Journal of Pure and Applied Algebra 21 (1981) 161-165
North-Holland Publishing Company

MODULE CATEGORIES OVER TOPOI

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Communicated by A. Heller
Received 25 March 1980

Introduction

In this paper we discuss some of the major functorial properties of module categories over topoi, and in particular their closed structure. The primary theorem is a general version of Morita's theorem classifying module categories. We conclude with semidirect products of small categories and some conjectures based on them.

1. Module categories over topoi

Let E be an elementary topos and R a (unitary) ring object in E. Denote by $\mathrm{Mod}(E;R)$ the category of (right) R module objects of E and R linear homomorphisms. Let $\mathrm{Ab}(E)$ denote the category of Abelian group objects of E and group homomorphisms.

Proposition 1. (a) $\mathrm{Ab}(E)$ *and* $\mathrm{Mod}(E;R)$ *both possess an* $\mathrm{Ab}(E)$ *valued internal hom.*

(b) *The forgetful functor* $U_R:\mathrm{Mod}(E;R)\to\mathrm{Ab}(E)$ *has a right adjoint.*

Proof. (a). The construction of the internal hom is the same for $\mathrm{Ab}(E)$ and $\mathrm{Mod}(E;R)$. We take $[A,B]$ (resp. $[A,B]_R$) to be the subobject of B^A consisting of the homomorphisms (resp. R linear homomorphisms), i.e., we interpret the equation $f(x+y)=f(x)+f(y)$ in the internal logic of the topos E.

(b). An R module in E can now be defined as an Abelian group of E together with a homomorphism $s:R\to[C,C]$. This yields, by repeated conversion, a homomorphism $s'':C\to[R,C]$. If A is an Abelian group of E, then $[R,A]$ is an R module since $R\times R\times[R,A]\to R\times[R,A]\to A$ yields $R\to[[R,A],[R,A]]$. The unit of R, $1\to R$, yields $[R,A]\to A^R\to A^1\simeq A$. The two maps $s'':C\to[R,C]$ and $[R,A]\to A$ easily provide the desired adjunction: $\mathrm{Mod}(E;R)(C,[R,A])\simeq\mathrm{Ab}(E)(U_R(C),A)$. $\square$

Neither does U_R have a left adjoint, nor are Ab(E) and Mod(E; R) closed categories in general. But if E possesses basic arithmetic, then both of these things are true.

Proposition 2. *If E has a natural numbers object (NNO), then:*
(a) $U : \mathrm{Ab}(E) \to E$ *has a left adjoint.*
(b) Ab(E) *and* Mod(E; R) *have* Ab(E) *closed structures.*
(c) U_R *has a left adjoint.*

Proof. (a) This is only a particular case of the well-known theorem that if E has an NNO, then it has free models for every finitary algebraic theory [3].

(b). The construction of a tensor product in Ab(E) is straight forward; we let $A \otimes B = F(A \times B)/\beta(A, B)$, where F is the left adjoint to U, and $\beta(A, B)$ is the subgroup of bilinear relations, that is, we interpret the rule $\langle a+b, c\rangle - \langle a, b\rangle - \langle b, c\rangle$, in the internal logic of the topos E. The same process gives a tensor product for Mod(E; R), but which is Ab(E) valued unless R is commutative.

(c). In view of the adjointness of $[\cdot, \cdot]$ and $(\cdot)\otimes(\cdot)$, it is then clear that the functor $C \to R \otimes C$ is the left adjoint to U_R. □

In particular, denote by Z_E the abelian group $F(1)$. Then Z_E is a tensor-hom identity, and thus $\mathrm{Ab}(E) \simeq \mathrm{Mod}(E; Z_E)$. Note, however, that if E does not have an NNO, then Ab(E) is not a module category over E. For if R were such a ring, then this R would have to be an integers numbers object, from which an NNO would be deducible.

2. Morita classification of module categories

The Morita Theorem, in part, classifies those Abelian categories equivalent to module categories (over Sets).

Morita Theorem [1]. *If A is an Abelian category, then the following are equivalent.*
(a) $A \simeq \mathrm{Mod}(R)$, *for some (unitary) ring R.*
(b) *A is cocomplete,* AB5, *and has a small projective generator.*
(c) *There is a faithful functor* $L : A \to \mathrm{Ab}$, *which has both adjoints,* L^* *(left), and* L_* *(right).*

The advantage of part (c) is that it makes no mention of objects, nor of cocompleteness and AB5-ness (these are deducible from the existence of both adjoints).

Given a pair of functors

$$\mathrm{Ab}(E) \underset{G}{\overset{H}{\rightleftarrows}} \mathrm{Ab}(E),$$

we will say that H is the *internal left adjoint* of G, iff there is a natural equivalence of bifunctors: $[H(A), B] \simeq [A, G(B)]$.

Proposition 3. *If E has an NNO, and H is the internal left adjoint of G, then H is also the left adjoint of G.*

Proof. For $(C,[H(A),B]) \simeq (C,[A,G(B)])$ implies $(C \otimes H(A), B) \simeq (C \otimes A, G(B))$. Now let $C = Z_E$. □

If $L: A \to \mathrm{Ab(E)}$, has left and right adjoints L^* and L_* respectively, then we shall say that these adjoints are internal adjoints if $L \cdot L^*$ is the internal left adjoint of $L \cdot L_*$.

Theorem 4. *Let A be an Abelian category, and E a topos with NNO. Then the following are equivalent.*

(a) $A \simeq \mathrm{Mod}(E; R)$ *for some (unitary) ring object R of E.*

(b) *There is a faithful functor* $L: A \to Ab(E)$, *which has internal left and right adjoints.*

Remark. The proof of this theorem is both long and cumbersome, for the major portion of it is devoted to showing that certain natural transformations are what they ought to be. This requires much tedious diagram chasing. We will outline the proof here, and leave the details in [2].

Outline of the proof of Theorem 4. (a) implies (b). U_R is clearly faithful, and has adjoints $R \otimes (\cdot)$, and $[R, (\cdot)]$. These adjoints are internal adjoints by their very nature.

(b) implies (a). The condition of internal adjointness is equivalent to saying that there are natural isomorphisms $LL^*(A \otimes B) \simeq LL^*(A) \otimes B$. This allows $LL^*(Z_E)$ to carry a ring structure as follows:

$$LL^*(Z_E) \otimes LL^*(Z_E) \simeq LL^*(LL^*(Z_E) \otimes Z_E) \simeq LL^*(LL^*(Z_E)) \simeq LL^*(Z_E).$$

The final map in this sequence is $L(d_{L^* Z_E})$, where d is the back adjunction map for L^* and L. Let $R = LL^*(Z_E)$. If B is an object of A, then $L(B)$ becomes an R module in E by

$$R \otimes L(B) \to LL^*(Z_E) \otimes L(B) \to LL^*(Z_E \otimes B) \to LL^*L(B) \to L(B),$$

where the final map, again, is provided by the back adjunction. Thus $L: A \to \mathrm{Mod}(E; R)$ is well defined and certainly must preserve all colimits, and be faithful as well.

Every module over R in E is a colimit of free R modules. A free R module is of the form $F_R(X) = R \otimes F(X)$, where $F(X)$ is the free abelian group on X. But then $F_R(X) \simeq LL^*(Z_E) \otimes F(X) \simeq LL^*(F(X))$, and therefore is in the image of $L: A \to$

Mod(E; R). Thus if the functor L, above, is full, then it must be dense and therefore an equivalence.

To prove that L is full, we note the following. If $f: L(A) \to L(B)$ is any map in Ab(E), then by adjointness this yields a map $f'': L^*L(A) \to B$ in $\mathbb{A}$. Since L is faithful and has both adjoints, the adjunction $L^* \dashv L$ must be tripleable, and therefore, for any A in $\mathbb{A}$, the diagram

$$L^*LL^*L(A) \underset{d_{L^*LA}}{\overset{L^*L(d_A)}{\rightrightarrows}} L^*L(A) \xrightarrow{d_A} A$$

is a coequalizer diagram. We will refer the reader to [2], for a proof that $f: L(A) \to L(B)$ is an R module homomorphism iff $f'' \cdot L^*L(d_A) = f'' \cdot d_{L^*L(A)}$. Thus if f is an R module homomorphism, there is a unique $g: A \to B$ in $\mathbb{A}$, such that $g \cdot d_A = f''$. Adjointness then gives that $f = L(g)$. □

Note that when E = Sets, then E has no internal structure, and thus the condition of internal adjointness is exactly the condition of adjointness. In this case, Theorem 4 is precisely the Morita classification theorem.

3. Semidirect products.

If $\mathscr{A}$ is a small site and R a sheaf of rings on $\mathscr{A}$, then we may form a new category $R \otimes_\theta \mathscr{A}$ as follows: Obj($R \otimes_\theta \mathscr{A}$) = Obj($\mathscr{A}$), and $R \otimes_\theta \mathscr{A}(A, B) = \bigoplus_{\mathscr{A}(A,B)} R(A)$. Composition is defined by the rule: $f \in \mathscr{A}(A, B)$, $g \in \mathscr{A}(B, C)$, $r \in R(A)$, and $s \in R(B)$, then $s_g \cdot r_f = (R(f)(s)r)_{g \cdot f}$. We put a topology on $R \otimes_\theta \mathscr{A}$ by extending the covers of $\mathscr{A}$ '*R-linearly*'. The category so constructed is a small additive site, and will be called the semi-direct product of $\mathscr{A}$ by R. The θ in the notation is purely symbolic and is used to denote that this construction is not a tensor product of categories.

Proposition 5. *Let R and $\mathscr{A}$ be as above, and let $E = \mathrm{Sh_{Set}}(\mathscr{A})$, the category of set-valued sheaves on $\mathscr{A}$. Then* Mod(E; R) $\simeq \mathrm{Sh_{Ab}}(R \otimes_\theta \mathscr{A})$.

Proof. If K is an Abelian group-valued sheaf on $R \otimes_\theta \mathscr{A}$, then for all A in $\mathscr{A}$, $K(A)$ is an Abelian group. If $r \in R(A)$, and $x \in K(A)$, then define $r \cdot x = K(r_{1_A})(x)$. It is easy to verify that this gives K the structure of an R module in E. Conversely, let M be an R module in E. If $r \in R(A)$, and $f \in \mathscr{A}(A, B)$, then define $M(r_f) = (r \cdot -) \cdot M(f)$, where $r \cdot - : K(A) \to K(A)$ is multiplication by r. Again, it is an easy computation to show that this makes M into an Abelian group valued sheaf over $R \otimes_\theta \mathscr{A}$. □

In particular, if we take $\mathscr{A}$ to be the group G and H any other group together with a homomorphism $\theta: G \to \mathrm{Aut}(H)$, then θ makes $\mathbb{Z}[H]$, the integral group ring, a ring object in S^G, the topos of G-sets. Proposition 5 in this case asserts that Mod(S^G; $\mathbb{Z}[H]$) $\simeq$ Mod($\mathbb{Z}[H \times_\theta G]$), where $H \times_\theta G$ is the semi-direct product of the group G by H.

More generally, if R is a ring object in S^G, then R is a ring (as usual), together with a homomorphism $\theta : G \to \mathrm{Aut}(R)$, the group of ring automorphisms of R. We have, by Proposition 5, that $\mathrm{Mod}(S^G; R) \simeq \mathrm{Mod}(R_\theta[G])$, where $R_\theta[G]$ is the twisted group ring of G over R. This is the ring whose underlying abelian group is $\bigoplus_G R$, and whose multiplication is given by: $r_g \cdot s_h = (r\theta(g)(s))_{gh}$. This is the integral group ring of G over R, iff θ is the trivial homomorphism. The twisted group ring does not seem to be appropriate to the study of groups by rings, i.e. group representation theory. Among other problems, there is no meaningful augmentation map $R_\theta[G] \to R$, and the notion of character is no longer well defined. The twisted group ring does fit in nicely to the situation of the study of rings by groups, i.e. Galois theory. For example, if $k \to K$ is a field extension, and G the Galois group, then there is a canonical $\theta : G \to \mathrm{Aut}(K)$, thus generating the ring $K_\theta[G]$. The forgetful functor $\mathrm{Mod}(K_\theta[G]) \to \mathrm{Ab}$ factors through $\mathrm{Mod}(k)$, and in fact $\mathrm{Mod}(k)$ is the largest abelian category through which the forgetful functor factors iff $k \to K$ is a Galois extension. In this fashion, all of Galois theory has a module theoretic interpretation.

Let us finally observe that every ring object in S^G generates a $\mathbb{Z}[G]$-algebra by the correspondence $R \to R_\theta[G]$. The converse is false as Theorem 4 tells us it must be. For example, the augmentation map $\varepsilon : \mathbb{Z}[G] \to \mathbb{Z}$ is a ring map, so gives rise to a functor $\varepsilon_0 : \mathrm{Ab} \to \mathrm{Ab}(S^G) = \mathrm{Mod}(\mathbb{Z}[G])$, which is faithful and possesses both adjoints. And yet it is easy to verify that ε_0^*, and ε_{0*} are internally adjoint iff $G = 1$. The condition of internal adjointness is strictly stronger than adjointness in general, and its presence in Theorem 4 may not be ignored. Thus not every $\mathbb{Z}[G]$-algebra arises as $R_\theta[G]$ for some ring object R in S^G. To say which $\mathbb{Z}[G]$-algebras do so arise, does not seem to be an easy thing to determine. The real question here is: Is there a cohomological characterization of such $\mathbb{Z}[G]$-algebras? This is what one would hope if the analogy between twisted group rings and semi-direct products of groups could be pushed far enough.

However, let us note that while not every $\mathbb{Z}[G]$-algebra arises as a ring object in $\mathrm{Mod}(\mathbb{Z}[G])$, every such algebra does arise as a ring object in Ab. We may conjecture: Is it the case that every (Grothendieck) Abelian category arises as $\mathrm{Mod}(E; R)$, for some (Grothendieck) topos E and ring object R in E? It is known that the above conjecture is false if one tries to always chose R to be Z_E (see [2]). If we are given the Grothendieck Abelian category $\mathbb{A}$, then it is not difficult to find a Grothendieck topos E and a functor $f : \mathbb{A} \to \mathrm{Ab}(E)$ which is faithful and possesses both adjoints. The nastiness is, of course, arranging things so that f^* and f_* are internally adjoint.

References

[1] P. Freyd, Abelian Categories (Harper and Row, New York, 1964).

[2] D. Howe, Thesis, University of Pennsylvania (1977).

[3] P.T. Johnstone, Topos Theory, L.M.S. Mathematical Monographs, No. 10 (Academic Press, New York, 1977).