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Note

Spanning trees in a cactus

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Abstract

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We prove a best possible lower bound for the number of isomorphism classes into which all rooted spanning trees of a rooted cactus partition. We announce a best possible lower bound for the number of isomorphism classes into which all spanning trees of a cactus partition.

1. Introduction

A *tree* T is a connected graph without circuits. A *forest* is a graph where each component is a tree. A *cactus* is a connected graph where each block is either an edge or a circuit. An *n-cactus* is a cactus with exactly n circuits. A *rooted graph* is a pair (G, R) of a graph G and a set R consisting of one specified vertex from each component of G.

Two rooted graphs are said to be *root isomorphic* if there exists an isomorphism under which the two root sets correspond. We write $(G', R') \doteqdot (G'', R'')$ or for short $G' \doteqdot G''$ if the context shows which root sets are intended.

$\lceil x \rceil$ denotes for $x \in \mathbb{R}$ the least integer not less than x. $|G| = |V(G)|$ denotes the number of vertices in G. The *girth* of a graph G, which contains at least one circuit, is the length of a shortest circuit in G.

Zelinka [3] proved that the spanning trees of an n-cactus are partitioned into at least $n+1$ isomorphism classes. Vestergaard [2] raised this lower bound to

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$2+(n-1)\lceil g/2\rceil$ for an n-cactus of girth g, and conjectured that the bound could further be raised to

$$\binom{n+\left\lceil \frac{g}{2}\right\rceil-1}{\left\lceil \frac{g}{2}\right\rceil-1}.$$

In this note we prove a rooted version of this conjecture. The case $k=1$ in Theorem 1 below states that a rooted n-cactus (G, r) of girth g has at least

$$\binom{n+\left\lceil \frac{g}{2}\right\rceil-1}{\left\lceil \frac{g}{2}\right\rceil-1}$$

root isomorphism classes of spanning trees, all rooted at r.

Let us now state the main theorem, its proof comes in the next section.

Theorem 1. *Let G have k components, $k\geq 1$, each of which is a rooted cactus (G_i, r_i), $1\leq i\leq k$. Let $g\geq 3$ be a number such that each circuit in G has length at least g, and for each i, $1\leq i\leq k$, let $n_i\geq 0$ denote the number of circuits in G_i. Let $n_1+n_2+\cdots+n_k=n$ and $\lceil g/2\rceil=q$. Then the rooted spanning forests of G, (each consisting of k rooted trees (T_i, r_i), $1\leq i\leq k$), are partitioned into at least $\binom{n+q-1}{q-1}$ root isomorphism classes.*

2. Proof of Theorem 1

We shall prove Theorem 1 by induction on $|V(G)|$. The main steps in the proof will be taken in Lemmas 1–3 below.

Obviously Theorem 1 is true if $|V(G)|=1$. Likewise, if $k\geq 2$ and $|V(G_i)|=1$ for some i, say for $i=k$, then we can simply delete G_k and apply the induction hypothesis to the resulting graph.

Suppose that $|V(G_i)|\geq 2$ for all i, and suppose that Theorem 1 holds for graphs with at most $|V(G)|-1$ vertices; we shall then prove that Theorem 1 holds for G.

We shall not apply the induction hypothesis until the proof of Lemma 3; in the meantime we shall do some preparations. Let $R=\{r_1, r_2, \ldots, r_k\}$, $k\geq 1$, be the set of roots of G. Let $\mathscr{C}=\{C_1, C_2, \ldots, C_m\}$, $m\geq 0$, be the set consisting of those circuits in G which contain a root. If $m=0$ the argument skips ahead and continues with Lemma 3 below.

Otherwise, $m\geq 1$ and we can choose a circuit C from $\mathscr{C}$. Let (G', r') be that rooted component of G which contains C. Define $L_C(G', r')$ to be the graph spanned in G' by r' together with that component of $G'-r'$ which contains $C-r'$.

For each edge e of C not incident with r' define $\alpha(e)$, $\beta(e)$, $\alpha(e) \leq \beta(e)$, to be the orders of the two components of $(L_C(G', r') - r') - e$. If e is incident with r', then define $\alpha(e)$ to be zero and $\beta(e)$ to be the order of $L_C(G', r') - r'$.

For each C in $\mathscr{C}$ there exist at least q distinct multisets $\{\alpha(e), \beta(e)\}$ as e ranges over $E(C)$, because C has length at least g.

In Lemma 1 below we shall use the fact that any spanning forest of G with root set R, i.e. a set of k rooted trees (T_i, r_i), $1 \leq i \leq k$, can be obtained from G by deletion of exactly one edge from each circuit in $\mathscr{C}$ followed by deletion of exactly one edge from each remaining circuit.

For each i, $i = 1, 2, \ldots, m$, let e_i be an edge in C_i and define the multiset

$$\Gamma(e_1, e_2, \ldots, e_m) = \{\alpha(e_1), \beta(e_1), \alpha(e_2), \beta(e_2), \ldots, \alpha(e_m), \beta(e_m)\}.$$

We let $\Gamma^*(e_1, e_2, \ldots, e_m)$ denote the sub-multiset of $\Gamma(e_1, e_2, \ldots, e_m)$ obtained by deleting all entries that are equal to zero. Further, let Δ denote the multiset consisting of the orders of those components of $G - R$, which have no vertex in common with any circuit from $\mathscr{C}$.

Lemma 1. *Let notation be as above and let $e_i \in E(C_i)$, $f_i \in E(C_i)$ for each i, $1 \leq i \leq m$. If the multisets $\Gamma(e_1, e_2, \ldots, e_m)$ and $\Gamma(f_1, f_2, \ldots, f_m)$ are distinct, then for any pair of rooted spanning forests T and S for $G - \{e_1, e_2, \ldots, e_m\}$ and $G - \{f_1, f_2, \ldots, f_m\}$, respectively, we have $(T, R) \not\doteq (S, R)$.*

Proof of Lemma 1. First note that from the assumption that $\Gamma(e_1, e_2, \ldots, e_m)$ and $\Gamma(f_1, f_2, \ldots, f_m)$ are distinct, it immediately follows that $\Gamma^*(e_1, e_2, \ldots, e_m)$ and $\Gamma^*(f_1, f_2, \ldots, f_m)$ are also distinct.

Suppose that $(T, R) \doteq (S, R)$. Then the multisets of orders of all components of $T - R$ and $S - R$, respectively, must be identical, but we see from above that these multisets are exactly $\Gamma^*(e_1, e_2, \ldots, e_m) \cup \Delta$ and $\Gamma^*(f_1, f_2, \ldots, f_m) \cup \Delta$, which by assumption are distinct. This contradiction proves Lemma 1. □

With the help of Lemma 1 we shall demonstrate that there exist sufficiently many distinct graphs $G - \{e_1, e_2, \ldots, e_m\}$, $e_i \in E(C_i)$, (Lemma 2) and combining this with an induction hypothesis that each $G - \{e_1, e_2, \ldots, e_m\}$ has sufficiently many spanning forests (Lemma 3) we shall obtain a proof of Theorem 1.

Lemma 2. *Let $q \geq 1$, $m \geq 1$. For each i, $1 \leq i \leq m$, let there be given q distinct pairs $(\alpha_{ij}, \beta_{ij})$ which satisfy*

$$0 \leq \alpha_{ij} \leq \beta_{ij} \quad \textit{for all } j, \quad 1 \leq j \leq q$$

$$\alpha_{i1} < \alpha_{i2} < \cdots < \alpha_{iq}.$$

Then there exist at least $\binom{m+q-1}{q-1}$ pairwise distinct multisets of the form

$$\{\alpha_{1j_1}, \beta_{1j_1}, \alpha_{2j_2}, \beta_{2j_2}, \ldots, \alpha_{mj_m}, \beta_{mj_m}\}.$$

Proof of Lemma 2. We shall use induction on q. Lemma 2 is trivially true for $q=1$. Let $q \geqslant 2$ and assume that Lemma 2 is true for $q-1$, we shall then prove that it is also true for q.

We may choose notation such that $\alpha_{11} \leqslant \alpha_{21} \leqslant \cdots \leqslant \alpha_{m1}$.

for each i, $0 \leqslant i \leqslant m$, define a set S_i of multisets:

$$S_i = \left\{ \begin{array}{l} \{\alpha_{11}, \beta_{11}, \alpha_{21}, \beta_{21}, \ldots, \alpha_{i1}, \beta_{i1}, \alpha_{i+1,j_{i+1}}, \beta_{i+1,j_{i+1}}, \ldots, \alpha_{mj_m}, \beta_{mj_m}\} \mid \\ j_k \geqslant 2, \textit{ for each } k \textit{ with } i+1 \leqslant k \leqslant m \end{array} \right\}.$$

S_0 has $j_k \geqslant 2$, for all k, and S_m has $j_k = 1$, for all k.

The proof of Lemma 2 follows from (i)–(iii) below:

(i) $S_i \cap S_j = \emptyset$, for all i, j with $0 \leqslant i < j \leqslant m$,

(ii) $|S_i| \geqslant \binom{m+i+q-2}{q-2}$, for all i with $0 \leqslant i \leqslant m$,

(iii) $\sum_{i=0}^{m} \binom{m-i+q-2}{q-2} = \binom{m+q-1}{q-1}$.

Proof of (i). Let $\sigma \in S_i$, $\tau \in S_j$. We shall see that $\sigma \neq \tau$ by demonstrating that the multisets σ and τ do not contain the same number of numbers less than or equal to $\alpha_{i+1,1}$.

The numbers from σ which do not exceed $\alpha_{i+1,1}$ lie in $\{\alpha_{11}, \beta_{11}, \alpha_{21}, \beta_{21}, \ldots, \alpha_{i1}, \beta_{i1}\}$ because for all $s \geqslant i+1$ and for all $t \geqslant 2$ we have that $\alpha_{i+1,1} \leqslant \alpha_{s1} < \alpha_{st} \leqslant \beta_{st}$.

The numbers from τ which do not exceed $\alpha_{i+1,1}$ include $\alpha_{i+1,1}$ in addition to the numbers from σ not exceeding $\alpha_{i+1,1}$. This proves that $\sigma \neq \tau$ and hence that $S_i \cap S_j = \emptyset$.

Proof of (ii). For each i, $i = 0, 1, 2, \ldots, m$, the following holds: Each multiset from S_i has i pairs fixed, namely $(\alpha_{11}, \beta_{11})$, $(\alpha_{21}, \beta_{21}), \ldots, (\alpha_{i1}, \beta_{i1})$ (with $j_k = 1$), while there are $q-1$ choices for each of the other $m-i$ pairs $(\alpha_{i+1,j_{i+1}}, \beta_{i+1,j_{i+1}}), \ldots, (\alpha_{mj_m}, \beta_{mj_m})$, $(2 \leqslant j_k \leqslant m)$. By the induction hypothesis of Lemma 2 with m replaced by $m-i$ and q replaced by $q-1$ we obtain $|S_i| \geqslant \binom{m-i+q-2}{q-2}$ as desired.

Proof of (iii). Use the identity

$$\binom{m+q-1-i}{q-1} = \binom{m+q-1-i-1}{q-2} + \binom{m+q-1-i-1}{q-1}$$

successively for $i = 0, 1, 2, \ldots, m$. This concludes the proof of Lemma 2. □

From Lemmas 1 and 2 follow that at least $\binom{m+q-1}{q-1}$ distinct sets $\{e_1, e_2, \ldots, e_m\}$ exist such that all the corresponding graphs $G - \{e_1, e_2, \ldots, e_m\}$ have pairwise disjoint families of root isomorphism classes of spanning forests with root set R.

We shall prove that each family is large enough to make Theorem 1 true. For the case $m = 0$ the argument is resumed here.

Lemma 3. *Let notation be as above. For any set of edges $e_1, e_2, \ldots, e_m$ with $e_i \in E(C_i)$, the graph $G - \{e_1, e_2, \ldots, e_m\}$ has at least $\binom{n-m+q-1}{q-1}$ spanning forests with root set R, no two of which are root isomorphic to each other.*

Proof of Lemma 3. In each component of $(G - \{e_1, e_2, \ldots, e_m\}) - R$ let the unique vertex which is adjacent to a vertex of R be designated as a new root r_j^*, $j = 1, 2, \ldots$.

Then $G^* = (G - \{e_1, e_2, \ldots, e_m\}) - R$ is a graph, whose components are rooted cacti (G_j^*, r_j^*), $j = 1, 2, \ldots$. G^* contains $n - m$ circuits, each of length at least g. Since $|V(G^*)| < |V(G)|$, we may apply the induction hypothesis of Theorem 1, and we see that G^* has at least $\binom{n-m+q-1}{q-1}$ distinct root isomorphism classes of spanning forests with root set $R^* = \{r_1^*, r_2^*, \ldots\}$.

From the construction of G^* it follows that $G - \{e_1, e_2, \ldots, e_m\}$ also must have at least $\binom{n-m+q-1}{q-1}$ root isomorphism classes of spanning forests with root set R. This is because there exist a surjection from the set of root isomorphism classes of $(G - \{e_1, e_2, \ldots, e_m\}, R)$ onto the set of root isomorphism classes of (G^*, R^*), and this implies that the first set is at least as large as the second set. This proves Lemma 3. □

The lower bounds of Lemmas 2 and 3 together with the inequality

$$\binom{m+q-1}{q-1}\binom{n-m+q-1}{q-1} \geq \binom{n+q-1}{q-1} \qquad (*)$$

finally prove Theorem 1. □

We can verify the truth of inequality (*) by observing that (*) is equivalent to

$$\binom{a+c}{c}\binom{b+c}{e} \geq \binom{a+b+c}{c}$$

(set $a = m$, $b = n - m$, $c = q - 1$), *which in turn is equivalent to*

$$\{(a+1)(a+2)\cdots(a+c)\}\{(b+1)(b+2)\cdots(b+c)\}$$
$$\geq \{(a+b+1)(a+b+2)\cdots(a+b+c)\}\{1\cdot 2\cdots c\}.$$

Taking the ith factor from each of the four products we see that $(a+i)(b+i) \geq (a+b+i)i$ hold for $i = 1, 2, \ldots, c$.

The bound of Theorem 1 is best possible. Let the rooted n-cactus (G, r) be obtained as disjoint union of n rooted circuits, each of length g, with all n roots identified into one vertex r. Then (G, r) has exactly $\binom{n+q-1}{q-1}$ root isomorphism classes of spanning trees.

In fact it follows from our proof that if G attains the lower bound in Theorem 1, then either $n = n_i$ for some i or $n_i \leq 1$ for all i. We shall say no more about the structure of a graph which attains the lower bound.

3. Spanning trees

For large q we can prove Theorem 2 below. We conjecture that Theorem 2 holds without this restriction. The details will hopefully be discussed in a subsequent paper.

Theorem 2. *Let $n \geqslant 2$, $g \geqslant 3$, $g = \lceil g/2 \rceil$. The spanning trees of a cactus with n circuits, each of length at least g, partition into at least*

$$\binom{n+q-1}{q-1} \text{ isomorphism classes.}$$

References

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