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On Ferri's characterization of the finite quadric Veronesean $\mathcal{V}_2^4$

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Abstract

We generalize and complete Ferri's characterization of the finite quadric Veronesean $\mathcal{V}_2^4$ by showing that Ferri's assumptions also characterize the quadric Veroneseans in spaces of even characteristic.
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1. Introduction

Let q be a fixed prime power. For any integer k, denote by $\mathbf{PG}(k, q)$ the k-dimensional projective space over the finite (Galois) field $\mathbf{GF}(q)$ of q elements. We choose coordinates in $\mathbf{PG}(2, q)$ and in $\mathbf{PG}(5, q)$. The *Veronesean map* maps a point of $\mathbf{PG}(2, q)$ with coordinates (x_0, x_1, x_2) onto the point of $\mathbf{PG}(5, q)$ with coordinates

$$(x_0^2, x_1^2, x_2^2, x_0x_1, x_0x_2, x_1x_2).$$

The *quadric Veronesean* $\mathcal{V}_2^4$ is the image of the Veronesean map. The set $\mathcal{V}_2^4$ is a cap of $\mathbf{PG}(5, q)$ and has a lot of other nice geometric and combinatorial properties, summarized in [2]. We also refer to [2] for characterizations of this cap, sometimes called a *Veronesean cap*. In particular, there exists a characterization of $\mathcal{V}_2^4$ in terms of the intersection numbers of a

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hyperplane which is valid for q odd. It was first considered and proved by Ferri [1]; the proof in [2] is much shorter because Hirschfeld and Thas make use of the other characterizations. Also, the proof of Ferri did not work for $q = 3$; see [1]. Recently, the authors proved a new characterization of the finite quadric Veroneseans, and they will use it here to generalize Ferri's result to all q.

We now prepare the statement of our Main result.

2. Main result

Recall from [2] that the quadric Veronesean $\mathcal{V}_2^4$ is a cap $\mathcal{K}$ in $\mathbf{PG}(5, q)$ satisfying the following two properties:

(VC1) For every hyperplane π of $\mathbf{PG}(5, q)$, we have $|\pi \cap \mathcal{K}| = 1, q + 1$ or $2q + 1$, and there exists some hyperplane π such that $|\pi \cap \mathcal{K}| = 2q + 1$.
(VC2) Any plane of $\mathbf{PG}(5, q)$ with four points in $\mathcal{K}$ has at least $q + 1$ points in $\mathcal{K}$.

It is also proved in [2] that these two properties characterize $\mathcal{V}_2^4$ for all odd q; Ferri [1] had proved this for all odd $q \neq 3$. In the present paper we will prove this for all q. In fact, we will be able to copy the proof in [2] for the general case (now relying on the Main results of [4]) except for $q = 4$, for which we produce a separate argument.

So we obtain the following general characterization:

Theorem 2.1. *Let $\mathcal{K}$ be a set of points of* $\mathbf{PG}(5, q)$, $q > 2$, *satisfying* (VC1) *and* (VC2). *Then $\mathcal{K}$ is projectively equivalent with the quadric Veronesean $\mathcal{V}_2^4$ in* $\mathbf{PG}(5, q)$. *For $q = 2$, a set of points in* $\mathbf{PG}(5, 2)$ *satisfying* (VC1) *and* (VC2) *is either a quadric Veronesean or an elliptic quadric in some subspace* $\mathbf{PG}(3, 2)$.

3. Proof of the main result

We now prove Theorem 2.1.

Let $\mathcal{K}$ be a set of points of $\mathbf{PG}(5, q)$, $q > 2$, satisfying (VC1) and (VC2) (see above). We first prove that $\mathcal{K}$ is a $(q^2 + q + 1)$-cap. This follows from the results in [2] if $q \neq 4$. So we first deal with the case $q = 4$.

In the next three lemmas, we assume that $q = 4$ and that $\mathcal{K}$ satisfies (VC1) and (VC2). We adopt the terminology of [2]: a *solid* is a 3-dimensional subspace of $\mathbf{PG}(5, 4)$, while a *prime* is a 4-dimensional subspace of $\mathbf{PG}(5, 4)$.

Lemma 3.1. *$\mathcal{K}$ generates* $\mathbf{PG}(5, 4)$.

Proof. By (VC1) the set $\mathcal{K}$ does not generate a line. Assume that $\mathcal{K}$ generates a plane π_2. By Lemma 25.3.5 of [2] there is a line L of π_2 with $|L \cap \mathcal{K}| \in \{2, 3\}$. Let π_4 be a prime which contains L but not π_2. Then $|\pi_4 \cap \mathcal{K}| \in \{2, 3\}$, contradicting (VC1). Next, assume that $\mathcal{K}$ generates a solid π_3. Then $|\mathcal{K}| = 9$ and each plane of π_3 has one or five points in $\mathcal{K}$. Let p and p' be distinct points of $\mathcal{K}$. Suppose that the line $pp' = L$ has $b \geqslant 2$ points in $\mathcal{K}$. Counting the points of $\mathcal{K}$ in the planes of π_3 through the line L, we obtain $5(5 - b) + b = 9$, whence $b = 4$. Let $L \cap \mathcal{K} = \{p, p', p'', p'''\}$ and let $\pi_2 \cap \mathcal{K} = \{p, p', p'', p''', r\}$, with π_2 some plane of π_3 through L. Then the line rp has only $2 \neq b$ points in $\mathcal{K}$, a contradiction. Finally, assume that $\mathcal{K}$ generates a prime π_4. By (VC1) we have again $|\mathcal{K}| = 9$ and each solid π_3 of π_4 has one or five points in $\mathcal{K}$. Let L be a line having at least 2 points in $\mathcal{K}$, and let π_2 be a plane of π_4 containing L. Further, let $|L \cap \mathcal{K}| = a$ and $|\pi_2 \cap \mathcal{K}| = b$. Counting the points of $\mathcal{K}$ in the solids of π_4 containing π_2, we obtain $5(5 - b) + b = 9$, whence $b = 4$. Counting the points of $\mathcal{K}$ in the planes of π_4 containing L, we obtain $21(4 - a) + a = 9$. Consequently $a = 15/4$, a contradiction. The lemma is proved. $\square$

Lemma 3.2. *$\mathcal{K}$ is a cap.*

Proof. Let L be a line. By Lemma 25.3.2 of [2] we have either $L \subseteq \mathcal{K}$ or $|L \cap \mathcal{K}| \leqslant 3$.

First assume that $L \cap \mathcal{K} = \{p, p', p''\}$. Choose points r_1, r_2, r_3 on $\mathcal{K} \setminus \{p, p', p''\}$ so that $\langle L, r_1, r_2, r_3\rangle$ is a prime π_4. Then $|\pi_4 \cap \mathcal{K}| = 9$. Necessarily $\langle L, r_i\rangle$ contains five points of $\mathcal{K}$, $i = 1, 2, 3$ (use (VC2)). The solid $\langle L, r_1, r_2\rangle$ contains either seven or eight points. If $\langle L, r_1, r_2\rangle$ contains eight points, then it contains the three planes $\langle L, r_i\rangle$, $i = 1, 2, 3$, so it contains nine points, a contradiction. Hence $|\mathcal{K} \cap \langle L, r_1, r_2\rangle| = 7$. Considering the primes containing $\langle L, r_1, r_2\rangle$ there arises $|\mathcal{K}| = 17$. Now we project $\mathcal{K} \setminus L$ from L onto a solid π_3 skew to L. There arises a set $\mathcal{K}'$ of size 7 in π_3 which intersects each plane of π_3 in either one or three points. By [3] such a set $\mathcal{K}'$ does not exist.

Next, assume that $\mathcal{K}$ contains a line L. Choose points $r_1, r_2, r_3 \in \mathcal{K} \setminus L$ such that $\langle L, r_1, r_2, r_3\rangle$ generates a prime π_4. Then $|\pi_4 \cap \mathcal{K}| = 9$. Let $(\mathcal{K} \cap \pi_4) \setminus L = \{r_1, r_2, r_3, r_4\}$. By the preceding paragraph $r_4 \notin \langle L, r_i\rangle$, $i = 1, 2, 3$, as otherwise there is a line containing exactly three points of $\mathcal{K}$. Now we project $\mathcal{K} \setminus L$ from L onto a solid π_3 skew to L. There arises a set $\mathcal{K}'$ which intersects each plane of π_3 in either one or four points. By [3] such a set $\mathcal{K}'$ does not exist.

The lemma is proved. $\square$

Lemma 3.3. *The cap $\mathcal{K}$ contains exactly* 21 *points.*

Proof. Put $|\mathcal{K}| = k$. Let $\pi_4^1, \pi_4^2, \ldots$ be the primes of $\mathbf{PG}(5, 4)$, and let s_i be the number of points of $\mathcal{K}$ in π_4^i. Counting in two ways the number of ordered pairs (p, π_4^i), with $p \in \mathcal{K} \cap \pi_4^i$, we obtain

$$\sum_{i=1}^{1365} s_i = 341k.$$

Counting in two ways the number of ordered triples (p, p', π_4^i), with $p, p' \in \mathcal{K} \cap \pi_4^i$, and $p \neq p'$, we obtain

$$\sum_{i=1}^{1365} s_i(s_i - 1) = 85k(k - 1).$$

The set $\mathcal{K}$ is a cap; so counting in two ways the number of ordered 4-tuples (p, p', p'', π_4^i), with $p, p', p'' \in \mathcal{K} \cap \pi_4^i$, and $p \neq p' \neq p'' \neq p$, we obtain

$$\sum_{i=1}^{1365} s_i(s_i-1)(s_i-2) = 21k(k-1)(k-2).$$

Since $s_i \in \{1, 5, 9\}$ for all i, we have

$$\sum_{i=1}^{1365} (s_i-1)(s_i-5)(s_i-9) = 0.$$

Hence

$$\sum_{i=1}^{1365} s_i(s_i-1)(s_i-2) - 12\sum_{i=1}^{1365} s_i(s_i-1) + 45\sum_{i=1}^{1365} s_i - 61425 = 0.$$

We obtain, substituting the previous equalities,

$$21k(k-1)(k-2) - 1020k(k-1) + 15345k - 61425 = 0.$$

Hence $7k^3 - 361k^2 + 5469k - 20475 = 0$. It follows that $k = 21$ or $k = 25$.

Assume that $k = 25$. If π_3 is a solid which contains $a \geqslant 6$ points of $\mathcal{K}$, then $|\mathcal{K}| = 25 = a + 5(9-a)$, so $a = 5$, a contradiction. If π_2 is a plane which contains at least four points of $\mathcal{K}$, then π_2 contains at least five points of $\mathcal{K}$ (by (VC2)), so there exists a solid which contains at least six points of $\mathcal{K}$, a contradiction. Hence any four points of $\mathcal{K}$ are linearly independent.

Let p be a fixed point of $\mathcal{K}$. Let c_i be the number of primes of $\mathbf{PG}(5, 4)$ which contain p and intersect $\mathcal{K}$ in i points, $i = 1, 5, 9$. Counting pairs $\{p', \pi_4\}$ with $p' \in \mathcal{K}$, $p \neq p'$, with π_4 a prime and $p, p' \in \pi_4$, we obtain $4c_5 + 8c_9 = 2040$. Counting triples $\{p', p'', \pi_4\}$ with $p', p'' \in \mathcal{K}$, $p \neq p' \neq p'' \neq p$, with π_4 a prime and $p, p', p'' \in \pi_4$, we obtain $6c_5 + 28c_9 = 5796$. Counting quadruples $\{p', p'', p''', \pi_4\}$ with $p', p'', p''' \in \mathcal{K}$, p, p', p'', p''' *distinct*, π_4 a prime and $p, p', p'', p''' \in \pi_4$, we obtain $4c_5 + 56c_9 = 10120$, clearly contradicting the previous equalities.

So we conclude that $k = 21$ and the lemma is proved. □

Now it is clear that Lemmas 25.3.10–25.3.13 of [2] hold for all $q \geqslant 3$. In particular, this means that there are exactly $q^2 + q + 1$ planes of $\mathbf{PG}(5, q)$ meeting $\mathcal{K}$ in an oval (which is a $(q+1)$-arc), and every pair of points of $\mathcal{K}$ is contained in exactly one such plane. Also, two such planes meet in exactly one point, which belongs to $\mathcal{K}$. Let $\mathcal{K}$ be as in Theorem 2.1 and suppose $q > 2$. By the proof of Theorem 25.3.14 of [2], we now also have that every three planes of $\mathbf{PG}(5, q)$ that intersect $\mathcal{K}$ in an oval generate $\mathbf{PG}(5, q)$. By Theorem 1.3 of [4], $\mathcal{K}$ either is the quadric Veronesean $\mathcal{V}_2^4$ or $q = 4$ and $\mathcal{K}$ is the unique 2-dimensional dual hyperoval of $\mathbf{PG}(5, 4)$. As in the latter case (VC2) is not satisfied, we proved Theorem 2.1 for all $q > 2$.

Finally suppose $q = 2$. We use similar terminology as before. Let π_4 be a prime of $\mathbf{PG}(5, 2)$ containing 5 points of $\mathcal{K}$. If these five points generate π_4, then, considering the three primes through a solid contained in π_4 and itself containing four points of $\mathcal{K}$, it is easily seen that $|\mathcal{K}| = 7$ and every six points of $\mathcal{K}$ generate $\mathbf{PG}(5, 2)$. In this case $\mathcal{K}$ is a skeleton and hence isomorphic to the quadric Veronesean $\mathcal{V}_2^4$. So we may assume that these five points do not generate π_4. Clearly this implies $|\mathcal{K}| = 5$. It is now an easy exercise to see that $\mathcal{K}$ generates a solid and is an elliptic quadric in that solid (because every plane of that solid contains either one or three points of $\mathcal{K}$).

The proof of Theorem 2.1 is complete.

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