpha)}(y) \tag{39}$$

for sufficiently large values of the integers M and N. Since $u(x,y)$ satisfies (37) we have approximately

$$\begin{aligned} \sum_{m=0}^{M} \Big\{ \sum_{n=0}^{N} a_{mn} \big[& A_1 D_x^2 C_m^{(\alpha)}(x) C_n^{(\alpha)}(y) + A_2 D_x C_m^{(\alpha)}(x) D_y C_n^{(\alpha)}(y) \\ & + A_3 C_m^{(\alpha)}(x) D_y^2 C_n^{(\alpha)}(y) + A_4 D_x C_m^{(\alpha)}(x) C_n^{(\alpha)}(y) + A_5 C_m^{(\alpha)}(x) D_y C_m^{(\alpha)}(y) \\ & + A_6 C_m^{(\alpha)}(x) C_n^{(\alpha)}(y) \big] \Big\} = f(x,y) \end{aligned} \tag{40}$$

On collocating equation (40) at $(M-1)(N-1)$ distinct points (x_i, y_j) $i = 1, 2, \ldots, M-1$, $j = 1, 2, \ldots, N-1$, in S, there results a set of $(M-1)(N-1)$ linear equations for the coefficients a_{mn}. If we now collocate equation (38) at $2(M+N)$ points on the sides of the square S, we find the remaining equations for the unique determination of the coefficients a_{mn}.

As in ordinary differential equations, the derivatives of ultraspherical polynomials occurring in (40) are computed by use of (24) and (25), and numbers x_i, y_j are chosen at the zeros of the appropriate ultraspherical polynomials.

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