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Quantum zonal spherical functions and Macdonald polynomials

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Abstract

A unified theory of quantum symmetric pairs is applied to q-special functions. Previous work characterized certain left coideal subalgebras in the quantized enveloping algebra and established an appropriate framework for quantum zonal spherical functions. Here a distinguished family of such functions, invariant under the Weyl group associated to the restricted roots, is shown to be a family of Macdonald polynomials, as conjectured by Koornwinder and Macdonald. Our results place earlier work for Lie algebras of classical type in a general context and extend to the exceptional cases.

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Keywords: Quantum groups; Zonal spherical functions; Macdonald polynomials

0. Introduction

A beautiful classical result shows that the zonal spherical functions associated to real compact symmetric spaces can be realized as Jacobi polynomials. An analogous result was proved for zonal spherical functions of p-adic symmetric spaces [22]. In the late 1980s, Macdonald [23] introduced his two-parameter family of orthogonal polynomials which provided a unified context for the polynomial families used in these parallel theories. With the discovery of quantum groups, also in the 1980s, both Koornwinder [15, Section 6.4] and Macdonald [23] asked whether one could obtain a

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similar description of quantum zonal spherical functions. This conjecture was investigated extensively in the 1990s by Noumi, Sugitani, Dijkhuizen, Stokman and others when the underlying Lie algebra is of classical type. Quantum zonal spherical functions are realized as particular q special functions in [24–26,28,3,4] using case-by-case computations. The main result of this paper is an answer to Macdonald and Koornwinder's question for all quantum symmetric pairs with reduced restricted root systems, thus generalizing this earlier work to include Lie algebras of exceptional type. Our methods are new and do not involve extensive case work. Instead, we rely on the unified theory of quantum symmetric pairs developed in [19–21]. As a consequence, we provide simple, uniform, formulas for the parameters which appear in the Macdonald polynomials corresponding to quantum zonal spherical functions.

The first problem faced in the quantum case was the actual definition of quantum symmetric spaces. Classically, a symmetric pair of Lie algebras is a pair $\mathfrak{g}, \mathfrak{g}^\theta$ where $\mathfrak{g}$ is a complex Lie algebra and θ is a Lie algebra involution. The initial breakthrough was made by Koornwinder [14], who constructed quantum 2 spheres and computed their zonal spherical functions by inventing quantum analogs of the symmetric pair $\mathfrak{sl}\ 2, \mathfrak{so}\ 2$. Noumi [24] extended Koornwinder's approach to two families of symmetric pairs for $\mathfrak{g}$ of type A_n, using two-sided coideal analogs of $\mathfrak{g}^\theta$ inside the quantized enveloping algebra $U_q(\mathfrak{g})$. He showed that the corresponding zonal spherical functions were, indeed, Macdonald polynomials. In [26], Noumi and Sugitani introduced one-sided coideal analogs of $U(\mathfrak{g}^\theta)$ inside of $U_q(\mathfrak{g})$ for other cases when $\mathfrak{g}$ is of classical type; analysis of the corresponding zonal spherical functions can be found in [26,28,3,25,4]. In his comprehensive survey of the early history of quantum symmetric spaces, Dijkhuizen [2] conjectures that coideals are the correct objects to use in order to develop a general theory of quantum symmetric pairs, the corresponding symmetric spaces, and their zonal spherical functions. In [19,20], a universal method was developed for constructing left coideal subalgebras of $U_q(\mathfrak{g})$, which are quantum analogs of $U(\mathfrak{g}^\theta)$. These analogs are further characterized as the appropriately unique maximal left coideal subalgebras of $U_q(\mathfrak{g})$ which specialize to $U(\mathfrak{g}^\theta)$ as q goes to 1. A complete list of the generators and relations of the coideal subalgebras for all possible symmetric pairs $\mathfrak{g}, \mathfrak{g}^\theta$ is given in [21]. Furthermore, although the symmetric pairs used in [24,26] are formally defined differently, it turns out that these examples are subsumed by this new theory (see [19, Section 6] and the last paragraph of [20]).

Using the new quantum analogs of $U(\mathfrak{g}^\theta)$ inside of $U_q(\mathfrak{g})$, one can define and study quantum symmetric spaces and their zonal spherical functions (see [20, Section 7] and [21]). In particular, fix an irreducible symmetric pair $\mathfrak{g}, \mathfrak{g}^\theta$. Let $\mathscr{B}_\theta$ denote the orbit of the analogs of $U(\mathfrak{g}^\theta)$ in $U_q(\mathfrak{g})$ under the group $\mathbf{H}$ of Hopf algebra automorphisms of $U_q(\mathfrak{g})$ fixing the Cartan part. For each pair B, B' in $\mathscr{B}$, one can associate the space ${}_{B'}\mathscr{H}_B$ of left B' and right B invariants inside the quantized function algebra $R_q[G]$ corresponding to $\mathfrak{g}$. Using the interpretation of elements of $R_q[G]$ as functions on $U_q(\mathfrak{g})$, restriction to the Cartan part of $U_q(\mathfrak{g})$ yields an algebra homomorphism Υ from ${}_{B'}\mathscr{H}_B$ into the character group ring associated to the restricted root system [21, Theorem 4.2]. Zonal spherical functions are joint eigenvectors in ${}_{B'}\mathscr{H}_B$ with respect to the action of the center of $U_q(\mathfrak{g})$. A zonal spherical family associated to $\mathscr{B}_\theta$ is the image of a specially chosen basis of ${}_{B'}\mathscr{H}_B$ consisting of zonal spherical functions for some B, B' in $\mathscr{B}_\theta$. Let W_Θ denote the Weyl group associated to the restricted roots. In [21, Theorem 6.5], it is shown that each $\mathbf{H}\times\mathbf{H}$ orbit of $\mathscr{B}_\theta\times\mathscr{B}_\theta$ contains a virtually unique W_Θ invariant zonal spherical family. Moreover, the natural map from $\mathscr{B}_\theta\times\mathscr{B}_\theta$ to the set of zonal spherical functions associated to $\mathscr{B}_\theta\times\mathscr{B}_\theta$ is $\mathbf{H}\times\mathbf{H}$ equivariant [21, Theorem 6.3]. In this context, the action of an element of $\mathbf{H}\times\mathbf{H}$ on a zonal spherical family corresponds to the image of this family under an automorphism of the restricted character group ring. Thus in order to compute the zonal spherical families associated to $\mathscr{B}_\theta$, it is only necessary to analyze the unique W_Θ invariant representative for each $\mathbf{H}\times\mathbf{H}$ orbit.

The set $\mathscr{B}_\theta$ contains a distinguished $\mathbf{H}$ orbit corresponding to the standard analogs of $U(\mathfrak{g}^\theta)$ in $U_q(\mathfrak{g})$. It follows from [21, Theorem 6.5 and subsequent discussion] (see Theorem 1.1 below) that there is a unique family of W_Θ invariant quantum zonal spherical functions associated to the standard analogs of $U(\mathfrak{g}^\theta)$. We further assume here that the restricted root space associated to $\mathfrak{g}, \mathfrak{g}^\theta$ is reduced. Our goal in this work is to show that this family of W_Θ invariant quantum zonal spherical functions is precisely a family of Macdonald polynomials where both parameters are equal to powers of q. Moreover, Theorem 8.2 provides a simple formula involving the multiplicity of restricted roots which relates one parameter to another. (We should mention that for most pairs $\mathfrak{g}, \mathfrak{g}^\theta$, the standard analogs are the only possible analogs. However, under special circumstances, there is a one parameter family of analogs of $U(\mathfrak{g}^\theta)$; the standard analogs appear when the parameter is set equal to zero. See [20, Section 7, Variation 2] for more information.)

Our overall method of determining the zonal spherical functions is inspired by the work of [24,26,2]. Indeed, we show that the radial components of certain central elements correspond to the difference operators, which then define Macdonald polynomials. However, the strategy we employ is quite different. We do not use L operators as in [24,26,2] to express the Casimir elements because, in part, this cannot be done for the exceptional Lie algebras. Moreover, our argument avoids an explicit expression of central elements inside $U_q(\mathfrak{g})$ for $\mathfrak{g}$ larger than $\mathfrak{sl}\ 2$, thus making a detour around difficult computations and extensive case work. Instead, we rely on a variety of representation theory techniques and draw upon the description of the center and the locally finite part of $U_q(\mathfrak{g})$ developed in [11,12].

It should be noted that many of the techniques of this paper extend to the case when Σ is not reduced and to the nonstandard analogs. In a future paper, we adapt the methods used here to identify the zonal spherical functions of these other cases as q hypergeometric functions. Ultimately, this will generalize work of [25,4] on nonstandard analogs of type AIII and work of [26,28,3,25,4] when the restricted root system is not reduced.

Since this paper is long, we present a detailed description of its organization. The first section sets notation and recalls background on the author's theory of quantum symmetric pairs and their zonal spherical functions. Section 2 presents four versions of an Iwasawa type tensor product decomposition (Theorem 2.2). Let B denote a quantum analog of $U(\mathfrak{g}^\theta)$ inside $U_q(\mathfrak{g})$ and write B_+ for the augmentation ideal of

$U_q(\mathfrak{g})$. Further analysis of projections of $U_q(\mathfrak{g})$ modulo $B_+U_q(\mathfrak{g})$ and $U_q(\mathfrak{g})B_+$ are obtained in Theorem 2.3. These maps are used in Section 3 (Theorem 3.2) to construct a function $\mathscr{X}$ from $U_q(\mathfrak{g})$ to the ring of endomorphisms of the restricted character group ring which corresponds to the action of $U_q(\mathfrak{g})$ on the quantum zonal spherical functions. Upon restriction to the center of $U_q(\mathfrak{g})$, $\mathscr{X}$ is exactly the function which computes the quantum radial components. Radial components of central elements, and more generally, radial components of elements in the centralizer of B in $U_q(\mathfrak{g})$, are shown to be invariant under the action of the restricted Weyl group W_Θ in Theorems 3.4 and 3.6.

Section 4 is devoted to analyzing the rank one radial components of central elements. We first prove that in the rank one case, the center of $U_q(\mathfrak{g})$ has a Casimir-like central element which looks remarkably like the Casimir element in the center of $U_q(\mathfrak{sl}\ 2)$ (Theorem 4.5). The radial component of this special central element is then computed using a straightforward $U_q(\mathfrak{sl}\ 2)$ calculation (Theorem 4.7).

In Section 5, a filtration, similar to the ad invariant filtration of $U_q(\mathfrak{g})$ studied in [12] and [8, Chapter 7], is introduced. The filtration used here is carefully chosen so that the algebra B lies in degree zero. In particular, the resulting graded algebra, $\mathrm{gr}\, U_q(\mathfrak{g})$, contains B as a subalgebra. A theory of graded zonal spherical functions is presented. It is shown that up to a shift of weight, the graded zonal spherical functions are all equal to each other (Lemma 5.7). Furthermore, the action of the graded image of a central element of $U_q(\mathfrak{g})$ on a graded zonal spherical function agrees with the action of the top degree of the corresponding radial component. This allows us to write the top degree terms of radial components in a simple form using the graded zonal spherical functions and the Harish-Chandra projection (Theorem 5.8). The resulting expression is similar to Harish-Chandra's formula and its generalizations for the radial components of central elements in $U(\mathfrak{g})$ with respect to the adjoint action of the corresponding Lie group on $\mathfrak{g}$ (see [29, 7.A.2.9 and 7.A.3.7]).

In Section 6, the graded zonal spherical functions, and thus the top degree terms of radial components, are determined first in the rank one case using Section 4 (see Lemma 6.6). This information is then glued together using the Weyl invariance of radial components in Theorem 6.7 and Corollary 6.8. The argument is delicate since the graded zonal spherical functions are elements of the formal power series ring corresponding to the restricted character group ring. Unfortunately, the restricted Weyl group does not act on this ring. We overcome this obstacle by finding the possible weights of highest weight vectors with respect to rank one subalgebras of $\mathrm{gr}\ U_q(\mathfrak{g})$ inside Verma-like $\mathrm{gr}\ U_q(\mathfrak{g})$ modules. This allows one to express the graded zonal spherical function as the product of all the rank one formulas times a W_Θ invariant term. Taking into account the highest weight summand of the graded zonal spherical function yields that this W_Θ invariant term must be 1. In the end, we show that the formula for the graded zonal spherical function is just the inverse of the element in the restricted character group ring used to define Macdonald's inner product (Theorem 6.7). It should be noted that the proof here is remarkably similar to an argument used in a completely different setting, the factorization of the affine PRV determinant [13]. The product formula combined with the rank one reduction used here is also reminiscent of Gindikin and Karpelevic's well known computation of Harish-Chandra's c function [6, Chapter IV, Section 6].

It is sometimes necessary to pass to the slightly larger simply connected quantized enveloping algebra $\check{U}$ (which is just a small extension of $U_q(\mathfrak{g})$). By construction, the Cartan subalgebra of $\check{U}$ is just the group algebra of the torus corresponding to the weight lattice. In Section 7, one finds an element in the centralizer $C_{\check{U}}(B)$ of B in $\check{U}$ whose top degree term with respect to the Harish-Chandra projection corresponds to a minuscule or pseudominuscule restricted weight. In most cases this element is in the center of $\check{U}$ and is easily determined using the description of the image of central elements under the Harish-Chandra map [11]. For a few exceptional irreducible symmetric pairs, the central elements turn out to be too "large" to correspond to minuscule or pseudominuscule restricted weights. This failure is related to the classical fact that the map from the center of $U(\mathfrak{g})$ to the set of invariant differential operators on the symmetric space corresponding to the pair $\mathfrak{g}, \mathfrak{g}^\theta$ is not always surjective [5]. An analysis of the locally finite part of $\check{U}$ is used in order to locate the suitable element in $C_{\check{U}}(B)$ in the problematic cases. The radial components of these "small" elements in $C_{\check{U}}(B)$ are then determined (Theorem 7.7) by taking the sum of the terms in the W_Θ orbit of the top degree part (found in Section 6) plus a possible zero degree term.

Section 8 recalls basic facts about Macdonald polynomials. The radial components studied in Section 7 are identified with difference operators associated to minuscule and pseudominuscule weights. This in turn establishes our main result that zonal spherical functions are particular Macdonald polynomials. We conclude the paper with two appendices. The first lists all irreducible symmetric pairs $\mathfrak{g}, \mathfrak{g}^\theta$ with reduced root system and the values of the parameters in the Macdonald polynomials corresponding to the quantum zonal spherical functions. The second appendix is an index of commonly used notation including definitions for objects not defined in Section 1.

1. Background and notation

Let $\mathbf{C}$ denote the complex numbers, $\mathbf{Q}$ denote the rational numbers, $\mathbf{Z}$ denote the integers, and $\mathbf{N}$ denote the nonnegative integers. If G is a multiplicative monoid and $\mathbf{F}$ is a field, then we write $\mathbf{F}[G]$ for the corresponding monoid algebra over $\mathbf{F}$. (This is the obvious generalization of "group algebra".) Unfortunately many monoids come to us additively. In the special case of the additive monoid $\mathbf{Q}$, we invent the symbol q and temporarily identify $\mathbf{Q}$ with the multiplicative monoid $\{q^r \mid r \in \mathbf{Q}\}$ where $q^r q^s = q^{r+s}$. Let $\mathscr{C}$ be the algebraic closure of the field of fractions for $\mathbf{C}[\mathbf{Q}]$. We write $\mathscr{C}^\times$ for the nonzero elements of $\mathscr{C}$.

Given a root system Φ, let Φ^+ denote the positive roots, $Q(\Phi)$ denote the root lattice, $P(\Phi)$ denote the weight lattice, $Q^+(\Phi)$ denote the $\mathbf{N}$ span of the elements in Φ^+, and $P^+(\Phi)$ denote the set of dominant integral weights. (Sometimes we will replace Φ with the symbol representing the subset of simple positive roots in the notation for the root and weight lattice and their subsets.) Write 2Φ for the root system $\{2\alpha \mid \alpha \in \Phi\}$ given the same inner product as Φ. If H is an *additive* submonoid of $\mathbf{C}\Phi$, we invent the formal variable z, so that $\mathscr{C}[H]$ consists of the $\mathscr{C}$ linear combinations of the basis elements z^λ for $\lambda \in H$.

Let $\mathfrak{g}$ denote a semisimple Lie algebra over the complex numbers $\mathbf{C}$ with a chosen triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. Let Δ be the set of roots for $\mathfrak{g}$ and let $\pi = \{\alpha_1, \dots, \alpha_n\}$ be the set of (positive) simple roots in Δ corresponding to the root vectors in $\mathfrak{n}^+$. Write $(\ ,\ )$ for the Cartan inner product on $\mathfrak{h}^*$ with respect to the root system Δ. Let W denote the Weyl group associated to the root system Δ. Set ρ equal to the half sum of the positive roots in Δ.

Let θ be a maximally split involution with respect to the fixed Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and triangular decomposition (see [20, (7.1), (7.2), and (7.3)]). We assume throughout the paper that $\mathfrak{g}, \mathfrak{g}^\theta$ is an irreducible pair in the sense of [1] (see also [21, Section 7]). Recall that a complex semisimple Lie algebra $\mathfrak{g}'$ with maximally split involution θ' can be written as a direct sum of semisimple Lie subalgebras $\mathfrak{r}_i$ such that each $\mathfrak{r}_i, \mathfrak{r}_i^{\theta'}$ is an irreducible pair. Using such a direct sum decomposition, the results of this paper easily extend to arbitrary symmetric pairs.

The involution θ on $\mathfrak{g}$ induces an involution on $\mathfrak{h}^*$ which we refer to as Θ. Furthermore, Θ restricts to an involution on Δ. Set $\pi_\Theta = \{\alpha_i \in \pi \mid \Theta(\alpha_i) = \alpha_i\}$. Recall [20, Section 7, (7.5)] that there is a permutation p on $1, \dots, n$ such that

$$-\Theta(\alpha_i) \in \alpha_{\mathrm{p}(i)} + Q^+(\pi_\Theta).$$

Set $\pi^* = \{\alpha_i \in \pi \backslash \pi_\Theta \mid i \leqslant \mathrm{p}(i)\}$.

Given $\alpha \in \mathfrak{h}^*$, set $\tilde{\alpha} = (\alpha - \Theta(\alpha))/2$. The subset

$$\Sigma = \{\tilde{\alpha} \mid \alpha \in \Delta \text{ and } \Theta(\alpha) \neq \alpha\} \tag{1.1}$$

of $\mathfrak{h}^*$ is the restricted root system associated to the pair $\mathfrak{g}, \mathfrak{g}^\theta$. Note that the set of (positive) simple restricted roots is just $\{\tilde{\alpha}_i \mid \alpha_i \in \pi^*\}$, while $\Sigma^+ = \{\tilde{\alpha} \mid \alpha \in \Delta^+$ and $\Theta(\alpha) \neq \alpha\}$. Let W_Θ denote the Weyl group associated to Σ.

We make the following assumption throughout this paper:

$$\Sigma \text{ is a reduced root system.}$$

A complete list of the possible irreducible pairs $\mathfrak{g}, \mathfrak{g}^\theta$ with Σ reduced using Araki's classification [1] can be found in the appendix of this paper.

Let $U = U_q(\mathfrak{g})$ be the quantized enveloping algebra generated by $x_1, \dots, x_n$, $y_1, \dots, y_n$, $t_1^{\pm 1}, \dots, t_n^{\pm 1}$ over $\mathscr{C}$. Recall that U is a Hopf algebra with coproduct, counit, and antipode. (See [8, 3.2.9] or [20, Section 1,(1.4)–(1.10)] for relations and Hopf algebra structure.) We write U^+ for the subalgebra of U generated by $x_1, \dots, x_n$ and U^- for the subalgebra of U generated by $y_1, \dots, y_n$. Let U_+ denote the augmentation ideal of U. Given a subalgebra S of U and a U module V, we write S_+ for $S \cap U_+$ and set $V^S = \{v \in V \mid sv = 0 \text{ for all } s \in S_+\}$.

Let T denote the group generated by the t_i for $1 \leqslant i \leqslant n$. Set U^0 equal to the group algebra $\mathscr{C}[T]$. Recall that there is an isomorphism $\tau : Q(\pi) \to T$ such that $\tau(\alpha_i) = t_i$ for each i. Given a T module N and a weight $\gamma \in \mathfrak{h}^*$, the γ weight subspace of N is just the set

$$\{m \in N \mid \tau(\beta)m = q^{(\beta,\gamma)}m \text{ for all } \tau(\beta) \in T\}.$$

Note that U is a module over itself via the (left) adjoint action denoted by ad (see for example [8, 1.3.1]). If N is a subspace of U, then N_γ is the γ weight space of N as an ad T module.

Consider a vector subspace F and a subset S of an algebra over $\mathscr{C}$. Note that products of the form as and sa for $a \in F$ and $s \in S$ exist and are elements of this $\mathscr{C}$ algebra. Set FS equal to the $\mathscr{C}$ span of the set $\{as \mid a \in F, s \in S\}$. Similarly, write SF for the vector space over $\mathscr{C}$ spanned by the set $\{sa \mid a \in F, s \in S\}$. Now suppose that S is a submonoid of T and F is both a subalgebra and ad S submodule of U. Note that the subalgebra of U generated by F and S is equal to the vector space FS.

Set $T_\Theta = \{\tau(\beta) \mid \beta \in Q(\pi) \text{ and } \Theta(\beta) = \beta\}$. Let $\mathscr{M}$ be the subalgebra of $U_q(\mathfrak{g})$ generated by $x_i, y_i, t_i^{\pm 1}$ for $\alpha_i \in \pi_\Theta$. By [20, Theorem 7.1], we can lift θ to a $\mathbf{C}$ algebra automorphism $\tilde{\theta}$ of U which sends q to q^{-1}. For each $\alpha_i \in \pi \setminus \pi_\Theta$, set

$$B_i = y_i t_i + \tilde{\theta}(y_i) t_i. \tag{1.2}$$

Let B_θ denote the subalgebra of U generated by the $B_i, \alpha_i \in \pi \setminus \pi_\Theta$, $\mathscr{M}$, and T_Θ. By [20, Theorem 7.2 and the discussion following the proof of Theorem 7.4], B_θ is a left coideal subalgebra which specializes to $U(\mathfrak{g}^\theta)$ as q goes to 1.

Let $\mathbf{H}$ denote the group of Hopf algebra automorphisms of U which fix elements of T. Note that $\mathbf{H}$ acts on the set of left coideal subalgebras of U. Set $\mathscr{B}$ equal to the orbit of B_θ under the action of $\mathbf{H}$. Of course, $\mathscr{B}$ depends on the choice of pair $\mathfrak{g}, \mathfrak{g}^\theta$, but this will be understood from context. For most irreducible pairs $\mathfrak{g}, \mathfrak{g}^\theta$, the orbit $\mathscr{B}$ equals the set $\mathscr{B}_\theta$ defined in [21, Section 2] and is just the orbit under the action of $\mathbf{H}$ of the quantum analogs of $U(\mathfrak{g}^\theta)$ inside of $U_q(\mathfrak{g})$. There are, however, a few cases of irreducible pairs for which the set $\mathscr{B}_\theta$ is strictly larger than $\mathscr{B}$. Since we are assuming that Σ is reduced, this occurs when $U(\mathfrak{g}^\theta)$ has nonstandard analogs—also referred to as analogs of Variation 2 see ([20, Section 7, Variation 2]). We only consider the standard analogs in this paper. A complete list of the quantum analogs of $U(\mathfrak{g}^\theta)$ in U associated to all possible irreducible symmetric pairs can be found in [21, Section 7]. (In the notation of [21], let $\mathscr{S}$ be the subset of π^* consisting of those α_i such that $\Theta(\alpha_i) = -\alpha_i$ and $2(\alpha_i, \alpha_j)/(\alpha_j, \alpha_j)$ is even for all α_j such that $\Theta(\alpha_j) = -\alpha_j$. By [21, Sections 2 and 7], since $\mathfrak{g}, \mathfrak{g}^\theta$ is irreducible, $\mathscr{S}$ is either empty or consists of exactly one root $\alpha_i \in \pi^*$. When this happens, the orbits of $\mathscr{B}_\theta$ are parametrized by one variable, s_i, and $\mathscr{B}$ is the $\mathbf{H}$ orbit in $\mathscr{B}_\theta$ associated to $s_i = 0$.)

Given $\lambda \in P^+(\pi)$, let $L(\lambda)$ denote the finite dimensional simple U module with highest weight λ. Write $L(\lambda)^*$ for the dual of $L(\lambda)$ given its natural right module structure. Recall the quantum Peter–Weyl theorem [8, 9.1.1 and 1.4.13], see also [20, (3.1)]: the quantized function algebra $R_q[G]$ is isomorphic as a U bimodule to a direct sum of the $L(\lambda) \otimes L(\lambda)^*$ as λ varies over the dominant integral weights $P^+(\pi)$. As explained in [21, Section 4], elements of $L(\lambda)\otimes L(\lambda)^*$, and thus of $R_q[G]$, can be thought of as functions on U. Restriction to the torus T yields an algebra homomorphism, denoted by Υ, from $R_q[G]$ into $\mathscr{C}[P(\pi)]$.

Let G denote the connected, simply connected Lie group with Lie algebra $\mathfrak{g}$ and let K be the compact Lie group corresponding to the Lie algebra $\mathfrak{g}^\theta$. In the classical case, zonal spherical functions are K invariant functions on the symmetric space G/K which are also eigenfunctions for the action of the center of $U(\mathfrak{g})$. The quantum symmetric space $R_q[G/K]_{B'}$, or more precisely, quantum analog of the ring of regular functions on G/K associated to $B'\in\mathscr{B}$, is the algebra of left B' invariants inside the quantized function algebra (see [20, Section 7 and Theorem 7.8]). Thus quantum zonal spherical functions at the pair (B,B') in $\mathscr{B}\times\mathscr{B}$ are right B invariant elements of $R_q[G/K]_{B'}$ which are eigenfunctions for the action of the center of U on $R_q[G]$. In particular, quantum zonal spherical functions live inside the space ${}_{B'}\mathscr{H}_B$ of left B' and right B invariants of $R_q[G]$. Moreover, the eigenspaces in $R_q[G]$ for the action of the center of U are just the subspaces $L(\lambda)\otimes L(\lambda)^*$ for λ dominant integral. Hence the quantum zonal spherical functions at λ associated to the pair (B,B') are the nonzero elements in the space ${}_{B'}\mathscr{H}_B(\lambda)$ defined by

$$ {}_{B'}\mathscr{H}_B(\lambda)=\{f\in L(\lambda)\otimes L(\lambda)^*\mid B'_+f=fB_+=0\}. $$

By [21, Theorem 3.4 and Section 4], ${}_{B'}\mathscr{H}_B(\lambda)$ is one dimensional if $\lambda\in P^+(2\Sigma)$ and zero otherwise. In particular, ${}_{B'}\mathscr{H}_B$ is a direct sum of the subspaces ${}_{B'}\mathscr{H}_B(\lambda)$ as λ runs over the elements in $P^+(2\Sigma)$ [21, (4.2)]. By [21, Theorem 4.2], the map Υ from $R_q[G]$ into $\mathscr{C}[P(\pi)]$ restricts to an injective algebra homomorphism from ${}_{B'}\mathscr{H}_B$ into $\mathscr{C}[P(2\Sigma)]$. Furthermore, [21, Lemma 4.1] ensures that $\Upsilon({}_{B'}\mathscr{H}_B)$ contains a distinguished basis $\{\varphi^\lambda_{B,B'}\mid\lambda\in P^+(2\Sigma)\}$ such that $\varphi^\lambda_{B,B'}\in\Upsilon({}_{B'}\mathscr{H}_B(\lambda))$ and

$$ \varphi^\lambda_{B,B'}\in z^\lambda+\sum_{\beta<\lambda}\mathscr{C}z^\beta \tag{1.3} $$

for all $\lambda\in P^+(2\Sigma)$.

We recall the notion of zonal spherical families introduced in [21, Section 6]. In particular, a function $\lambda\mapsto\psi_\lambda$ from $P^+(2\Sigma)$ to $\mathscr{C}[P(2\Sigma)]$ is called a zonal spherical family associated to $\mathscr{B}$ if there exists B and B' in $\mathscr{B}$ such that $\psi_\lambda=\varphi^\lambda_{B,B'}$ for all $\lambda\in P^+(2\Sigma)$. With a slight abuse of notation, we generally identify the zonal spherical family above with its image, $\{\psi_\lambda\mid\lambda\in P^+(2\Sigma)\}$. We have the following modification of [21, Theorem 6.5].

Theorem 1.1. *Let $\lambda\in P^+(2\Sigma)$. There exists a unique W_Θ invariant zonal spherical family $\{\varphi_\lambda\mid\lambda\in P^+(2\Sigma)\}$ associated to $\mathscr{B}$.*

Proof. In the case when $U(\mathfrak{g}^\theta)$ does not admit a nonstandard analog, then this is a consequence of [21, Theorem 4.2 and Theorem 6.5] as explained in the discussion following the proof of the [21, Theorem 6.5]. (This case corresponds to (i) of [21].) Now assume that $U(\mathfrak{g}^\theta)$ does admit nonstandard analogs and in particular, $\mathcal{S} = \{\alpha_i\}$. Let $B \in \mathcal{B}$. Set

$$N = \mathbf{Z}\tilde{\alpha}_i + \sum_{\alpha_j \in \pi^* \setminus \mathcal{S}} \mathbf{Z}2\tilde{\alpha}_j.$$

Consider a spherical vector $\xi_\lambda \in L(\lambda)$ with respect to B. By [21, Theorem 3.6 and its proof], ξ_λ is a sum of weight vectors of weight $\lambda - \beta$ where $\beta \in N \cap Q^+(\Sigma)$. Now B is the image under an element in $\mathbf{H}$ of the analog in $\mathcal{B}_\theta$ associated to $s_i = 0$. Hence the proof of [21, Theorem 3.6] actually shows that ξ_λ is a sum of weight vectors of weight $\lambda - \beta$ where β is in the smaller set $Q^+(2\Sigma)$. Thus arguing as in [21, Lemma 4.1], the space ${}_{B'}\mathcal{H}_B(\lambda)$ is a subspace of $z^\lambda \mathcal{C}[Q(2\Sigma)]$ for all pairs B and B' in $\mathcal{B}$.

Recall that $\mathcal{B}$ is the single orbit of B_θ under the action of $\mathbf{H}$. Thus $\mathcal{B} \times \mathcal{B}$ is a single $\mathbf{H} \times \mathbf{H}$ orbit contained in $\mathcal{B}_\theta \times \mathcal{B}_\theta$. Note that elements of $\mathrm{Hom}(N, \mathcal{C}^\times)$ act on $\mathcal{C}[N]$ as linear transformations where $g(z^\beta) = g(\beta)z^\beta$ for all $\beta \in N$ and $g \in \mathrm{Hom}(N, \mathcal{C}^\times)$. Set $\mathcal{Z}_\lambda$ equal to the set of W_Θ invariant zonal spherical functions at λ associated to $\mathcal{B}$ with top degree term equal to z^λ. By [21, Theorem 6.5] there exists a W_Θ invariant zonal spherical family $\{\varphi_\lambda \mid \lambda \in P^+(2\Sigma)\}$ associated to $\mathcal{B}$ such that the

$$\mathcal{Z}_\lambda = \{z^\lambda g(z^{-\lambda}\varphi_\lambda) \mid g \in \mathrm{Hom}(N, \mathcal{C}^\times) \text{ and } g \text{ acts trivially on } Q(2\Sigma)\}.$$

However, by the previous paragraph, $z^{-\lambda}\varphi_\lambda \in \mathcal{C}[Q(2\Sigma)]$. Hence $\mathcal{Z}_\lambda$ contains exactly one element φ_λ. Therefore $\{\varphi_\lambda \mid \lambda \in P^+(2\Sigma)\}$ is the unique W_Θ invariant zonal spherical family associated to $\mathcal{B}$. $\square$

Using Theorem 1.1, we write $\{\varphi_\lambda \mid \lambda \in P^+(2\Sigma)\}$ for the unique W_Θ invariant zonal spherical family associated to $\mathcal{B}$. Of course this family depends on the choice of $\mathfrak{g}, \mathfrak{g}^\theta$, but this will be understood from context. As an immediate consequence of Theorem 1.1, if $\Upsilon({}_{B'}\mathcal{H}_B)$ is W_Θ invariant then $\varphi^\lambda_{B,B'} = \varphi_\lambda$ for all $\lambda \in P^+(2\Sigma)$. By [21, Corollary 5.4], we can choose $B'_\theta \in \mathcal{B}$ such that $\Upsilon({}_{B'_\theta}\mathcal{H}_{B_\theta})$ is W_Θ invariant. We drop the subscript θ and abbreviate B_θ as B and B'_θ as B' after Section 2 is completed.

2. Decompositions and related projections

In this section, we present tensor product decompositions of U with respect to a subalgebra $B \in \mathcal{B}_\theta$ similar to the quantum Iwasawa decomposition of [18,19]. This, in turn, is used to analyze various projections of elements in U modulo B_+U and UB_+.

Let T' be the subgroup of T generated by $\{t_i \mid \alpha_i \in \pi^*\}$. Note that $T_\Theta \times T' = T$ and so the multiplication map defines a vector space isomorphism

$$U^0 \cong \mathcal{C}[T_\Theta] \otimes \mathcal{C}[T']. \tag{2.1}$$

Let G^- be the subalgebra of U generated by $y_i t_i, 1 \leqslant i \leqslant n$. Set $\mathcal{M}^- = \mathcal{M} \cap G^-$ and $\mathcal{M}^+ = \mathcal{M} \cap U^+$. Let N^- be the subalgebra of G^- generated by the $(\mathrm{ad}\,\mathcal{M}^-)$ module $(\mathrm{ad}\,\mathcal{M}^-)\mathcal{C}[y_i t_i \mid \alpha_i \notin \pi_\Theta]$. Similarly, let N^+ be the subalgebra of U^+ generated by the

$(\text{ad}\,\mathcal{M}^+)$ module $(\text{ad}\,\mathcal{M}^+)\mathcal{C}[x_i \mid \alpha_i \notin \pi_\Theta]$. Note that both N^- and N^+ can be written as a direct sum of weight spaces.

Given a subset S of U and a weight $\beta \in Q(\pi)$, we write $S_{\beta,r}$ for the restricted weight space of S corresponding to β. In particular,

$$S_{\beta,r} = \sum_{\{\beta' \mid \tilde{\beta}' = \tilde{\beta}\}} S_{\beta'}.$$

Recall the standard partial ordering on $\mathfrak{h}^*$: For all distinct pairs of elements α and β in $\mathfrak{h}^*$, $\alpha \leqslant \beta$ provided that $\beta - \alpha \in Q^+(\pi)$. Now suppose α and β are in $\mathfrak{h}^*$ and $\tilde{\alpha} < \tilde{\beta}$. It follows that $\tilde{\beta} - \tilde{\alpha}$ is an element of $Q^+(\pi) \cap (\sum_{\alpha \in \pi^*} \mathbf{C}\tilde{\alpha})$. This latter set is contained in $Q^+(\Sigma)$. In particular, the partial ordering on $\mathfrak{h}^*$ restricts to the standard partial ordering on the restricted weights.

Set $T'_{\geqslant}$ equal to the multiplicative monoid generated by the t_i^2 for $\alpha_i \in \pi^*$. Note that $\mathcal{C}[T'_{\geqslant}]$ is just the polynomial ring $\mathcal{C}[t_i^2 \mid \alpha_i \in \pi^*]$.

Lemma 2.1. *For each $B \in \mathcal{B}$, all $\beta, \gamma \in Q^+(\pi)$, and $Y \in U^+_\gamma G^-_{-\beta}$, we have*

$$Y \in N^+_{\beta+\gamma,r} B + \sum_{\tilde{\beta}' < \tilde{\beta} + \tilde{\gamma}} N^+_{\beta',r} T'_{\geqslant} B \tag{2.2}$$

and

$$Y \in B N^-_{-\beta-\gamma,r} + \sum_{\tilde{\beta}' < \tilde{\beta} + \tilde{\gamma}} B T'_{\geqslant} N^-_{-\beta',r}. \tag{2.3}$$

Proof. Let $B \in \mathcal{B}$. Note that any Hopf algebra automorphism which fixes T restricts to the identity on $T'_{\geqslant}$ and an automorphism of N^- and N^+. Hence, without loss of generality, we may assume that $B = B_\theta$.

Choose $\beta \in Q^+(\pi)$. By construction, N^+ is an ad $\mathcal{M}^+$ module. Note that

$$(\text{ad}\,x_i)a = x_i a - t_i a t_i^{-1} x_i \tag{2.4}$$

for all i and for all $a \in U$. Hence

$$\mathcal{M}^+ N^+_\beta \subset N^+_{\beta,r} \mathcal{M}^+.$$

Now if $\alpha_i \notin \pi_\Theta$, it follows that $x_i \in N^+$. Hence

$$U^+_\gamma N^+_\beta \subset N^+_{\beta+\gamma,r} \mathcal{M}^+$$

for all $\gamma \in Q^+(\pi)$. Thus it is sufficient to prove (2.2) for $Y \in G^-_{-\beta}$.

It follows from the defining relations of U that

$$G^-_{-\beta} x_i \subset x_i G^-_{-\beta} + G^-_{-\beta+\alpha_i} + G^-_{-\beta+\alpha_i} t_i^2$$

for each $1 \leqslant i \leqslant n$. If $\alpha_i \in \pi_\Theta$ then $t_i^2 \in T_\Theta$. If $\alpha_i \in \pi^*$ then $t_i^2 \in T'_{\geqslant}$. Finally, if $\alpha_i \notin \pi_\Theta \cup \pi^*$, then $t_i^2 = t^2_{\mathrm{p}(i)}(t_i^2 t^{-2}_{\mathrm{p}(i)})$. In particular, t_i^2 is an element of $T'_{\geqslant} T_\Theta$ for each i, $1 \leqslant i \leqslant n$.

Hence

$$G^{-}_{-\beta}x_i \subset x_i G^{-}_{-\beta} + G^{-}_{-\beta+\alpha_i} T'_{\geqslant} T_{\Theta} \tag{2.5}$$

for all $\alpha_i \in \pi$.

Now consider a weight vector Y of weight $-\beta$ in G^-. Without loss of generality, we may assume that Y is a monomial in the $y_i t_i$, say $y_{i_1} t_{i_1} \cdots y_{i_m} t_{i_m}$. Note that if the restricted weight of Y is zero, or if $m = 0$, then Y is an element of $\mathcal{M}^-$, and hence of B. Thus (2.2) holds in these cases. We proceed by induction on both m and the restricted weight of Y. In particular, we assume that (2.2) holds for all monomials in the $y_i t_i$ of length strictly smaller than m as well as for all elements in $U^+_{\gamma} G^-_{-\lambda}$ with $\gamma \in Q^+(\pi)$ and $\tilde{\lambda} < \tilde{\beta}$.

Note that if $\alpha_{i_m} \in \pi_{\Theta}$ then $y_{i_m} t_{i_m} \in B_+$. By the inductive hypothesis (2.2) holds for $y_{i_1} t_{i_1} \cdots y_{i_{m-1}} t_{i_{m-1}}$. It follows that (2.2) holds whenever $\alpha_{i_m} \in \pi_{\Theta}$. Thus, we may assume that $\alpha_{i_m} \notin \pi_{\Theta}$.

Recall the definition of B_i (1.2) and note that $B_i \in B_+$ (see for example [21, (2.1) and (2.2)]). Hence

$$y_{i_1} t_{i_1} \cdots y_{i_m} t_{i_m} + y_{i_1} t_{i_1} \cdots y_{i_{m-1}} t_{i_{m-1}} \tilde{\theta}(y_{i_m}) t_{i_m} \in y_{i_1} t_{i_1} \cdots y_{i_{m-1}} t_{i_{m-1}} B_+.$$

Applying the inductive hypothesis again to $y_{i_1} t_{i_1} \cdots y_{i_{m-1}} t_{i_{m-1}}$ shows that Y is an element of

$$-y_{i_1} t_{i_1} \cdots y_{i_{m-1}} t_{i_{m-1}} \tilde{\theta}(y_{i_m}) t_{i_m} + \sum_{\tilde{\gamma} \leqslant \tilde{\beta} - \tilde{\alpha}_{i_m}} N^+_{\gamma,r} T'_{\geqslant} B.$$

By [21, (2.1) and (2.2)], $\tilde{\theta}(y_{i_m}) t_{i_m} \in \mathcal{M}^+ x_{\mathrm{p}(i_m)} \mathcal{M}^+ T_{\Theta}$. Note that $\tilde{\beta} = 0$ for all $\beta \in Q(\pi_{\Theta})$. Thus by (2.5), Y is contained in

$$\mathcal{M}^+ x_{\mathrm{p}(i_m)} \mathcal{M}^+ T_{\Theta} G^-_{-\beta+\alpha_{i_m},r} + \mathcal{M}^+ T'_{\geqslant} T_{\Theta} G^-_{-\beta+2\alpha_{i_m},r} + \sum_{\tilde{\gamma} \leqslant \tilde{\beta} - \tilde{\alpha}_{i_m}} N^+_{\gamma,r} T'_{\geqslant} B.$$

Note that both $\tilde{\beta} - \tilde{\alpha}_{i_m}$ and $\tilde{\beta} - 2\tilde{\alpha}_{i_m}$ are strictly smaller than $\tilde{\beta}$. The result (2.2) now follows by induction on $\tilde{\beta}$. It follows from [21, Theorem 3.1] (see also [21, (3.3) and the proof of Theorem 3.4]) that B contains elements $x_i + Y_i$, for $\alpha_i \notin \pi_{\Theta}$, where $Y_i \in G^- T_{\Theta}$ is a weight vector of weight $\Theta(\alpha_i)$. The proof of (2.3) is similar to that of (2.2) using the elements $x_i + Y_i$ instead of B_i. □

Using the above lemma, we obtain four tensor product decompositions of U.

Theorem 2.2. *For all $B \in \mathcal{B}$, there are isomorphisms of vector spaces via the multiplication map*

(i) $N^+ \otimes \mathcal{C}[T'] \otimes B \cong U$,
(ii) $B \otimes \mathcal{C}[T'] \otimes N^+ \cong U$,
(iii) $N^- \otimes \mathcal{C}[T'] \otimes B \cong U$,
(iv) $B \otimes \mathcal{C}[T'] \otimes N^- \cong U$.

Proof. We prove the theorem for $B = B_\theta$. The general result follows from the fact that N^+, N^-, $\mathscr{C}[T']$ and U are all preserved by Hopf algebra automorphisms in $\mathbf{H}$.

Recall that U admits a triangular decomposition [27] or equivalently, an isomorphism of vector spaces via the multiplication map:

$$U \cong G^- \otimes U^0 \otimes U^+. \tag{2.6}$$

By [16], (see also [20, Section 6 and (6.2)]),

$$U^+ \cong \mathscr{M}^+ \otimes N^+ \tag{2.7}$$

as vector spaces using the multiplication map. Combining (2.6) and (2.7) with (2.1) yields the following vector space isomorphism

$$U \cong G^- \otimes \mathscr{M}^+ \otimes \mathscr{C}[T_\Theta] \otimes \mathscr{C}[T'] \otimes N^+ \tag{2.8}$$

induced by the multiplication map.

Set $B_i = y_i t_i$ for $\alpha_i \in \pi_\Theta$. Give an m-tuple $J = (j_1, \dots, j_m)$, set $y_J = y_{j_1} t_{j_1} \cdots y_{j_m} t_{j_m}$ and $B_J = B_{j_1} \cdots B_{j_m}$. Let $\mathscr{J}$ be a set of m-tuples, where m varies, such that the set $\{y_J \mid J \in \mathscr{J}\}$ is a basis for G^-. By the proof of [20, Theorem 7.4], we have

$$B = \bigoplus_{J \in \mathscr{J}} (B_J \mathscr{M}^+ T_\Theta). \tag{2.9}$$

Note that when B_J is written as a direct sum of weight vectors, the lowest weight term is just y_J. Hence (2.9) ensures the lowest weight term of an element of B is contained in $G^- \mathscr{M}^+ T_\Theta$. It follows from (2.8) that $Bv \cap Bv' = 0$ for any two linearly independent elements of $T'N^+$. This forces the map induced by multiplication from $B \otimes \mathscr{C}[T'] \otimes N^+$ to U to be injective.

Let $\imath$ denote the $\mathbf{C}$ algebra antiautomorphism of U defined by $\imath(x_i) = x_i$, $\imath(y_i) = y_i$, $\imath(t_i) = t_i$, and $\imath(q) = q^{-1}$. We use Δ to denote the coproduct of U. It is straightforward to check using the Hopf algebra relations of U that $(\imath \otimes \imath) \circ \Delta(a) = \Delta(\imath(a))$ for all $a \in U$. Hence $\imath(B)$ is also a left coideal subalgebra of U. We recall briefly the notion of specialization at $q = 1$ (see [20, Section 1]). Let $\hat{U}$ denote the $\mathbf{C}[q, q^{-1}]_{(q-1)}$ subalgebra of U generated by x_i, y_i, $t_i^{\pm 1}$, and $(t_i - 1)/(q - 1)$ for $1 \leqslant i \leqslant n$. Recall that $\hat{U}/(q-1)\hat{U}$ is isomorphic to $U(\mathfrak{g})$. Note that $\imath(a) = a + (q-1)\hat{U}$ and hence the images of $\imath(a)$ and a are equal in $\hat{U}/(q-1)\hat{U}$ for all $a \in \hat{U}$. It follows that both B and $\imath(B)$ specializes at $q = 1$ to the same subalgebra, $U(\mathfrak{g}^\theta)$, of $U(\mathfrak{g})$. Now the algebra $\imath(B)$ cannot be an analog of $U(\mathfrak{g})$ of Variation 1 [20, Section 7] since Σ is reduced. A check of the generators of $\imath(B)$ shows that $\imath(B)$ cannot be an analog of $U(\mathfrak{g})$ of Variation 2 [20, Section 7]. Hence, by [20, Theorem 7.5], $\imath(B) \in \mathscr{B}$. In particular, there exists a Hopf algebra automorphism $\psi \in \mathbf{H}$ such that $\psi\imath(B) = B$. Set $\imath' = \psi\imath$. Now $\imath$, and hence $\imath'$, restricts to an antiautomorphism of $\mathscr{C}[T']$. Furthermore, a straightforward computation yields that $\imath'((\operatorname{ad} x_i)x_j)$ is a scalar multiple of $(\mathrm{ad}\,x_i)x_j$ for all i and j. It follows that $\iota'(N^+) = N^+$. Similarly $\iota'(N^-) = N^-$. Hence applying ι' to U and using the previous paragraph, we obtain that multiplication induces an injection from $N^+ \otimes \mathscr{C}[T'] \otimes B$ to U.

Using the triangular decomposition (2.6) and the relations of U, we have that $U = U^0 U^+ G^-$. By assertion (2.2) of Lemma 2.1, any element of U^+G^- is contained in $N^+T'B$. This fact combined with (2.1) yields

$$U = U^0 U^+ G^- \subseteq N^+ T' B \subseteq U.$$

Hence $U = N^+T'B$. Furthermore, $U = \iota'(U) = \iota'(N^+T'B) = BT'N^+$. Thus the multiplication map induces a surjection from $B \otimes \mathscr{C}[T'] \otimes N^+$ onto U and a surjection $N^+ \otimes \mathscr{C}[T'] \otimes B$ onto U which proves (i) and (ii).

By [21, Theorem 3.1], there is a $\mathscr{C}$ algebra anti-involution κ of U which fixes elements of T, sends each x_i to $c_i y_i t_i$ and y_i to $c_i^{-1} t_i^{-1} x_i$ for some nonzero scalar c_i, and restricts to a $\mathscr{C}$ algebra antiautomorphism of B. It follows that $\kappa((\mathrm{ad}\,y_i)b) = -c_i^{-1}(\mathrm{ad}\,x_i)\kappa(b)$ for all $b \in U$. In particular, $\kappa(N^+) = N^-$. Thus assertion (iii) follows from applying κ to (ii). Similarly, assertion (iv) follows from applying κ to (i). □

Let $\mathscr{A}$ denote the subgroup of T generated by $\tau(2\tilde{\alpha})$ as α ranges over π^*. Alternatively, we can view $\mathscr{A}$ as the image under τ of the group $Q(2\Sigma)$. Let $\mathscr{A}_\geqslant$ be the semigroup generated by the $\tau(2\tilde{\alpha}_i)$ for $\alpha_i \in \pi^*$. Note that for $\alpha_i \in \pi^*$,

$$t_i^2 = \tau(2\tilde{\alpha}_i)\tau(\alpha_i + \Theta(\alpha_i)) \in \mathscr{A}_\geqslant T_\Theta.$$

Hence

$$\mathscr{C}[T'_\geqslant] \subseteq \mathscr{C}[\mathscr{A}_\geqslant]\mathscr{C}[T_\Theta] = \mathscr{C}[\mathscr{A}_\geqslant] + \mathscr{C}[\mathscr{A}_\geqslant]\mathscr{C}[T_\Theta]_+. \tag{2.10}$$

The following direct sum decompositions of vector spaces follow immediately from Theorem 2.2:

$$U = (UB_+ + U\mathscr{C}[T']_+) \oplus N^+ \tag{2.11}$$

and

$$U = (B_+U + \mathscr{C}[T']_+U) \oplus N^- \tag{2.12}$$

for all B in $\mathscr{B}$. Given $B \in \mathscr{B}$, let P_B be the projection of U onto N^+ using (2.11) and R_B be the projection of U onto N^- using (2.12). In the next theorem, the projections P_B and R_B are used to construct particular linear isomorphisms between the restricted weight spaces of N^+ and N^-.

Theorem 2.3. *For each $B \in \mathscr{B}$ and each $\tilde{\beta}$, with $\beta \in Q^+(\pi)$, there exist linear isomorphisms*

$$P_{\tilde{\beta},B} : N^-_{-\beta,r} \to N^+_{\beta,r}$$

and

$$R_{\tilde{\beta},B} : N^+_{\beta,r} \to N^-_{-\beta,r}$$

such that

$$Y - P_{\tilde{\beta},B}(Y) \in N^+_{\beta,r}B_+ + \sum_{\tilde{\gamma}<\tilde{\beta}} N^+_{\gamma,r}\mathscr{A}_{\geqslant}B \tag{2.13}$$

and

$$X - R_{\tilde{\beta},B}(X) \in B_+N^-_{-\beta,r} + \sum_{\tilde{\gamma}<\tilde{\beta}} B\mathscr{A}_{\geqslant}N^-_{-\gamma,r}$$

for all $Y \in N^-_{-\beta,r}$ *and* $X \in N^+_{\beta,r}$.

Proof. Let $B \in \mathscr{B}$. Fix $\beta \in Q^+(\pi)$. By Lemma 2.1,

$$P_B\left(\sum_{\tilde{\gamma}<\tilde{\beta}} N^-_{-\gamma,r}\right) \subseteq \sum_{\tilde{\gamma}<\tilde{\beta}} N^+_{\gamma,r}.$$

Theorem 2.2(iii) ensures that $N^- \cap UB_+ = 0$. Thus P_B is injective. Let $\mathfrak{n}^-_{\Theta}$ be the Lie algebra generated by the root vectors in $\mathfrak{g}$ corresponding to the set $\{-\gamma \mid \gamma \in \Delta^+ \backslash Q(\pi_\Theta)\}$. By [20, Section 6], we have the equality of formal characters: ch N^- = ch $U(\mathfrak{n}^-_\Theta)$. Similarly, ch $N^+ = \mathrm{ch}\, U(\mathfrak{n}^+_\Theta)$ where $\mathfrak{n}^+_\Theta$ is the Lie algebra generated by the root vectors in $\mathfrak{g}$ corresponding to the set $\{\gamma \mid \gamma \in \Delta^+ \backslash Q(\pi_\Theta)\}$. Hence both $\sum_{\tilde{\gamma}<\tilde{\beta}} N^-_{-\gamma,r}$ and $\sum_{\tilde{\gamma}<\tilde{\beta}} N^+_{\gamma,r}$ are finite-dimensional and have the same dimension. It follows that P_B restricted to the former subspace is a bijection. Thus

$$P_B\left(\sum_{\tilde{\gamma}<\tilde{\beta}} N^-_{-\gamma,r}\right) = \sum_{\tilde{\gamma}<\tilde{\beta}} N^+_{\gamma,r}. \tag{2.14}$$

Note that there is a projection of N^+ onto $N^+_{\beta,r}$ with respect to the direct sum decomposition of N^+ into restricted weight spaces. Set $P_{\tilde{\beta},B}$ equal to the composition of P_B with this projection. By (2.14), $P_{\tilde{\beta},B}$ is an isomorphism of $N^-_{-\tilde{\beta}}$ onto $N^+_{\tilde{\beta}}$. Furthermore, Lemma 2.1 ensures that $Y - P_{\tilde{\beta},B}(Y) \in N^+_{\beta,r}B_+ + \sum_{\tilde{\gamma}<\tilde{\beta}} N^+_{\gamma,r}\mathscr{A}_{\geqslant}B$. Thus (2.13) follows. A similar argument constructs $R_{\tilde{\beta},B}$. $\square$

3. Action of the center on spherical functions

Set $Q_\Sigma = Q(\Sigma)$. Note that Q_Σ is a subset of $P(\Sigma)$ and hence $\mathscr{C}[Q_\Sigma]$ is a subring of $\mathscr{C}[P(\Sigma)]$. Thus $\mathscr{C}[P(\Sigma)]$ is a right $\mathscr{C}[Q_\Sigma]$ module where elements of $\mathscr{C}[Q_\Sigma]$ act as right multiplication. It follows that we may embed $\mathscr{C}[Q_\Sigma]$ into the (right) endomorphism ring $\mathrm{End}_r\ \mathscr{C}[P(\Sigma)]$ of $\mathscr{C}[P(\Sigma)]$.

Note that $\mathscr{C}[P(\Sigma)]$ is also a right $\mathscr{C}[\mathscr{A}]$ module where

$$z^\lambda * \tau(\mu) = q^{(\lambda,\mu)} z^\lambda$$

for all $z^\lambda \in \mathscr{C}[P(\Sigma)]$ and $\tau(\mu) \in \mathscr{A}$. Since the Cartan inner product restricts to a nondegenerate bilinear form on $P(\Sigma) \times Q_\Sigma$, it follows that the action of $\mathscr{C}[\mathscr{A}]$ on $\mathscr{C}[P(\Sigma)]$ is faithful. Hence $\mathscr{C}[\mathscr{A}]$ also embeds in $\mathrm{End}_r\ \mathscr{C}[P(\Sigma)]$. Let $\mathscr{C}[Q_\Sigma]\mathscr{A}$ denote the subring of $\mathrm{End}_r\ \mathscr{C}[P(\Sigma)]$ generated by $\mathscr{C}[Q_\Sigma]$ and $\mathscr{A}$. Note that

$$z^\lambda \tau(\mu) = q^{(\lambda,\mu)} \tau(\mu) z^\lambda \tag{3.1}$$

for all $\lambda \in Q_\Sigma$ and $\tau(\mu) \in \mathscr{A}$. Furthermore (3.1) implies that the nonzero elements of $\mathscr{C}[Q_\Sigma]$ form an Ore set in $\mathscr{C}[Q_\Sigma]\mathscr{A}$. Write $\mathscr{C}(Q_\Sigma)$ for the quotient ring of $\mathscr{C}[Q_\Sigma]$ and set $\mathscr{C}(Q_\Sigma)\mathscr{A}$ equal to the localization of $\mathscr{C}[Q_\Sigma]\mathscr{A}$ at the Ore set $\mathscr{C}[Q_\Sigma]\backslash\{0\}$. In this section, we obtain a homomorphism of the center $Z(U)$ of U into $\mathscr{C}(Q_\Sigma)[\mathscr{A}]$ which corresponds to the action of $Z(U)$ on the zonal spherical functions.

We give $\mathscr{C}[\mathscr{A}]$ the structure of a left $\mathscr{C}[Q_\Sigma]\mathscr{A}$ module as follows. Elements of $\mathscr{A}$ act by left multiplication while

$$z^\lambda \cdot \tau(\mu) = q^{(\lambda,\mu)} \tau(\mu)$$

for each $\lambda \in Q_\Sigma$ and $\tau(\mu) \in \mathscr{A}$. In particular, we may also view $\mathscr{C}[Q_\Sigma]\mathscr{A}$ as the subring of the (left) endomorphism ring $\mathrm{End}_l\ \mathscr{C}[\mathscr{A}]$ of $\mathscr{C}[\mathscr{A}]$ generated by $\mathscr{C}[Q_\Sigma]$ and $\mathscr{A}$.

The action of $\mathscr{C}[Q_\Sigma]\mathscr{A}$ on $\mathscr{C}[\mathscr{A}]$ can be extended to an action of elements in $\mathscr{C}(Q_\Sigma)\mathscr{A}$ on certain elements of $\mathscr{A}$ as long as we avoid denominator problems. In particular, consider $f \in \mathscr{C}[Q_\Sigma]\mathscr{A}$ and $g \in \mathscr{C}[Q_\Sigma]$. Suppose that $\tau(\mu) \in \mathscr{A}$ such that $g \cdot \tau(\mu) \neq 0$. Note that $(g \cdot \tau(\mu))\tau(\mu)^{-1}$ is just an element of $\mathscr{C}$. We denote $(f \cdot \tau(\mu))(g \cdot \tau(\mu)\tau(\mu)^{-1})^{-1}$ by $(fg^{-1}) \cdot \tau(\mu)$.

Note that the algebra $\mathscr{C}[P(2\Sigma)]$ can be identified with a subspace of the dual of $\mathscr{C}[\mathscr{A}]$ where

$$z^\lambda(\tau(\mu)) = q^{(\lambda,\mu)}$$

for all $z^\lambda \in \mathscr{C}[P(2\Sigma)]$ and $\tau(\mu) \in \mathscr{C}[\mathscr{A}]$. Moreover, the above two actions $\mathscr{C}[Q_\Sigma]\mathscr{A}$ are compatible with the pairing between $\mathscr{C}[P(2\Sigma)]$ and $\mathscr{C}[\mathscr{A}]$. In particular, given $a' \in \mathscr{C}[P(2\Sigma)], a \in \mathscr{C}[\mathscr{A}]$, and $b \in \mathscr{C}(Q_\Sigma)\mathscr{A}$, we obtain

$$d'(b \cdot a) = (a' * b)(a).$$

Recall that $B'_\theta \in \mathscr{B}$ has been chosen so that the image of ${}_{B'_\theta}\mathscr{H}_{B_\theta}$ in $\mathscr{C}[P(2\Sigma)]$ is W_Θ invariant (see the end of Section 1). For the remainder of the paper, we will drop the θ subscript, setting $B = B_\theta$ and $B' = B'_\theta$.

Let $\mathscr{C}(Q_\Sigma)\mathscr{A}_\geqslant$ denote the subalgebra of $\mathscr{C}(Q_\Sigma)\mathscr{A}$ generated by $\mathscr{C}(Q_\Sigma)$ and $\mathscr{A}_\geqslant$. Set $U_\geqslant = U^+ G^- \mathscr{A}_\geqslant$.

Lemma 3.1. *For each* $\tau(\gamma)\in\mathcal{A}$ *and* $X\in U^+G^-\tau(\gamma)$, *there exists* $p_X\in\mathcal{C}(Q_\Sigma)\mathcal{A}_{\geqslant}\tau(\gamma)$ *such that*

$$X\tau(\lambda)-(p_X\cdot\tau(\lambda))\in B_+(U_{\geqslant}\tau(\gamma+\lambda))+(U_{\geqslant}\tau(\gamma+\lambda))B'_+ \tag{3.2}$$

for all $\tau(\lambda)\in\mathcal{A}$ *such that* $p_X\cdot\tau(\lambda)$ *is defined.*

Proof. Given $\beta\in Q^+(\pi)$, set $P_{\tilde{\beta},B'}=P_{\tilde{\beta}}$ and $R_{\tilde{\beta},B}=R_{\tilde{\beta}}$. By Lemma 2.1 and (2.10), we may reduce to the case when $X\in N^+\mathcal{A}_{\geqslant}\tau(\gamma)$. Note that if X and X' both satisfy (3.2), then so does $X+X'$. Hence we may assume that there exists a $\beta\in Q^+(\pi)$ such that $X\in N^+_{\beta,r}\tau(\gamma')$ for some $\tau(\gamma')\in\mathcal{A}_{\geqslant}\tau(\gamma)$.

Given $\alpha\in Q^+(\pi)$, set $\mathrm{ht}_r(\alpha)=\sum_{\alpha_i\in\pi^*}m^i_\alpha$ where $\tilde{\alpha}=\sum_{\alpha_i\in\pi^*}m^i_\alpha\tilde{\alpha}_i$. We prove the lemma by induction on $\mathrm{ht}_r(\beta)$. In particular, assume first that $\mathrm{ht}_r(\beta)=0$. Hence $\tilde{\beta}=0$. Since $N^+T\cap\mathcal{M}T=T$, it further follows that $X\in T$ and (3.2) holds with $p_X=1$. Now assume that $\mathrm{ht}_r(\tilde{\beta})>0$ and (3.2) is true for all elements in $N^+_{\gamma,r}T$ with $\mathrm{ht}_r(\gamma)<\mathrm{ht}_r(\beta)$.

By Theorem 2.3, $P_{\tilde{\beta}}\circ R_{\tilde{\beta}}$ is an isomorphism of $N^+_{\beta,r}$ onto itself. Let X_i, $1\leqslant i\leqslant m$, be a basis for $N^+_{\beta,r}$ considered as a vector space over $\mathcal{C}$ so that $P_{\tilde{\beta}}\circ R_{\tilde{\beta}}$ is an upper triangular $m\times m$ matrix. The fact that $P_{\tilde{\beta}}\circ R_{\tilde{\beta}}$ is an isomorphism ensures that the diagonal entries of this matrix, say c_{ii}, are nonzero.

By Theorem 2.3, $X_i\tau(\gamma')$ is an element of

$$R_{\tilde{\beta}}(X_i)\tau(\gamma')+\sum_{\tilde{\xi}<\tilde{\beta}}N^-_{-\xi,r}\mathcal{A}_{\geqslant}\tau(\gamma)+B_+U_{\geqslant}\tau(\gamma).$$

Note that $\tilde{\xi}<\tilde{\beta}$ implies that $\mathrm{ht}_r(\xi')<\mathrm{ht}_r(\beta)$ for all $\xi'\in Q^+(\pi)$ satisfying $\tilde{\xi}'=\tilde{\xi}$. Thus by the inductive hypothesis, there exists $p_1\in\mathcal{C}(Q_\Sigma)\mathcal{A}_{\geqslant}\tau(\gamma)$ such that $X_i\tau(\gamma')\tau(\lambda)$ is an element of

$$R_{\tilde{\beta}}(X_i)\tau(\gamma')\tau(\lambda)+p_1\cdot\tau(\lambda)+B_+U_{\geqslant}\tau(\gamma+\lambda)+U_{\geqslant}\tau(\gamma+\lambda)B'_+ \tag{3.3}$$

for all λ such that $p_1\cdot\tau(\lambda)$ is defined. Since $R_{\tilde{\beta}}(X_i)\in N^-_{-\beta,r}$, we further have that

$$\tau(\gamma')R_{\tilde{\beta}}(X_i)\in\tau(\gamma')P_{\tilde{\beta}}(R_{\tilde{\beta}}(X_i))+\sum_{\tilde{\xi}<\tilde{\beta}}\mathcal{A}_{\geqslant}N^+_{\xi,r}\tau(\gamma)+U_{\geqslant}\tau(\gamma)B'_+.$$

Applying induction to elements of $N^+_{\xi,r}$, we can find $p_2\in\mathcal{C}(Q_\Sigma)\mathcal{A}_{\geqslant}\tau(\gamma)$ such that

$$\begin{aligned}&\tau(\lambda)\tau(\gamma')R_{\tilde{\beta}}(X_i)-\tau(\lambda)\tau(\gamma')P_{\tilde{\beta}}(R_{\tilde{\beta}}(X_i))-p_2\cdot\tau(\lambda)\\&\quad\in B_+U_{\geqslant}\tau(\gamma+\lambda)+U_{\geqslant}\tau(\gamma+\lambda)B'_+\end{aligned} \tag{3.4}$$

for all λ such that $p_2 \cdot \tau(\lambda)$ is defined. Set $p_2' = q^{(\gamma',\tilde{\beta})} z^{\tilde{\beta}} p_2$. Combining (3.3) and (3.4) yields

$$X_i\tau(\gamma')\tau(\lambda) - q^{(2\gamma'+2\lambda,\tilde{\beta})} P_{\tilde{\beta}} \circ R_{\tilde{\beta}}(X_i)\tau(\lambda)\tau(\gamma') - (p_1 + p_2') \cdot \tau(\lambda)$$
$$\in B_+ U_{\geqslant}\tau(\gamma+\lambda) + U_{\geqslant}\tau(\gamma+\lambda)B'_+$$

for all λ such that $(p_1 + p_2') \cdot \tau(\lambda)$ is defined.

By the choice of the $\{X_i\}$, it follows that

$$P_{\tilde{\beta}}(R_{\tilde{\beta}}(X_i)) \in c_{ii}X_i + \sum_{j<i} \mathscr{C}X_j.$$

Hence

$$X_i\tau(\gamma')(1 - c_{ii}z^{2\tilde{\beta}}q^{(2\gamma',\tilde{\beta})}) \cdot \tau(\lambda) - (p_1 + p_2') \cdot \tau(\lambda)$$
$$\in \sum_{j<i} \mathscr{C}X_j\tau(\gamma'+\lambda) + B_+U_{\geqslant}\tau(\gamma+\lambda) + U_{\geqslant}\tau(\gamma+\lambda)B'_+.$$

By induction on i, there exist s_j in $\mathscr{C}$ and $p_3 \in \mathscr{C}(Q_\Sigma)\mathscr{A}_{\geqslant}\tau(\gamma)$ such that

$$X_i\tau(\gamma') \prod_{1\leqslant j\leqslant i} (1 - s_jz^{2\tilde{\beta}}) \cdot \tau(\lambda) - p_3 \cdot \tau(\lambda) \in B_+U_{\geqslant}\tau(\gamma+\lambda) + U_{\geqslant}\tau(\gamma+\lambda)B'_+$$

for all λ such that $p_3 \cdot \tau(\lambda)$ is defined. The lemma follows by setting $p_{X_i} = p_3 \prod_{1\leqslant j\leqslant i} (1 - s_jz^{2\tilde{\beta}})^{-1}$ and noting that p_X is a linear combination of the p_{X_i}. □

Let $\check{U}$ denote the simply connected quantized enveloping algebra corresponding to $\mathfrak{g}$ [8, Section 3.2.10]. Recall that $\check{U}$ is generated by U and the torus $\check{T} = \{\tau(\lambda) \mid \lambda \in P(\pi)\}$ corresponding to the weight lattice. Set

$$\check{\mathscr{A}} = \{\tau(\tilde{\mu}) \mid \mu \in P(\pi)\}.$$

The ring $\mathscr{C}(Q_\Sigma)\check{\mathscr{A}}$ is defined in an analogous way to $\mathscr{C}(Q_\Sigma)\mathscr{A}$ using the fact that the right action of $\mathscr{A}$ on $\mathscr{C}[P(\Sigma)]$ extends to $\check{\mathscr{A}}$.

Let $\check{U}^0$ denote the group algebra $\mathscr{C}[\check{T}]$ and set $\check{T}_\Theta = \{\tau(\mu) \mid \tau(\mu) \in \check{T} \text{ and } \Theta(\mu) = \mu\}$. The definition of $\check{\mathscr{A}}$ and $\check{T}_\Theta$ yields the following inclusion:

$$\check{U}^0 \subset \mathscr{C}[\check{\mathscr{A}}] \oplus \check{U}^0\mathscr{C}[\check{T}_\Theta]_+. \tag{3.5}$$

Hence

$$\tau(\gamma) = \tau(\tilde{\gamma}) + \tau(\tilde{\gamma})\left(\tau(\frac{1}{2}(\gamma + \Theta(\gamma)) - 1\right) \in \tau(\tilde{\gamma}) + U^0\mathscr{C}[\check{T}_\Theta]_+$$

for all $\tau(\gamma) \in \check{T}$. It follows that Lemma 3.1 extends to elements $X \in U^+G^-\tau(\gamma)$ for any $\tau(\gamma) \in \check{T}$, where $\mathscr{C}(Q_\Sigma)\mathscr{A}_{\geqslant}\tau(\gamma)$ is replaced by $\mathscr{C}(Q_\Sigma)\mathscr{A}_{\geqslant}\tau(\tilde{\gamma})$, B_+ is replaced by $(B\check{T}_\Theta)_+$, and B'_+ is replaced by $(B'\check{T}_\Theta)_+$.

Set $T_{\geqslant}$ equal to the submonoid of T generated by t_i^2, for $i = 1, \dots, n$. Consider X and p_X defined as in the previous lemma. Let g_λ be the zonal spherical function in ${}_{B'}\mathscr{H}_B(\lambda)$ with image $\varphi_\lambda \in \mathscr{C}[P(2\Sigma)]$ where φ_λ is chosen as in the end of Section 1. Assume further that $p_X \cdot \tau(\beta)$ is defined. Then

$$g_\lambda(X\tau(\beta)) = g_\lambda(p_X \cdot \tau(\beta)) = \varphi_\lambda(p_X \cdot \tau(\beta)).$$

Hence $(\varphi_\lambda * p_X)(\tau(\beta)) = g_\lambda(X\tau(\beta))$ for all β such that $p_X \cdot \tau(\beta)$ is defined. We have established the following result.

Theorem 3.2. *There is a linear map* $\mathscr{X} : \check{U} \to \mathscr{C}(Q_\Sigma)\check{\mathscr{A}}$ *such that*

$$g_\lambda(u\tau(\beta)) = (\varphi_\lambda * \mathscr{X}(u))(\tau(\beta))$$

for all $u \in \check{U}$, $\lambda \in P^+(2\Sigma)$ *and* $\tau(\beta) \in \mathscr{A}$ *such that* $\mathscr{X}(u) \cdot \tau(\beta)$ *is defined. Furthermore, if* $u \in U^+G^-T_{\geqslant}\tau(\gamma)$, *then* $\mathscr{X}(u) \in \mathscr{C}(Q_\Sigma)\mathscr{A}_{\geqslant}\tau(\tilde{\gamma})$.

Let $Z(\check{U})$ denote the center of $\check{U}$. The restriction of $\mathscr{X}$ to $Z(\check{U})$ is particularly nice. Recall that $\check{U}$ admits a direct sum decomposition

$$\check{U} = \check{U}^0 \oplus (G_+^- \check{U} + \check{U} U_+^+).$$

Let $\mathscr{P}$ denote the quantum Harish-Chandra projection of $\check{U}$ onto $\check{U}^0$ using this decomposition. A central element c acts on elements of $L(\lambda)$, and hence on the zonal spherical function g_λ, as multiplication by the scalar $z^\lambda(\mathscr{P}(c))$. In particular

$$g_\lambda(c\tau(\gamma)) = z^\lambda(\mathscr{P}(c))(\varphi_\lambda(\tau(\gamma)))$$

for all $c \in Z(\check{U})$ and $\tau(\gamma) \in \mathscr{A}$. It follows that

$$(\varphi_\lambda * \mathscr{X}(c) - z^\lambda(\mathscr{P}(c))\varphi_\lambda) \cdot \tau(\gamma) = 0 \tag{3.6}$$

for all $\tau(\gamma)$ such that $\mathscr{X}(c) \cdot \tau(\gamma)$ is defined. The next result shows that (3.6) holds for all $\tau(\gamma)$.

Corollary 3.3. *The restriction of* $\mathscr{X}$ *to* $Z(\check{U})$ *is an algebra homomorphism from* $Z(\check{U})$ *to* $\mathscr{C}(Q_\Sigma)\check{\mathscr{A}}$ *such that*

$$\varphi_\lambda * \mathscr{X}(c) = z^\lambda(\mathscr{P}(c))\varphi_\lambda \tag{3.7}$$

and thus

$$g_\lambda(c\tau(\beta)) = (\varphi_\lambda * \mathscr{X}(c))(\tau(\beta)) \tag{3.8}$$

for all $c \in Z(\check{U})$, $\lambda \in P^+(2\Sigma)$, *and* $\tau(\beta) \in \mathscr{A}$. *Furthermore, if* $z \in Z(\check{U})$ *and* $z \in U^+G^-T_{\geqslant}\tau(\gamma)$, *then* $\mathscr{X}(z) \in \mathscr{C}(Q_\Sigma)\mathscr{A}_{\geqslant}\tau(\tilde{\gamma})$.

Proof. Let $c \in Z(\check{U})$. Note that we can find a nonzero element $p = \sum p_\beta z^\beta$ in $\mathscr{C}[Q_\Sigma]$ such that $p\mathscr{X}(c)$ is in the subring of $\mathscr{C}(Q_\Sigma)\check{\mathscr{A}}$ generated by $\mathscr{C}[Q_\Sigma]$ and $\check{\mathscr{A}}$. It follows that there exists $f = \sum f_\beta z^\beta$ in $\mathscr{C}[Q_\Sigma]$ such that $p^{-1}f = \varphi_\lambda * \mathscr{X}(c) - z^\lambda(\mathscr{P}(c))\varphi_\lambda$. Hence $\sum f_\beta q^{(\beta,\gamma)} = 0$ for all γ such that $\sum_\beta p_\beta q^{(\beta,\gamma)} \neq 0$.

Assume that $f \neq 0$. Choose γ such that $(\gamma, \beta) \neq 0$ for at least one β with $f_\beta \neq 0$ and $(\gamma, \beta') \neq 0$ for at least one β' with $p_{\beta'} \neq 0$. A standard Vandermonde determinant argument shows that there exists $N \geqslant 0$ such that $\sum_\beta f_\beta q^{(\beta, m\gamma)} \neq 0$ for all $m \geqslant N$. By the previous paragraph, $\sum p_\beta q^{(\beta, m\gamma)} = 0$ for all $m \geqslant N$. Another application of the Vandermonde determinant argument yields $p = 0$, a contradiction. Hence $f = 0$. This proves (3.7) and (3.8) immediately follows. The last assertion of the corollary is a direct consequence of Theorem 3.2. $\square$

The restricted Weyl group W_Θ acts on $\mathscr{C}(Q_\Sigma)\check{\mathscr{A}}$ by

$$s \cdot \tau(\mu) = \tau(s\mu)$$

and

$$s \cdot z^\mu = z^{s\mu}$$

for any $s \in W_\Theta$ and $\tau(\mu) \in \check{\mathscr{A}}$. In the classical case, elements of the center of the classical enveloping algebra of $U(\mathfrak{g})$ can be realized as W_Θ invariant elements of the classical analog of $\mathscr{C}(Q_\Sigma)\check{\mathscr{A}}$ with respect to their action on spherical functions. The next theorem is a quantum version of this result.

The group algebra $\mathscr{C}[Q_\Sigma]$ is just the Laurent polynomial ring corresponding to the polynomial ring $\mathscr{C}[z^{-\tilde{\alpha}_i} \mid \alpha_i \in \pi^*]$. Let $\mathscr{C}((Q_\Sigma))$ denote the formal Laurent series ring $\mathscr{C}((z^{-\tilde{\alpha}_i} \mid \alpha_i \in \pi^*))$. In particular, the ring $\mathscr{C}((Q_\Sigma))$ consists of finite linear combinations of possibly infinite sums of the form $\sum_{\gamma \leqslant \beta} a_\gamma z^\gamma$ where γ and β are elements of Q_Σ and each $a_\gamma \in \mathscr{C}$. Note that $\check{\mathscr{A}}$ embeds inside $\mathrm{End}_r \mathscr{C}((Q_\Sigma))$ where

$$\sum_{\gamma \leqslant \beta} a_\gamma z^\gamma * \tau(v) = \sum_{\gamma \leqslant \beta} a_\gamma q^{(\gamma, v)} z^\gamma \tag{3.9}$$

for all $\sum_{\gamma \leqslant \beta} a_\gamma z^\gamma \in \mathscr{C}((Q_\Sigma))$ and $\tau(v) \in \check{\mathscr{A}}$. Let $\mathscr{C}((Q_\Sigma))\check{\mathscr{A}}$ denote the subring of $\mathrm{End}_r \mathscr{C}((Q_\Sigma))$ generated by $\mathscr{C}((Q_\Sigma))$ and $\check{\mathscr{A}}$. The quotient ring $\mathscr{C}(Q_\Sigma)$ embeds inside of $\mathscr{C}((Q_\Sigma))$ in a standard way. It follows that $\mathscr{C}(Q_\Sigma)\check{\mathscr{A}}$ is a subring of $\mathscr{C}((Q_\Sigma))\check{\mathscr{A}}$.

The relations in (3.9) ensures that the multiplication map yields vector space isomorphisms

$$\mathscr{C}((Q_\Sigma))\check{\mathscr{A}} \cong \mathscr{C}((Q_\Sigma)) \otimes \mathscr{C}[\check{\mathscr{A}}] \cong \mathscr{C}[\check{\mathscr{A}}] \otimes \mathscr{C}((Q_\Sigma)). \tag{3.10}$$

In particular, elements of $\mathscr{C}((Q_\Sigma))\check{\mathscr{A}}$ are finite sums of the form $\sum_i a_i b_i$ where $a_i \in \mathscr{C}[\check{\mathscr{A}}]$ and $b_i \in \mathscr{C}((Q_\Sigma))$. Grouping together the coefficients of each z^β, we can write any element in $\mathscr{C}((Q_\Sigma))\check{\mathscr{A}}$ as a finite linear combination of possibly infinite sums of the form $\sum_{\gamma < \beta} a_\gamma z^\gamma$ where γ and β are in Q_Σ and each $a_\gamma \in \mathscr{C}[\check{\mathscr{A}}]$. However, the reader should be aware that not all such sums are elements of $\mathscr{C}((Q_\Sigma))\check{\mathscr{A}}$.

Let ω_i' be the fundamental weight in $P^+(\Sigma)$ corresponding to the restricted root $\tilde{\alpha}_i$. Since $Q(\Sigma)$ is a subset of $P(\Sigma)$, it follows that $\mathscr{C}((Q_\Sigma))$ is a subring of the Laurent power series ring $\mathscr{C}((z^{-\omega_i'} \mid \alpha_i \in \pi^*))$. Thus we may view $\mathscr{C}((Q_\Sigma))\check{\mathscr{A}}$ as a subring of $\mathrm{End}_r\mathscr{C}((z^{-\omega_i'} \mid \alpha_i \in \pi^*))$. This interpretation will be needed in the proof below where elements of $\mathscr{C}((Q_\Sigma))\check{\mathscr{A}}$ act on zonal spherical functions.

Theorem 3.4. *The image of $Z(\check{U})$ under $\mathscr{X}$ is contained in $(\mathscr{C}(Q_\Sigma)\check{\mathscr{A}})^{W_\Theta}$.*

Proof. Let $a \in Z(\check{U})$ and $s = s_\alpha$ be the reflection in W_Θ corresponding to a simple root $\alpha \in \Sigma$. Assume that $\mathscr{X}(a) \neq s \cdot \mathscr{X}(a)$. We can think of $\mathscr{X}(a) - s \cdot \mathscr{X}(a)$ as an element of $\mathscr{C}((Q_\Sigma))\check{\mathscr{A}}$. It follows from (3.10) that there exists a finite set $\{\gamma_1, \dots, \gamma_r\}$ of noncomparable elements in Q_Σ and elements $a_{\beta_i} \in \mathscr{C}[\check{\mathscr{A}}]$ for $\beta_i \leqslant \gamma_i$ and $1 \leqslant i \leqslant r$ such that

$$\mathscr{X}(a) - s \cdot \mathscr{X}(a) = \sum_i \sum_{\beta_i \leqslant \gamma_i} a_{\beta_i} z^{\beta_i}.$$

We may further assume that $a_{\gamma_i} \neq 0$ for each $1 \leqslant i \leqslant r$.

Recall that the zonal spherical function φ_λ can be written as a sum of the form

$$z^\lambda + \sum_{\beta < \lambda} y_\beta z^\beta,$$

where each $\beta \in Q_\Sigma$ and each $y_\beta \in \mathscr{C}$ (see (1.3)). Hence

$$0 = \varphi_\lambda * (\mathscr{X}(a) - s \cdot \mathscr{X}(a)) \in \sum_i (z^\lambda(a_{\gamma_i}))z^{\lambda+\gamma_i} + \sum_i \sum_{\beta < \lambda + \gamma_i} \mathscr{C} z^\beta.$$

Hence $z^\lambda(a_{\gamma_i}) = 0$ for all $1 \leqslant i \leqslant r$ and all $\lambda \in P^+(2\Sigma)$. This forces each $a_{\gamma_i} = 0$, a contradiction. □

We wish to extend Corollary 3.3 and Theorem 3.4 to B invariant elements of $\check{U}$. In particular, let $\check{U}^B$ denote the subalgebra of $\check{U}$ consisting of $\mathrm{ad}_r B$ invariant elements, where ad_r refers to the right adjoint action. (For more information about the right adjoint action, see [21, (1.1) and (1.2)].) Since B is not a Hopf subalgebra of U, it is not obvious that the centralizer $C_{\check{U}}(B) = \{c \in \check{U} \mid bc = cb \text{ for all } b \in B\}$ of B in $\check{U}$ is equal to the set $\check{U}^B$. Nevertheless, the next lemma shows that this is indeed true.

Lemma 3.5. $\check{U}^B = C_{\check{U}}(B)$. *In particular, $Z(\check{U})$ is a subset of $\check{U}^B$.*

Proof. Note that the second assertion is an immediate consequence of the first. It is straightforward to check that $C_{\check{U}}(B) \subseteq \check{U}^B$. (The argument follows as in [8, Lemma 1.3.3] using the right adjoint action instead of the left adjoint action.)

Recall that $\sum \sigma(a_{(1)})a_{(2)} = \varepsilon(a)$ for all $a \in U$ where ε is the counit for U and the coproduct is given in Sweedler notation, $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$. Recall further that B is a left coideal and so $\Delta(B) \subset U \otimes B$. Suppose that $c \in \check{U}^B$. Then

$$\begin{aligned} ac &= \sum a_{(1)}\varepsilon(a_{(2)})c = \sum a_{(1)}\sigma(a_{(2)})ca_{(3)} \\ &= \sum \varepsilon(a_{(1)})ca_{(2)} = c\sum \varepsilon(a_{(1)})a_{(2)} = ca \end{aligned}$$

for all $a \in B$. Hence $c \in C_{\check{U}}(B)$ and $\check{U}^B \subseteq C_{\check{U}}(B)$. $\square$

Theorem 2.2(ii) and (3.5) imply the following inclusion

$$\check{U} \subseteq ((B\check{T}_\Theta)_+\check{U} + N^+_+\check{\mathscr{A}}) \oplus \mathscr{C}[\check{\mathscr{A}}]. \tag{3.11}$$

Let $\mathscr{P}_{\mathscr{A}}$ denote the projection of $\check{U}$ onto $\mathscr{C}[\check{\mathscr{A}}]$ using this decomposition.

Let $\lambda \in P^+(2\Sigma)$. Recall [21, Theorem 3.2] that $(L(\lambda)^*)^B$ is one dimensional. Choose a nonzero generating vector v^*_λ of weight λ for $L(\lambda)^*$. By [21, Lemma 3.3], we can choose a nonzero vector ξ^*_λ in $(L(\lambda)^*)^B$ such that $\xi^*_\lambda = v^*_\lambda + v^*_\lambda N^+_+$. Suppose that $c \in \check{U}^B$. By the previous lemma, $\xi^*_\lambda cu = \xi^*_\lambda uc = 0$ for all $u \in B_+$. Hence $\xi^*_\lambda c$ is a scalar multiple of ξ^*_λ. It follows from (3.11) and the definition of $\mathscr{P}_{\mathscr{A}}$ that

$$\xi^*_\lambda c \in \xi^*_\lambda(\mathscr{P}_{\mathscr{A}}(c) + N^+_+) \subseteq z^\lambda(\mathscr{P}_{\mathscr{A}}(c))v^*_\lambda + v^*_\lambda N^+_+.$$

Hence $\xi^*_\lambda c = z^\lambda(\mathscr{P}_{\mathscr{A}}(c))\xi^*_\lambda$.

Consider the special case when c is an element in $Z(\check{U})$. Now $\xi^*_\lambda \in v^*_\lambda + v^*_\lambda B_+$ (see the proof of [20, Theorem 7.7]). In particular, we can find a b such that $b - 1 \in B_+$ and $\xi^*_\lambda = v^*_\lambda b$. It follows that $v^*_\lambda bc = v^*_\lambda cb$. Therefore, $\xi^*_\lambda c = z^\lambda(\mathscr{P}(c))\xi^*_\lambda$. By the previous paragraph, we see that $z^\lambda(\mathscr{P}_{\mathscr{A}}(c)) = z^\lambda(\mathscr{P}(c))$ for all $c \in Z(\check{U})$. In particular, $\mathscr{P}_{\mathscr{A}}(c)$ agrees with the image of $\mathscr{P}(c)$ under projection onto $\mathscr{C}[\check{\mathscr{A}}]$ using (3.5). Thus arguing as in Corollary 3.3 and Theorem 3.4, we have the following generalization of Theorem 3.4.

Theorem 3.6. *The restriction of $\mathscr{X}$ to $\check{U}^B$ is an algebra homomorphism from $\check{U}^B$ to $(\mathscr{C}(Q_\Sigma)\check{\mathscr{A}})^{W_\Theta}$ such that*

$$\varphi_\lambda * \mathscr{X}(c) = z^\lambda(\mathscr{P}_{\mathscr{A}}(c))\varphi_\lambda$$

and

$$g_\lambda(c\tau(\beta)) = (\varphi_\lambda * \mathscr{X}(c))(\tau(\beta))$$

for all $c \in \check{U}^B$, $\lambda \in P^+(2\Sigma)$, and $\tau(\beta) \in \mathscr{A}$.

Consider $c \in \check{U}^B$. As in the classical case, we refer to the image $\mathscr{X}(c)$ in $(\mathscr{C}(Q_\Sigma)\check{\mathscr{A}})^{W_\Theta}$ as the *radial component* of c.

4. Central elements: the rank one case

In this section, we study certain central elements of $\check{U}$ and compute their radial components when the restricted root system has rank one. In particular, we assume that π^* contains a single root α_i and so $\Sigma^+ = \{\tilde{\alpha}_i\}$. Assume for the moment that $\pi = \pi^*$ and so $\Theta(\alpha_i) = -\alpha_i$. Then U is just $U_q(\mathfrak{sl}\ 2)$ and is generated by x_i, y_i, t_i, t_i^{-1}. Set $q_i = q^{(\alpha_i,\alpha_i)/2}$. Note that

$$(q_i t_i + q_i^{-1} t_i^{-1}) + (q_i - q_i^{-1})^2 y_i x_i \tag{4.1}$$

is central in U. We show that the other rank one cases contain a similar central element.

Note that $\tilde{\alpha}_i - \widetilde{\Theta(\alpha_i)} = 2\tilde{\alpha}_i$. Since we are assuming that Σ is reduced, it follows that $\alpha_i + \Theta(-\alpha_i)$ is not a root in Δ. Hence $(\alpha_i, \Theta(-\alpha_i)) = 0$. Using Araki's classification of irreducible symmetric pairs, we have the following possibilities for $\Theta(\alpha_i)$.

$$\mathfrak{g} \text{ is of type } A_1 \text{ with } \pi = \{\alpha_i\} \text{ and } \Theta(\alpha_i) = -\alpha_i, \tag{4.2}$$

$$\mathfrak{g} \text{ is of type } A_1 \times A_1 \text{ with } \pi = \{\alpha_i, \alpha_{\mathrm{p}(i)}\} \text{ and } \Theta(\alpha_i) = -\alpha_{\mathrm{p}(i)}, \tag{4.3}$$

$$\mathfrak{g} \text{ is of type } A_3 \text{ with } \pi = \{\alpha_1, \alpha_2, \alpha_3\},\ \alpha_i = \alpha_2, \text{ and} \tag{4.4}$$

$$\Theta(\alpha_2) = -\alpha_1 - \alpha_3 - \alpha_2,$$

$$\mathfrak{g} \text{ is of type } B_r \text{ with } \pi = \{\alpha_1, \dots, \alpha_r\},\ \alpha_i = \alpha_1, \text{ and} \tag{4.5}$$

$$\Theta(\alpha_1) = -\alpha_1 - 2\alpha_2 - \cdots - 2\alpha_r,$$

$$\mathfrak{g} \text{ is of type } D_r \text{ with } \pi = \{\alpha_1, \dots, \alpha_r\},\ \alpha_i = \alpha_1, \text{ and} \tag{4.6}$$

$$\Theta(\alpha_1) = -\alpha_1 - 2\alpha_2 - \cdots - 2\alpha_{r-2} - \alpha_{r-1} - \alpha_r.$$

Recall that the $(\operatorname{ad} U)$ module $(\operatorname{ad} U)\tau(-2\mu)$ for $\mu \in P^+(\pi)$ contains a one-dimensional subspace of $Z(\check{U})$. (See [8, 7.1.16–7.19 and 7.1.25] or [11] for more information about $Z(\check{U})$.) Moreover, there exists a (unique) nonzero vector c_μ in $(\operatorname{ad} U)\tau(-2\mu) \cap Z(\check{U})$ such that

$$c_\mu \in \tau(-2\mu) + (\operatorname{ad} U_+)\tau(-2\mu).$$

We find a "small" weight $\mu \in P^+(\pi)$ such that c_μ looks like (4.1) modulo $(\mathscr{M}\check{T}_\Theta)_+ \check{U} + \check{U}(\mathscr{M}\check{T}_\Theta)_+$. In particular, μ will satisfy the conditions of the following lemma. Let w_0 denote the longest element of the Weyl group W associated to the root system of $\mathfrak{g}$.

Lemma 4.1. *There exists $\mu \in P^+(\pi)$ such that*

$$\Theta(\mu - \alpha_i/2) = \mu - \alpha_i/2 \quad \textit{and} \quad \Theta(-w_0\mu - \alpha_i/2) = -w_0\mu - \alpha_i/2. \tag{4.7}$$

Proof. Suppose first that $\mathfrak{g}$ satisfies the conditions of (4.2) or (4.3) above. Then $\alpha_i/2$ is in $P^+(\pi)$ and $w_0\alpha_i/2 = -\alpha_i/2$. Hence we set $\mu = \alpha_i/2$ in these cases.

Let ω_j be the fundamental weight corresponding to α_j, for each j. The remaining cases are handled below.

Case (4.4): Set $\mu = \omega_1$. We have $\omega_1 = 1/4(\alpha_1 + 2\alpha_2 + 3\alpha_3)$ while $-w_o\omega_1 = \omega_3 = 1/4(3\alpha_1 + 2\alpha_2 + \alpha_3)$. Thus (4.7) follows since $i = 2$.

Case (4.5): Set $\mu = \omega_r$. In this case $\omega_r = -w_0\omega_r = 1/2(r\alpha_r + (r-1)\alpha_{r-1} + \cdots + 3\alpha_3 + 2\alpha_2 + \alpha_1)$. Thus (4.7) follows since $i = 1$.

Case (4.6): In this case, μ can be either ω_r or ω_{r-1}. Note that $\omega_r = 1/2(\frac{r}{2}\alpha_r + \frac{(r-2)}{2}\alpha_{r-1} + (r-2)\alpha_{r-2} + \cdots 2\alpha_2 + \alpha_1)$ and $\omega_{r-1} = 1/2(\frac{r}{2}\alpha_{r-1} + \frac{(r-2)}{2}\alpha_r + (r-2)\alpha_{r-2} + \cdots 2\alpha_2 + \alpha_1)$. Furthermore, $-w_o(\omega_r) = \omega_{r-1}$ and $i = 1$. □

Recall the definition of the Harish-Chandra projection $\mathscr{P}$ given in Section 3. We have the following description of the Harish-Chandra projection of central elements c_μ of $\check{U}$ [8, 7.1.19 and 7.1.25]: up to a nonzero scalar,

$$\mathscr{P}(c_\mu) = \sum_{v \in P^+(\pi)} \hat{\tau}(-2v) \dim L(\mu)_v \tag{4.8}$$

where

$$\hat{\tau}(\beta) = \sum_{w \in W} \tau(w\beta) q^{(\rho, w\beta)}.$$

The next result is the first step in understanding the central element c_μ when μ satisfies the conditions of Lemma 4.1.

Proposition 4.2. *Suppose that $\mu \in P^+(\pi)$ is chosen as in* Lemma 4.1 *to satisfy condition* (4.7). *Then*

$$c_\mu \in y_i \mathscr{M}^- \mathscr{M}^+ x_i + \mathscr{P}(c_\mu) + (\mathscr{M}\check{T}_\Theta)_+ \check{U} + \check{U}(\mathscr{M}\check{T}_\Theta)_+.$$

Moreover, $\mathscr{P}(c_\mu) \in \mathscr{C}[T_\geqslant]\tau(-2\mu)$.

Proof. The last statement is an immediate consequence of (4.8). Note that c_μ is a sum of zero weight vectors in $(\mathrm{ad}\ U)\tau(-2\mu)$. By [10, Theorem 3.3], we can construct a basis for the zero weight space of $(\mathrm{ad}\ U)\tau(-2\mu)$ consisting of vectors in sets of the form

$$a_{-\beta} b_\beta \tau(-2\mu) + \sum_{\gamma \in Q^+(\pi)} \sum_{0 < \gamma' < \beta - \gamma} G^-_{-\gamma'} U^+_{\gamma'} \tau(-2\mu + 2\gamma),$$

where $a_{-\beta}$ is a weight vector of weight $-\beta$ in $(\text{ad } U^-)\tau(-2\mu)$ and b_β is a weight vector of weight β in $(\text{ad } U^+)\tau(-2\mu)$. It further follows from [8, 7.1.20], that the weights β appearing in the above expression satisfy $\beta \leqslant \mu - w_0\mu$. Hence

$$c_\mu \in \sum_{\gamma \in Q^+(\pi_\Theta)} G^-_{-\mu+w_0\mu+\gamma} U^+_{\mu-w_0\mu-\gamma} T_\Theta \tau(-2\mu) + \sum_{\{\beta \mid 0 \leqslant \tilde{\beta} < \tilde{\mu} - \widetilde{w_0\mu}\}} G^-_{-\beta} U^+_\beta T_{\geqslant} \tau(-2\mu). \tag{4.9}$$

By choice of μ, we have that $\mu - w_0\mu - \alpha_i \in Q(\pi_\Theta)$. Hence $0 \leqslant \tilde{\beta} < \tilde{\mu} - \widetilde{w_0\mu}$ forces $\beta \in Q^+(\pi_\Theta)$. Thus (4.9) implies that

$$c_\mu \in y_i t_i \mathscr{M}^- \mathscr{M}^+ x_i T_\Theta \tau(-2\mu) + \mathscr{P}(c_\mu) + \mathscr{M}_+ \check{U} + \check{U} \mathscr{M}_+. \tag{4.10}$$

The assumption on μ further implies that $t_i\tau(-2\mu) = \tau(-2\mu + \alpha_i) \in \check{T}_\Theta$. Hence

$$t_i\tau(-2\mu) = 1 + (t_i\tau(-2\mu) - 1) \in \mathscr{C} + U\mathscr{C}[\check{T}_\Theta]_+.$$

The lemma now follows from this expression and (4.10). $\square$

The next lemma simplifies the component of c_μ coming from $y_i \mathscr{M}^- \mathscr{M}^+ x_i$.

Lemma 4.3. $y_i \mathscr{M}^- \mathscr{M}^+ x_i \subseteq \mathscr{C} y_i x_i + \mathscr{M}_+ U + U \mathscr{M}_+$.

Proof. Consider an element $y_i b c x_i$ of U where $b \in \mathscr{M}^-$ and $c \in \mathscr{M}^+$. Using the relations of U, we can rewrite this element as a sum of terms of the form $y_i c' b' d' x_i$ where $c' \in \mathscr{M}^+$, $b' \in \mathscr{M} \cap U^0$, and $d' \in \mathscr{M}^-$. Since $\alpha_i \notin \pi_\Theta$, we have $y_i c' b' d' x_i = c' y_i b' x_i d'$ up to a nonzero scalar. So if either c' is in $\mathscr{M}^+_+$ or d' is in $\mathscr{M}^-_+$, then $y_i c' b' d' x_i \in \mathscr{M}^+_+ U + U \mathscr{M}^-_+ \subseteq \mathscr{M}_+ U + U \mathscr{M}_+$. If neither of these conditions hold, we may assume that $c' b' d' = b'$ which is an element of $\mathscr{M} \cap U^0$. Then $y_i b' x_i \in \mathscr{C} y_i x_i + y_i x_i (\mathscr{M} \cap U^0)_+ \subseteq \mathscr{C} y_i x_i + \mathscr{M}_+ U + U \mathscr{M}_+$. $\square$

The tensor product decomposition (2.1) implies the following direct sum decomposition:

$$U^0 = \mathscr{C}[T'] \oplus U^0 \mathscr{C}[T_\Theta]_+. \tag{4.11}$$

Recall the direct sum decomposition (3.5) using $\check{\mathscr{A}}$ instead of T'. Let $\mathscr{P}'_{\mathscr{A}}$ denote the map from $\check{U}$ to $\mathscr{C}[\check{\mathscr{A}}]$ defined by composing the Harish-Chandra map $\mathscr{P}$ with the projection of $\check{U}^0$ into $\mathscr{C}[\check{\mathscr{A}}]$ using (3.5). Similarly, let $\mathscr{P}'$ denote the map from U onto $\mathscr{C}[T']$ which is the composition of $\mathscr{P}$ with the projection onto $\mathscr{C}[T']$ using (4.11). Note that

$$\mathscr{P}'_{\mathscr{A}}(a) = \sum_m a_m \tau(\tilde{\alpha}_i)^m \quad \text{if and only if } \mathscr{P}'(a) = \sum_m a_m t_i^m \tag{4.12}$$

for all $a \in U$.

Recall the dotted Weyl group action on $\check{T}$ defined by

$$w.\tau(\mu)q^{(\rho,\mu)} = \tau(w\mu)q^{(\rho,w\mu)} \tag{4.13}$$

for all $\tau(\mu)\in\check{T}$ and $w\in W$. By [8, 7.1.17 and 7.1.25], the image of $Z(\check{U})$ under $\mathscr{P}$ is contained in $\mathscr{C}[\check{T}]^{W\cdot}$. Define the dotted action of W_Θ on $\check{\mathscr{A}}$ using the same formula as in (4.13) where now w is an element of W_Θ and $\tau(\mu)\in\check{\mathscr{A}}$.

Lemma 4.4. *For all $\mu\in P^+(2\Sigma)$, the image of c_μ under $\mathscr{P}'_{\mathscr{A}}$ is invariant under the dotted action of W_Θ. Moreover, if μ satisfies the conditions of* (4.7) *then $\mathscr{P}'(c_\mu)$ is a scalar multiple of $q^{(\rho,\tilde{\alpha}_i)}t_i + q^{-(\rho,\tilde{\alpha}_i)}t_i^{-1}$.*

Proof. Recall that w_0 is the longest element of W and let w_0' denote the longest element of the Weyl group $\langle s_\alpha \,|\, \alpha\in\pi_\Theta\rangle$. Set $w = w_0'w_0$. By [20, Section 7] or checking (4.2)–(4.6) directly, we see that $w_0'\alpha_i = \Theta(-\alpha_i)$ and $w_0'\Theta(\alpha_i) = -\alpha_i$. Further checking the possibilities for w_0 in (4.2)–(4.6) yields that $w\alpha_i = \Theta(\alpha_i)$ and $w\Theta(\alpha_i) = \alpha_i$. Hence $w\tilde{\alpha}_i = -\tilde{\alpha}_i$ and we may identify W_Θ with the subgroup $\langle w\rangle$ of W.

It is straightforward to check using (4.2) through (4.6) that w_0 sends α_i to $-\alpha_i$. Furthermore, w_0 sends a simple root in π_Θ to the negative of a simple root in π_Θ. It follows that w permutes the elements of π_Θ. Thus $q^{(\rho,\gamma)} = q^{(\rho,w\gamma)}$ for all γ such that $\Theta(\gamma) = \gamma$. Hence $w.\tau(\mu) = \tau(w\mu)$ for all μ such that $\Theta(\mu) = \mu$. It follows that $\mathscr{C}[\check{T}_\Theta]_+$ is invariant under the dotted action of W_Θ. Thus

$$\mathscr{P}'_{\mathscr{A}}(\mathscr{C}[\check{T}]^{W\cdot})\subseteq\mathscr{C}[\check{\mathscr{A}}]^{W_\Theta\cdot}$$

which proves the first assertion of the lemma.

Now assume that μ satisfies (4.7). It follows that $\mathscr{P}'_{\mathscr{A}}(\tau(-2\mu)) = \tau(-\tilde{\alpha}_i)$. By Proposition 4.2, $\mathscr{P}(c_\mu)\in\mathscr{C}[T_\geqslant]\tau(-2\mu)$. Hence (2.1) and (2.10) imply that $\mathscr{P}'_{\mathscr{A}}(c_\mu)\in\mathscr{C}[\mathscr{A}_\geqslant]\tau(-\tilde{\alpha}_i)$. The second assertion now follows from (4.12) and the fact that the only elements of $\mathscr{C}[\mathscr{A}_\geqslant]\tau(-\tilde{\alpha}_i)$ invariant under the dotted action of W_Θ are scalar multiples of $q^{(\rho,\tilde{\alpha}_i)}\tau(\tilde{\alpha}_i) + q^{-(\rho,\tilde{\alpha}_i)}\tau(-\tilde{\alpha}_i)$. $\square$

An immediate consequence of (4.8) is that $z^\lambda(\mathscr{P}(c_\mu))\neq 0$, and more importantly, $\mathscr{P}'(c_\mu)\neq 0$ for any choice of λ and μ. This fact is used in the next result. In particular, we show that when μ is chosen as in Lemma 4.1, then c_μ looks like the central element of $U_q(\mathfrak{sl}\,2)$ described in (4.1).

Theorem 4.5. *Assume that μ satisfies the conditions of* (4.7). *Let a be the nonzero scalar guaranteed by Lemma* 4.4 (*and the above comments*) *such that $\mathscr{P}'(c_\mu) = a(q^{(\rho,\tilde{\alpha}_i)}t_i + q^{-(\rho,\tilde{\alpha}_i)}t_i^{-1})$. Then*

$$c_\mu\in a[q^{(\rho,\tilde{\alpha}_i)}t_i + q^{-(\rho,\tilde{\alpha}_i)}t_i^{-1} + (q_i - q_i^{-1})(q^{(\rho,\tilde{\alpha}_i)} - q^{-(\rho,\tilde{\alpha}_i)})y_ix_i]$$
$$+\ (\mathscr{M}\check{T}_\Theta)_+\check{U} + \check{U}(\mathscr{M}\check{T}_\Theta)_+.$$

Proof. By Proposition 4.2 and Lemmas 4.3 and 4.4, there exists $c \in \mathscr{C}$ such that

$$c_\mu - (cy_ix_i + b) \in (\mathscr{M}\check{T}_\Theta)_+ \check{U} + \check{U}(\mathscr{M}\check{T}_\Theta)_+, \tag{4.14}$$

where

$$b = a(q^{(\rho,\tilde{\alpha}_i)}t_i + q^{-(\rho,\tilde{\alpha}_i)}t_i^{-1}). \tag{4.15}$$

Let $\check{U}_{\mathbf{C}(q)}$ denote the $\mathbf{C}(q)$ subalgebra of $\check{U}$ generated by x_i, y_i, for $1 \leqslant i \leqslant n$ and $\check{T}$. Write $U_{\mathbf{C}(q)}$ for $\check{U}_{\mathbf{C}(q)} \cap U$. Note that $U_{\mathbf{C}(q)}$ and $\check{U}_{\mathbf{C}(q)}$ are the versions of the quantized enveloping algebra and the simply quantized enveloping algebra studied in [8,11,12]. Thus c_μ is actually an element of $(\text{ad } U_{\mathbf{C}(q)})\tau(-2\mu)$, and so both a and c are elements of $\mathbf{C}(q)$.

Consider the $\mathbf{C}$ algebra automorphism ψ of $\check{U}_{\mathbf{C}(q)}$ defined by $\psi(x_j) = y_jt_j$, $\psi(y_j) = t_j^{-1}x_j$, $\psi(t) = t$ and $\psi(q) = q^{-1}$ for all $1 \leqslant j \leqslant n$ and $t \in \check{T}$. A straightforward check shows that $\psi((\text{ad } x_i)b) = (\text{ad } y_it_i)\psi(b)$ and $\psi((\text{ad } y_i)b) = (\text{ad } t_i^{-1}x_i)\psi(b)$ for all $b \in \check{U}_{\mathbf{C}(q)}$. In particular, if $x \in (\text{ad } (U_{\mathbf{C}(q)})_+)\tau(\mu)$, then so is $\psi(x)$.

Recall that c_μ has been scaled so that $c_\mu \in \tau(-2\mu) + (\text{ad } U_+)\tau(-2\mu)$. Hence $\psi(c_\mu) \in \tau(-2\mu) + (\text{ad } U_+)\tau(-2\mu)$. Since $(\text{ad } U_+)\check{U} \cap Z(\check{U}) = 0$, it follows that $\psi(c_\mu) = c_\mu$. Therefore, applying ψ to (4.14) using the form of b given in (4.15) yields c_μ is an element of

$$\psi(a)(q^{-(\rho,\tilde{\alpha}_i)}t_i + q^{(\rho,\tilde{\alpha}_i)}t_i^{-1}) + \psi(c)t_i^{-1}x_iy_it_i + (\mathscr{M}\check{T}_\Theta)_+\check{U} + \check{U}(\mathscr{M}\check{T}_\Theta)_+.$$

Thus

$$c_\mu \in \psi(a)(q^{-(\rho,\tilde{\alpha}_i)}t_i + q^{(\rho,\tilde{\alpha}_i)}t_i^{-1}) + \psi(c)(q_i - q_i)^{-1}(t_i - t_i^{-1}) + \psi(c)y_ix_i$$
$$+ (\mathscr{M}\check{T}_\Theta)_+\check{U} + \check{U}(\mathscr{M}\check{T}_\Theta)_+.$$

A comparison with (4.14) yields $\psi(c) = c$. Note that $q^{-(\rho,\tilde{\alpha}_i)}t_i + q^{(\rho,\tilde{\alpha}_i)}t_i^{-1}$ is not invariant with respect to the dotted W_Θ action. Hence c must be nonzero. Moreover, in order for $\psi(a)(q^{-(\rho,\tilde{\alpha}_i)}t_i + q^{(\rho,\tilde{\alpha}_i)}t_i^{-1}) + c(q_i - q_i)^{-1}(t_i - t_i^{-1})$ to be invariant under the dotted W_Θ action, we must have $c = (q_i - q_i^{-1})(q^{(\rho,\tilde{\alpha}_i)} - q^{-(\rho,\tilde{\alpha}_i)})\psi(a)$. The fact that $\psi(c) = c$ and $\psi((q_i - q_i^{-1})(q^{(\rho,\tilde{\alpha}_i)} - q^{-(\rho,\tilde{\alpha}_i)})) = (q_i - q_i^{-1})(q^{(\rho,\tilde{\alpha}_i)} - q^{-(\rho,\tilde{\alpha}_i)})$ now forces $\psi(a) = a$. $\square$

Let χ be the Hopf algebra automorphism (which restricts to the identity on $\mathscr{M}T$) defined by

$$\chi(x_i) = q^{-1/2(\rho,\Theta(\alpha_i)-\alpha_i)}x_i$$

and

$$\chi(y_i) = q^{1/2(\rho,\Theta(\alpha_i)-\alpha_i)}y_i.$$

By [21, Section 5], we may assume that $B' = \chi(B)$.

Note that B_+ contains

$$B_i = y_i t_i + \tilde{\theta}(y_i) t_i$$

and B'_+ contains

$$B'_i = y_i t_i + q^{-(\rho,\Theta(\alpha_i)-\alpha_i)} \tilde{\theta}(y_i) t_i.$$

By [21, Lemma 5.1], we have

$$q^{(\rho,\Theta(\alpha_i)+\alpha_i)} \tilde{\theta}(y_{\mathrm{p}(i)}) t_{\mathrm{p}(i)}^{-1} x_{\mathrm{p}(i)} \in t_i^{-1} x_i \tilde{\theta}(y_i) + \mathcal{M}_+^+ U + U \mathcal{M}_+^+.$$

When $i = \mathrm{p}(i)$, it follows that

$$q^{(\rho,\Theta(\alpha_i)+\alpha_i)} \tilde{\theta}(y_i) t_i^{-1} x_i \in t_i^{-1} x_i \tilde{\theta}(y_i) + \mathcal{M}_+^+ U + U \mathcal{M}_+^+. \tag{4.16}$$

Assume for the moment that $i \neq \mathrm{p}(i)$. The assumption that Σ is reduced ensures that $\Theta(\alpha_i) = -\alpha_{\mathrm{p}(i)}$ and $(\alpha_i, \alpha_{\mathrm{p}(i)}) = 0$. Moreover, checking cases (4.2)–(4.6) yields that $\tilde{\theta}(y_i) = t_{\mathrm{p}(i)}^{-1} x_{\mathrm{p}(i)}$. Hence $t_i^{-1} x_i \tilde{\theta}(y_i) = t_i^{-1} x_i t_{\mathrm{p}(i)}^{-1} x_{\mathrm{p}(i)} = t_{\mathrm{p}(i)}^{-1} x_{\mathrm{p}(i)} t_i^{-1} x_i = \tilde{\theta}(y_i) t_i^{-1} x_i$. It follows that (4.16) holds when $i \neq \mathrm{p}(i)$ as well.

The next lemma will allow us to compute $\mathcal{X}(y_i x_i)$. This, in turn, will be used to compute the image of c_μ under $\mathcal{X}$ where μ satisfies the conditions of (4.7).

Lemma 4.6. *Let $\tau(\lambda) \in T$ such that $2s = (\lambda, \alpha_i) = (\lambda, \Theta(-\alpha_i)) \neq 0$. Then*

$$y_i x_i \tau(\lambda) + q^{-4s} \frac{(t_i - t_i^{-1})}{(q^{-4s} - 1)(q_i - q_i^{-1})} \tau(\lambda)$$

is an element in $B_+ U + U B'_+$.

Proof. Set $a_i = (\rho, \Theta(\alpha_i) - \alpha_i)$. Note that

$$B_i t_i^{-1} x_i \tau(\lambda) = y_i x_i \tau(\lambda) + \tilde{\theta}(y_i) t_i t_i^{-1} x_i \tau(\lambda).$$

Now $(\rho, \Theta(\alpha_i) + \alpha_i) = (\alpha_i, \alpha_i) + a_i$. Hence by (4.16),

$$y_i x_i \tau(\lambda) + q^{-a_i} t_i^{-1} x_i \tilde{\theta}(y_i) t_i \tau(\lambda) - B_i t_i^{-1} x_i \tau(\lambda) \in \mathcal{M}_+ U + U \mathcal{M}_+.$$

On the other hand

$$\begin{aligned} t_i^{-1} x_i \tau(\lambda) B'_i &= q^{-2s} x_i y_i \tau(\lambda) + q^{2s-a_i} t_i^{-1} x_i \tilde{\theta}(y_i) t_i \tau(\lambda) \\ &= q^{-2s} y_i x_i \tau(\lambda) + q^{-2s} \frac{(t_i - t_i^{-1})}{(q_i - q_i^{-1})} \tau(\lambda) + q^{2s-a_i} t_i^{-1} x_i \tilde{\theta}(y_i) t_i \tau(\lambda). \end{aligned}$$

Thus

$$(q^{-4s}-1)y_ix_i\tau(\lambda)+q^{-4s}\frac{(t_i-t_i^{-1})}{(q_i-q_i^{-1})}\tau(\lambda)\in B_+U+UB'_+. \qquad \square$$

Set $\tilde{t}_i=\tau(\tilde{\alpha}_i)$. We are now ready to compute the radial components of the central elements described in Theorem 4.5.

Theorem 4.7. *Let μ satisfy the conditions of* (4.7). *Let a be the nonzero scalar such that $\mathscr{P}'(c_\mu)=a(q^{(\rho,\tilde{\alpha}_i)}t_i+q^{-(\rho,\tilde{\alpha}_i)}t_i^{-1})$. Then*

$$\mathscr{X}(c_\mu)=a[q^{-(\rho,\tilde{\alpha}_i)}\tilde{t}_i(q^{2(\rho,\tilde{\alpha}_i)}z^{2\tilde{\alpha}_i}-1)+q^{(\rho,\tilde{\alpha}_i)}\tilde{t}_i^{-1}(q^{-2(\rho,\tilde{\alpha}_i)}z^{2\tilde{\alpha}_i}-1)](z^{2\tilde{\alpha}_i}-1)^{-1}.$$

Proof. By Theorem 4.5 and Lemma 4.6, $a^{-1}c_\mu\tau(\lambda)$ is an element in

$$(q^{(\rho,\tilde{\alpha}_i)}t_i+q^{-(\rho,\tilde{\alpha}_i)}t_i^{-1})\tau(\lambda)+\frac{(q^{(\rho,\tilde{\alpha}_i)}-q^{-(\rho,\tilde{\alpha}_i)})(t_i-t_i^{-1})\tau(\lambda)}{(q^{2(\lambda,\tilde{\alpha}_i)}-1)}+B_+\check{U}+\check{U}B'_+$$

for all $\tau(\lambda)\in\mathscr{A}$. (Here $(\lambda,\tilde{\alpha}_i)$ corresponds to $2s$ in the notation of the previous lemma.) This set simplifies to

$$\frac{(q^{2(\lambda,\tilde{\alpha}_i)+2(\rho,\tilde{\alpha}_i)}-1)q^{-(\rho,\tilde{\alpha}_i)}\tilde{t}_i+(q^{2(\lambda,\tilde{\alpha}_i)-2(\rho,\tilde{\alpha}_i)}-1)q^{(\rho,\tilde{\alpha}_i)}\tilde{t}_i^{-1}}{(q^{2(\lambda,\tilde{\alpha}_i)}-1)}\tau(\lambda)+B_+\check{U}+\check{U}B'_+.$$

Set Y to be the right-hand expression in the statement of the theorem. The desired formula now follows from the fact that Y is the unique element in $\mathscr{C}(Q_\Sigma)\check{\mathscr{A}}$ such that

$$a^{-1}Y\cdot\tau(\lambda)=\left(\frac{(q^{2(\lambda,\tilde{\alpha}_i)+2(\rho,\tilde{\alpha}_i)}-1)q^{-(\rho,\tilde{\alpha}_i)}\tilde{t}_i+(q^{2(\lambda,\tilde{\alpha}_i)-2(\rho,\tilde{\alpha}_i)}-1)q^{(\rho,\tilde{\alpha}_i)}\tilde{t}_i^{-1}}{(q^{2(\lambda,\tilde{\alpha}_i)}-1)}\right)\tau(\lambda)$$

for all $\tau(\lambda)\in\mathscr{A}$. $\square$

5. Graded zonal spherical functions

Recall the ad U filtration on U defined in [8, 5.3.1] (see also [12, Section 2.2]). In this section, we use a modified version of this filtration that is chosen so that the associated graded ring of U contains B as a subalgebra.

Define a degree function on U by

$$\deg x_i=\deg y_it_i=0 \quad \text{for all } i, 1\leqslant i\leqslant n, \tag{5.1}$$

$$\deg t_i^{-1}=1 \quad \text{for all } i \text{ such that } \alpha_i\in\pi\backslash\pi_\Theta, \tag{5.2}$$

$$\deg t=0 \quad \text{for all } t\in T_\Theta. \tag{5.3}$$

This can be made more precise as follows. Let $\tau(\mu)$ be an element of T. By (2.1), we can write $\mu = \mu_1 + \mu_2$ where $\mu_1 \in T'$ and $\mu_2 \in T_\Theta$. Thus $\tau(\mu) = \tau(\mu_1)\tau(\mu_2)$ with degree of $\tau(\mu_2) = 0$. Furthermore, by the definition of T', there exist integers m_i such that

$$\mu_1 = \sum_{\alpha_i \notin \pi_\Theta} m_i \alpha_i.$$

It follows that $\tau(\mu_1)$, and hence $\tau(\mu)$, has degree m where $m = \sum_{\alpha_i \notin \pi_\Theta} m_i$. Moreover, by (5.1), every element of $G^-\tau(\mu)U^+$ has degree m. Let $\mathscr{F}$ denote the filtration on U defined by the above degree function. In particular, given $u \in U$, we can write $u = u_1 + \cdots u_s$ where $u_i \in G^-\tau(\eta_i)U^+$ for each i. Then $u \in \mathscr{F}^r U$ provided that the degree of $\tau(\eta_i)$ is less than or equal to r for each i, $1 \leqslant i \leqslant s$.

Write $\operatorname{gr} U$ for the associated graded algebra with respect to this filtration. Note that elements of U^+, G^-, and $\mathscr{M}$ are all in degree zero. Moreover the relations satisfied by the elements of U^+ (resp. G^-, $\mathscr{M}$) are homogeneous of degree 0. Therefore, the map $a \mapsto \operatorname{gr} a$ defines an isomorphism between U^+ and $\operatorname{gr} U^+$, between G^- and $\operatorname{gr} G^-$, and between $\mathscr{M}$ and $\operatorname{gr} \mathscr{M}$. Using this isomorphism, we write U^+ for $\operatorname{gr} U^+$, G^- for $\operatorname{gr} G^-$, and $\mathscr{M}$ for $\operatorname{gr} \mathscr{M}$. Furthermore, if S is a subset and a is an element of U^+, G^-, or $\mathscr{M}$ then we simply write a for $\operatorname{gr} a$ and S for $\operatorname{gr} S$.

The next lemma shows that a similar identification holds for B. In particular, we may identify B with $\operatorname{gr} B$ as a subalgebra of $\operatorname{gr} U$.

Lemma 5.1. *For all $b \in B$, $\deg b = 0$. Moreover, the filtration $\mathscr{F}$ restricts to the trivial filtration on B and the map $a \mapsto \operatorname{gr} a$ defines an isomorphism between B and* $\operatorname{gr} B$.

Proof. Let $\tilde{B}$ denote the algebra generated freely over $\mathscr{M}^+ T_\Theta$ by elements $\tilde{B}_i$, $1 \leqslant i \leqslant n$. By [20, Theorem 7.4], there is a homomorphism from $\tilde{B}$ onto B which is the identity on $\mathscr{M} T_\Theta$, sends $\tilde{B}_i$ to B_i for $\alpha_i \notin \pi_\Theta$, and sends $\tilde{B}_i$ to $y_i t_i$ for $\alpha_i \in \pi_\Theta$.

By (5.1) and (5.3), we have that $\deg a = 0$ for all $a \in \mathscr{M} T_\Theta$. Consider $B_i = y_i t_i + \tilde{\theta}(y_i) t_i$ for some $\alpha_i \notin \pi_\Theta$. We have $\deg y_i t_i = 0$. Recall that $\tilde{\theta}$ is a particular lift of the involution θ to a $\mathbf{C}$ algebra automorphism of U. It follows from the explicit description of $\tilde{\theta}$ given in [20, Theorem 7.1] that $\tilde{\theta}(y_i) \in U^+ \tau(\Theta(\alpha_i))$. Since $\tau(\Theta(\alpha_i)) t_i = \tau(\Theta(\alpha_i) + \alpha_i) \in T_\Theta$, we have that $\deg \tilde{\theta}(y_i) t_i = 0$ as well. So the generators of B are all in degree 0. By the previous paragraph, the relations satisfied by these generators are all homogeneous of degree 0. Hence all the elements of B are in degree 0 and the generators of $\operatorname{gr} B$ satisfy exactly the same relations as B. $\square$

Note that not all elements of $\mathscr{C}[T]$ are in degree 0. So $\mathscr{C}[T]$ does not naturally identify with its graded image in the same way as the subalgebras discussed above. However, the algebra map induced by $t_i \mapsto \operatorname{gr} t_i$ for $1 \leqslant i \leqslant n$ does define an isomorphism from $\mathscr{C}[T]$ to $\operatorname{gr} \mathscr{C}[T]$. Thus any $\operatorname{gr} T$ module inherits the structure of a T module via this isomorphism.

Set $\mathfrak{h}^*_\Theta = \{\lambda \in \sum_{1 \leqslant i \leqslant n} \mathbf{Q}\alpha_i \mid \Theta(\lambda) = -\lambda\}$. Consider $\lambda \in \mathfrak{h}^*_\Theta$ and let v_λ be a (left) T weight vector of weight λ. Note that $tv_\lambda = v_\lambda$ for all $t \in T_\Theta$. Make $\mathscr{C}v_\lambda$ into a gr $\mathscr{M}TU^+$ module by insisting that $\mathscr{M}_+ v_\lambda = U^+_+ v_\lambda = 0$. Define the left gr U module $\bar{M}(\lambda)$ by

$$\bar{M}(\lambda) = \text{gr}\, U \otimes_{(\text{gr}\, \mathscr{M}TU^+)} v_\lambda.$$

By Lemma 2.1 and Theorem 2.2, $\bar{M}(\lambda) = \text{gr}\, N^- \otimes v_\lambda$ as gr N^- modules. Since N^- is a subalgebra of G^-, N^- can be identified with gr N^- via the obvious map.

The algebra G^- can be given the structure of a U^+ module as in [8, Sections 5.3 and 7.1]. In particular, let x_i' and x_i'' be functions on G^- such that

$$(\text{ad}\, x_i)m = x_i'(m) + x_i''(m)t_i^2$$

for all $m \in G^-$ and for all i such that $1 \leqslant i \leqslant n$. Given i such that $1 \leqslant i \leqslant n$, the action of x_i on the element $m \in G^-$ is defined by

$$x_i * m = x_i'(m).$$

Lemma 5.2. *N^- is a U^+ submodule of G^-. Moreover, $\bar{M}(\lambda) \cong N^-$ as U^+ modules for all $\lambda \in \mathfrak{h}^*_\Theta$.*

Proof. First, note that by [20, Section 6], N^- is an ad $\mathscr{M}^+$ module. In particular, $(\text{ad}\, x)n \in N^-$ for all $x \in \mathscr{M}^+$ and $n \in N^-$. It follows that $x_i''(n) = 0$ and $x_i * n = (\text{ad}\, x_i)n$ is an element of N^- for all $\alpha_i \in \pi_\Theta$ and $n \in N^-$. Thus $\mathscr{M}^+ * N^- \subseteq N^-$.

Now N^- is generated by elements of the form $(\text{ad}\, y)y_j t_j$ [20, Section 6] where $y \in \mathscr{M}^-$ and $\alpha_j \notin \pi_\Theta$. By the defining relations of U, $(\text{ad}\, x_k)((\text{ad}\, y)y_j t_j) = (\text{ad}\, y)(\text{ad}\, x_k) y_j t_j)$ for all k such that $\alpha_k \notin \pi_\Theta$. It follows that

$$x_k'((\text{ad}\, y)y_j t_j) = (\text{ad}\, y)\left(\frac{-\delta_{kj}}{q_j - q_j^{-1}}\right).$$

Thus $x_k'((\text{ad}\, y)y_j t_j) = 0$ for $y \in \mathscr{M}^-_+$ while $x_k'(y_j t_j)$ is a scalar. Therefore $x_k'(n) \in N^-$ for all $n \in N^-$ and $\alpha_k \notin \pi_\Theta$. The fact that G^- is generated by N^- and $\mathscr{M}^-$ [20, Section 6] yields that N^- is a U^+ submodule of G^-.

Fix $n \in N^-$. By (2.4), $(\text{gr}\, x_i)(n \otimes v_\lambda) = \text{gr}\,((\text{ad}\, x_i)n) \otimes v_\lambda$ since $x_i v_\lambda = 0$. Furthermore, gr $(\text{ad}\, x_i)n = x_i'(n)$ for all $\alpha_i \notin \pi_\Theta$ by the definition of the filtration $\mathscr{F}$. Now consider $\alpha_i \in \pi_\Theta$. It follows that $(\text{ad}\, x_i)n = x_i'(n)$ and so $(\text{gr}\, x_i)(n \otimes v_\lambda) = x_i'(n) \otimes v_\lambda$ in this case as well. Thus, the map $n \mapsto n \otimes v_\lambda$ is an isomorphism of U^+ modules under the identification of U^+ with gr U^+. $\square$

Recall that N^- is an ad T module. Since v_λ is a T weight vector, it follows that $\bar{M}(\lambda)$ is a direct sum of its T weight spaces with highest weight equal to λ. Moreover, suppose b is a weight vector in either G^- or U^+, and m is a weight vector in $\bar{M}(\lambda)$. Note that the weight of $(\text{gr}\, b)m$ is equal to the sum of the weights of gr b and m.

Lemma 5.3. *$\bar{M}(\lambda)$ is a simple* gr *U module for each $\lambda \in \mathfrak{h}^*_\Theta$.*

Proof. Observe that $\bar{M}(\lambda)$ is a cyclic gr U module generated by v_λ. So it is sufficient to show that $1 \otimes v_\lambda$ is in the gr U module generated by $n \otimes v_\lambda$ for any weight vector $n \in N^-$.

By the discussion preceding the lemma, $(\text{gr}\, b)m$ has higher weight than m for any nonzero weight vector gr b in gr U^+_+ and nonzero weight vector m in $\bar{M}(\lambda)$. Hence, it is enough to show that the only vectors in $\bar{M}(\lambda)$ annihilated by U^+_+ are scalar multiples of $1 \otimes v_\lambda$. This follows from Lemma 5.2 and the fact that the only U^+ invariant vectors of G^- are scalar multiples of 1 [12, Lemma 4.7(i)]. $\square$

Now let v^r_λ be a right T weight vector of weight λ. We can make $\mathscr{C}v^r_\lambda$ into a right gr U module as follows. Set $v^r_\lambda \mathscr{M}_+ = v^r_\lambda G^-_+ = 0$ and set

$$\bar{M}(\lambda)^r = \mathscr{C}v^r_\lambda \otimes_{\text{gr}\, \mathscr{M}TG^-} \text{gr}\, U.$$

Replacing x_i with $y_i t_i$ and N^- with N^+, we can give N^+ a G^- module structure analogous to the U^+ module structure of N^-. Furthermore, as in Lemma 5.2, $\bar{M}(\lambda)^r$ is isomorphic to N^+ as a G^- module. Moreover, as in Lemma 5.3, $\bar{M}(\lambda)^r$ is a simple right gr U module.

Lemma 5.4. *Let λ and λ' be elements in $\mathfrak{h}^*_\Theta$. The map which sends*

$$m \otimes v_\lambda \mapsto m \otimes v_{\lambda'}$$

for all $m \in N^-$ defines an isomorphism from $\bar{M}(\lambda)$ onto $\bar{M}(\lambda')$ as U^+ modules, G^- modules, and B modules. Similarly, the map which sends

$$v^r_\lambda \otimes m \mapsto v^r_{\lambda'} \otimes m$$

for all $m \in N^+$ defines an isomorphism from $\bar{M}(\lambda)^r$ onto $\bar{M}(\lambda')^r$ as U^+ modules, G^- modules, and B modules.

Proof. We prove the first assertion. The second follows in a similar fashion. The proof of Lemma 5.2 shows that the map $n \mapsto n \otimes v_\lambda$ is a U^+ module isomorphism from N^- onto $\bar{M}(\lambda)$. This isomorphism is independent of λ. Hence $\bar{M}(\lambda)$ is isomorphic to $\bar{M}(\lambda')$ as U^+ modules for all $\lambda, \lambda' \in \mathfrak{h}^*_\Theta$.

Now N^- is a $\mathscr{C}[T_\Theta]$ module via the adjoint action. Recall that $tv_\lambda = v_\lambda$ for all $t \in T_\Theta$. Thus, $(\text{gr}\, t)(n \otimes v_\lambda) = (\text{gr}\, tnt^{-1}) \otimes v_\lambda$ for all $t \in T_\Theta$ and $n \in N^-$. Hence $n \mapsto n \otimes v_\lambda$ is an isomorphism of $\mathscr{C}[T_\Theta]$ modules. This isomorphism is independent of the choice of $\lambda \in \mathfrak{h}^*_\Theta$. Therefore $\bar{M}(\lambda) \cong \bar{M}(\lambda')$ as $\mathscr{C}[T_\Theta]$ modules for all $\lambda, \lambda' \in \mathfrak{h}^*_\Theta$.

Recall that $\bar{M}(\lambda) = \text{gr}\, U \otimes v_\lambda = N^- \otimes v_\lambda$ as left N^- modules. In particular, the action of an element in N^- on $\bar{M}(\lambda)$ just corresponds to left multiplication by that element. Thus the action of N^- on $\bar{M}(\lambda)$ is independent of λ.

Now consider $y_i t_i$ where $\alpha_i \in \pi_\Theta$. In particular, $y_i t_i$ is an element of $\mathscr{M}$. Recall that N^- is $\operatorname{ad} y_i t_i$ invariant while $y_i t_i v_\lambda = 0$. Hence, given $n \in N^-$, we have

$$(y_i t_i n) \otimes v_\lambda = ((\operatorname{ad} y_i t_i) n) \otimes v_\lambda.$$

Thus the action of $y_i t_i$ on $\bar{M}(\lambda)$ corresponds to the action of $(\operatorname{ad} y_i t_i)$ on N^- for all $\alpha_i \in \pi_\Theta$. As mentioned earlier [20, Section 6] G^- is generated by $\mathscr{M}^-$ and N^-. It follows that the action of G^- on $\bar{M}(\lambda)$ is independent of λ.

Recall the identification of B with $\operatorname{gr} B$. It follows from the proof of Lemma 5.1 that $\operatorname{gr} B$ is contained in the subalgebra of $\operatorname{gr} U$ generated by $\operatorname{gr} U^+$, $\operatorname{gr} G^-$, and $\operatorname{gr} \mathscr{C}[T_\Theta]$. Thus the isomorphism of $\bar{M}(\lambda)$ to $\bar{M}(\lambda')$ as B modules follows from their isomorphism as U^+, G^-, and $\mathscr{C}[T_\Theta]$ modules. $\square$

Let $\bar{M}(\lambda)^*$ denote the dual of $\bar{M}(\lambda)$ given its natural right $\operatorname{gr} U$ module structure. The locally finite T part, $F_T(\bar{M}(\lambda)^*)$, of $\bar{M}(\lambda)^*$ is the direct sum of its T weight spaces. Note further that the β weight space of $F_T(\bar{M}(\lambda)^*)$ is the dual of the β weight space of $\bar{M}(\lambda)$. In particular,

$$\dim F_T(\bar{M}(\lambda)^*)_\beta = \bar{M}(\lambda)_\beta$$

for all β. Let v_λ^* be a nonzero vector in $F_T(\bar{M}(\lambda)^*)_\lambda$. Then v_λ^* generates a simple $\operatorname{gr} U$ module isomorphic to $\bar{M}(\lambda)^r$. A comparison of the dimension of the weight spaces yields $F_T(\bar{M}(\lambda)^*) = v_\lambda^* \operatorname{gr} U$.

Let $\hat{M}(\lambda)^r$ denote the completion of $\bar{M}(\lambda)^r$ consisting of possibly infinite sums of distinct weight vectors $\sum_\gamma a_\gamma$ for $a_\gamma \in (\bar{M}(\lambda)^r)_\gamma$. We can identify $\bar{M}(\lambda)^*$ with $\hat{M}(\lambda)^r$. Similar considerations allow us to identify $\bar{M}(\lambda)^{r*}$ with the completion $\hat{M}(\lambda)$ consisting of possibly infinite sums of distinct weight vectors in $\bar{M}(\lambda)$.

For the remainder of the paper, given $u \in \operatorname{gr} U$ we write uv_λ for the element $u(1 \otimes v_\lambda)$ of $\bar{M}(\lambda)$ and $v_\lambda^r u$ for the element $(v_\lambda^r \otimes 1)u$ of $\bar{M}(\lambda)^r$. In light of the isomorphisms of Lemma 5.4, we often abbreviate $(\operatorname{gr} a)w$ as aw and $w'(\operatorname{gr} a)$ as $w'a$ for $a \in G^- \cup B \cup U^+$, $w \in \bar{M}(\lambda)$, and $w' \in \bar{M}(\lambda)^r$.

Lemma 5.5. *Let V be a finite dimensional simple right B-module and W be a finite dimensional simple left B module. There are vector space isomorphisms*

$$\operatorname{Hom}_B(V, \bar{M}(\lambda)^*) \cong \operatorname{Hom}_{\mathscr{M}}(\mathscr{C}v_\lambda, V^*)$$

and

$$\operatorname{Hom}_B(W, \bar{M}(\lambda)^{r*}) \cong \operatorname{Hom}_{\mathscr{M}}(\mathscr{C}v_\lambda^r, W^*).$$

Proof. Given $\psi \in \operatorname{Hom}_B(V, \bar{M}(\lambda)^*)$, define a linear map $\tilde{\psi}$ from $\mathscr{C}v_\lambda$ to V^* by $\tilde{\psi}(v_\lambda)(w) = \psi(w)(v_\lambda)$. Note that $\psi(w)(mv_\lambda) = (\psi(w)m)v_\lambda = \psi(wm)v_\lambda$ for all $m \in \mathscr{M}$. Hence $\tilde{\psi}(mv_\lambda)(w) = \tilde{\psi}(v_\lambda)(wm)$ for $m \in \mathscr{M}$. Thus $\tilde{\psi} \in \operatorname{Hom}_{\mathscr{M}}(\mathscr{C}v_\lambda, V^*)$.

Using Lemma 2.1, we obtain a graded version of Theorem 2.2. In particular, there is an isomorphism of vector spaces via the (graded) multiplication map:

$$\mathrm{gr}\, U \cong B \otimes \mathrm{gr}\, \mathscr{C}[T'] \otimes N^-.$$

It follows that $\bar{M}(\lambda) = Bv_\lambda$. Hence $\psi(w)$ is completely determined by its action on v_λ. Thus the map from ψ to $\tilde{\psi}$ is one-to-one. The first isomorphism now follows from the fact that this map is clearly invertible. A similar argument verifies the second isomorphism. □

Note that Lemmas 5.1, 5.4 and 5.5 hold when we replace B by any subalgebra in $\mathscr{B}$. In particular, these lemmas apply to B'. Let V_1 denote the trivial one-dimensional left B' module. It follows that V_1 is annihilated by B'_+. Let V_1^r denote the trivial one-dimensional right B module. Then by Lemma 5.5,

$$\dim \mathrm{Hom}_{B'}(V_1, \hat{M}(\lambda)) = \dim \mathrm{Hom}_{\mathscr{M}}(\mathscr{C}v_\lambda, V_1) = 1.$$

Similarly, $\dim \mathrm{Hom}_B(V_1^r, \hat{M}(\lambda)^r) = 1$. In particular, the space of B' invariants in $\hat{M}(\lambda)$ is one dimensional and the space of B invariants in $\hat{M}(\lambda)^r$ is one dimensional. Let ζ_λ^r be a nonzero vector in $(\hat{M}(\lambda)^r)^B$ and ζ_λ be a nonzero vector in $\hat{M}(\lambda)^{B'}$.

Let $\hat{N}^-$ be the space consisting of possibly infinite sums of the form $\sum_{\gamma \leqslant 0} a_\gamma$ where a_γ is a weight vector of weight γ in N^-. Similarly, let $\hat{N}^+$ be the space consisting of possibly infinite sums of the form $\sum_{\gamma \geqslant 0} a_\gamma$ where a_γ is a weight vector of weight γ in N^+. Note that $\hat{M}(\lambda)^r = v_\lambda^r \hat{N}^+$ and $\hat{M}(\lambda) = \hat{N}^- v_\lambda$.

Lemma 5.6. *There exists $b \in \hat{N}^-$ and $b^r \in \hat{N}^+$ such that $\zeta_\lambda = bv_\lambda$ and $\zeta_\lambda^r = v_\lambda^r b^r$ for all $\lambda \in \mathfrak{h}_\Theta^*$. Moreover, both b and b^r have nonzero constant terms.*

Proof. The fact that there is a universal element b which satisfies $\zeta_\lambda = bv_\lambda$ for any choice of $\lambda \in \mathfrak{h}_\Theta^*$ follows immediately from Lemma 5.4. Similarly, Lemma 5.4 ensures the existence of a unique element b^r satisfying $\zeta_\lambda^r = v_\lambda^r b^r$ for all $\lambda \in \mathfrak{h}_\Theta^*$. We prove the last statement of the lemma. Fix $\lambda \in \mathfrak{h}_\Theta^*$ and write $b = \sum_{\gamma \leqslant \beta} b_\gamma$ where each b_γ is an element of $\hat{N}^-$ of weight γ and $b_\beta \neq 0$. If $x_j b_\gamma v_\lambda \neq 0$, then $x_j b_\gamma v_\lambda$ has weight $\alpha_j + \gamma + \lambda$. Hence

$$x_j b v_\lambda = x_j b_\beta v_\lambda + \text{terms of weight lower than } \alpha_j + \beta + \lambda.$$

Now if $\alpha_j \in \pi_\Theta$, then $x_j b v_\lambda = 0$. It follows that $x_j b_\beta v_\lambda = 0$ for all $\alpha_j \in \pi_\Theta$. On the other hand, as in the proof of Lemma 2.1, if $\alpha_i \notin \pi_\Theta$, then B_+ contains an element of the form $x_i + Y_i$, where Y_i is a weight vector in $G^- T_\Theta$ of weight $\Theta(\alpha_i)$. In particular, $Y_i b_\gamma v_\lambda$ has weight strictly lower than γ. So for $\alpha_i \notin \pi_\Theta$, we have

$$0 = (x_i + Y_i) b v_\lambda = x_i b_\beta v_\lambda + \text{terms of weight lower than } \alpha_i + \beta + \lambda.$$

Hence $x_i b_\beta v_\lambda = 0$ for all i such that $1 \leqslant i \leqslant n$. Since the only highest weight vectors in $\bar{M}(\lambda)$ are scalar multiples of v_λ, it follows that b_β is a nonzero scalar. The same argument works for b^r using $y_it_i + \tilde{\theta}(y_i)t_i$ instead of $x_i + Y_i$ and y_jt_j instead of x_j. $\square$

Note that we can consider elements of $\bar{M}(\lambda)^* \otimes \bar{M}(\lambda)$ as functions on T where $(m^* \otimes m)(t) = m^*(tm)$ for all $m \in \bar{M}(\lambda)$, $m^* \in \bar{M}^*(\lambda)$, and $t \in T$. This gives rise to a linear map $\bar{\Upsilon}$ from $\bar{M}(\lambda)^* \otimes \bar{M}(\lambda)$ to $\mathscr{C}[P(\pi)]$ such that $(\bar{\Upsilon}(m^* \otimes m))(t) = (m^* \otimes m)(t)$. Recall the identification of $\hat{M}(\lambda)^r$ with $\bar{M}(\lambda)^*$. Consider weight vectors $m_\gamma^r \in \bar{M}(\lambda)_\gamma^r$ and $m_{\gamma'} \in \bar{M}(\lambda)_{\gamma'}$. Note that $m_\gamma^r(tm_{\gamma'})$ is zero if $\gamma \neq \gamma'$. It follows that the map $\bar{\Upsilon}$ can be extended to a linear map, which we also refer to as $\bar{\Upsilon}$, from $\hat{M}(\lambda)^r \otimes \hat{M}(\lambda)$ to the formal Laurent series ring $\mathscr{C}((z^{-\alpha} \mid \alpha \in \pi))$ such that (with the obvious interpretations)

$$\bar{\Upsilon}\left(\sum_{\gamma \leqslant \lambda} m_\gamma^r \otimes \sum_{\gamma \leqslant \lambda} m_\gamma\right) = \sum_{\gamma \leqslant \lambda} \bar{\Upsilon}(m_\gamma^r \otimes m_\gamma).$$

Let $\mathscr{C}[[z^{-\tilde{\alpha}_i} \mid \alpha_i \in \pi^*]]$ denote the subring of $\mathscr{C}((z^{-\alpha} \mid \alpha \in \pi))$ consisting of elements of the form $\sum_{\tilde{\gamma} \geqslant 0} a_{\tilde{\gamma}} z^{-\tilde{\gamma}}$ where each $\tilde{\gamma} \in Q^+(\Sigma)$ and $a_{\tilde{\gamma}} \in \mathscr{C}$.

Note that gr U inherits a triangular decomposition from U. Let $\overline{\mathscr{P}}$ be the projection of gr U onto gr U^0 using the direct sum decomposition

$$\mathrm{gr}\, U = \mathrm{gr}\, U^0 \oplus \mathrm{gr}\, (G_+^- U + UU_+^+).$$

Now $\mathrm{gr}\, x_iy_jt_j = \delta_{ij}(q_i - q_i^{-1})^{-1} + \mathrm{gr}\, q^{(-\alpha_j,\alpha_i)} y_jt_jx_i$ for all $\alpha_i \notin \pi_\Theta$. Hence

$$\overline{\mathscr{P}}(\mathrm{gr}\, U^+G^-) \subseteq \mathscr{C}[T_\Theta].$$

Lemma 5.7. *There exists $p \in \mathscr{C}[[z^{-\tilde{\alpha}_i} \mid \alpha_i \in \pi^*]]$ such that $\bar{\Upsilon}(\zeta_\lambda^r \otimes \zeta_\lambda) = z^\lambda p$ for all $\lambda \in \mathfrak{h}_\Theta^*$. Moreover, p has a nonzero constant term.*

Proof. Let b and b^r be as in Lemma 5.6. We can write $b = \sum_\gamma b_{-\gamma}$ where each $b_{-\gamma}$ is a weight vector of weight $-\gamma$ in N^-. Similarly, we can write $b^r = \sum_\gamma b_\gamma^r$ where each b_γ^r is a weight vector of weight γ in N^+. Let b_γ'' be the scalar such that $\overline{\mathscr{P}}(\mathrm{gr}\, b_\gamma^r b_{-\gamma}) \in b_\gamma'' + \mathscr{C}[T_\Theta]_+$. Note that $sv_\lambda = v_\lambda^r s = 0$ for all $s \in \mathscr{C}[T_\Theta]_+$ and $\lambda \in \mathfrak{h}_\Theta^*$. It follows that

$$\begin{aligned}(v_\lambda^r b_\gamma^r)(\tau(\beta)b_{-\gamma}v_\lambda) &= (v_\lambda^r)((\mathrm{gr}\, b_\gamma^r \tau(\beta) b_{-\gamma})v_\lambda)\\ &= q^{(\beta,-\gamma+\lambda)} v_\lambda^r((\mathrm{gr}\, b_\gamma^r b_{-\gamma})v_\lambda)\\ &= q^{(\beta,-\gamma+\lambda)} v_\lambda^r(b_\gamma'' v_\lambda)\end{aligned}$$

for all $\tau(\beta) \in T$ and $\lambda \in \mathfrak{h}_\Theta^*$. Thus

$$z^{-\lambda} \bar{\Upsilon}(b_{-\gamma}v_\lambda \otimes v_\lambda^r b_\gamma^r) = b_\gamma'' z^{-\gamma}$$

for each γ and for all $\lambda \in \mathfrak{h}^*_\Theta$. Therefore, by Lemma 5.6, $z^{-\lambda}\bar{\Upsilon}(\zeta^r_\lambda \otimes \zeta_\lambda) = z^{-\lambda'}\bar{\Upsilon}(\zeta^r_{\lambda'} \otimes \zeta_{\lambda'})$ for all λ and λ' in $\mathfrak{h}^*_\Theta$. This proves the first assertion. The second assertion follows from the fact that both b and b^r have nonzero constant terms (Lemma 5.6). □

We can give $\mathscr{C}(Q_\Sigma)\mathscr{A}$ a filtration by setting $\deg f = 0$ for all $f \in \mathscr{C}(Q_\Sigma)$ and setting the degree of an element $t \in \mathscr{A}$ equal to its degree in the $\mathscr{F}$ filtration of U. Note that $\mathscr{C}(Q_\Sigma)\mathscr{A}$ is isomorphic to its associated graded ring under this filtration. Given a homogeneous element $g\tau(\beta)$ where $g \in \mathscr{C}(Q_\Sigma)$ and $\tau(\beta) \in \mathscr{A}$, we write $\operatorname{gr} g\tau(\beta)$ as just $g\tau(\beta)$.

Note that $\mathscr{F}$ extends in an obvious way to a filtration on $\check{U}$. In particular, suppose that $\tau(\mu) \in \check{T}$. We can write $\mu = \mu_1 + \mu_2$ where $\Theta(\mu_2) = \mu_2$ and $\mu_1 = \sum_{\alpha_i \in \pi^*} m_i\alpha_i$ for some rational numbers m_i. Then the degree of elements in $G^-\tau(\mu)U^+$ is just $\sum_{\alpha_i \in \pi^*} m_i$. (Since the quotient group $\check{T}/T$ is finite, there exists a positive integer m such that $m(\deg t)$ is an integer for all $t \in \check{T}$. In particular, it is possible to rescale this degree function on $\check{U}$ so that it is an integer valued degree function.) The above filtration extends in a similar fashion to $\mathscr{C}(Q_\Sigma)\check{\mathscr{A}}$. Now suppose $z \in Z(\check{U})$. Then $\operatorname{gr} z$ is in the center of $\operatorname{gr} \check{U}$. As explained in Section 4, given $\mu \in P^+(\pi)$, there exists a central element c_μ in $(\operatorname{ad} U)\tau(-2\mu)$ such that $c_\mu \in \tau(-2\mu) + (\operatorname{ad} U_+)\tau(-2\mu)$. Furthermore, by (4.8) (which applies in general and not just to the rank one cases), there exists a nonzero scalar multiple c'_μ of c_μ such that

$$c'_\mu \in \tau(-2\mu) + G^-_+ U_\geqslant U^+_+ \tau(-2\mu). \tag{5.4}$$

Hence by Lemma 3.1, there exists $p_\mu \in \mathscr{C}(Q_\Sigma)$ such that $\operatorname{gr} c'_\mu\tau(\beta)$ is an element of

$$(B\check{T}_\Theta)_+ \operatorname{gr}(G^- U^+ \tau(\beta - 2\mu)) + \operatorname{gr}(G^- U^+ \tau(\beta - 2\mu))(B'\check{T}_\Theta)_+ + \tau(-2\tilde{\mu})(p_\mu \cdot \tau(\beta)).$$

It follows that

$$\operatorname{gr}(\mathscr{X}(c'_\mu)) = \tau(-2\tilde{\mu})p_\mu$$

and

$$\begin{aligned}(\zeta^r_\lambda \otimes \zeta_\lambda)(\operatorname{gr} c'_\mu\tau(\beta)) &= \bar{\Upsilon}(\zeta^r_\lambda \otimes \zeta_\lambda)((\operatorname{gr} \mathscr{X}(c'_\mu)) \cdot \tau(\beta)) \\ &= \bar{\Upsilon}(\zeta^r_\lambda \otimes \zeta_\lambda)((\tau(-2\tilde{\mu})p_\mu) \cdot \tau(\beta))\end{aligned}$$

for all $\tau(\beta) \in \mathscr{A}$.

Theorem 5.8. *There exists $p \in \mathscr{C}[[z^{-\tilde{\alpha}_i} \mid \alpha_i \in \pi^*]]$ such that $\bar{\Upsilon}(\zeta^r_\lambda \otimes \zeta_\lambda) = z^\lambda p$ for all $\lambda \in \mathfrak{h}^*_\Theta$ and* $\operatorname{gr}(\mathscr{X}(c'_\mu)) = p^{-1}\tau(-2\tilde{\mu})p$ *for all $\mu \in P^+(\pi)$.*

Proof. The first assertion is simply Lemma 5.7. By (5.4), it follows that $(\operatorname{gr} c'_\mu)v_\lambda = q^{-2(\lambda,\mu)}v_\lambda$. Furthermore, $\lambda \in \mathfrak{h}^*_\Theta$ ensures that $(\lambda, \mu) = (\lambda, \tilde{\mu})$. Since $(\operatorname{gr} c'_\mu)$ is central in $\operatorname{gr} U$, it follows that $\operatorname{gr} c'_\mu$ acts on $\bar{M}(\lambda)$ as multiplication by the scalar $q^{-2(\lambda,\tilde{\mu})}$.

A similar argument yields that $\text{gr}\, c'_\mu$ acts on $\bar{M}(\lambda)^r$ as multiplication by the same scalar $q^{-2(\lambda,\tilde{\mu})}$. Hence $\text{gr}\, c'_\mu$ acts on elements of both $\hat{M}(\lambda)^r$ and $\hat{M}(\lambda)$ as multiplication by the scalar $q^{-2(\lambda,\tilde{\mu})}$. It follows that

$$\begin{aligned}\bar{\Upsilon}(\zeta_\lambda^r \otimes \zeta_\lambda) * p^{-1}\tau(-2\tilde{\mu})p &= z^\lambda p * p^{-1}\tau(-2\tilde{\mu})p \\ &= q^{-2(\lambda,\tilde{\mu})} z^\lambda p \\ &= z^\lambda p * (\text{gr}\, \mathscr{X}(c'_\mu)).\end{aligned}$$

In particular,

$$\text{gr}\,(\mathscr{X}(c'_\mu)) - p^{-1}\tau(-2\tilde{\mu})p \tag{5.5}$$

acts as zero on $z^\lambda p$ for all $\lambda \in \mathfrak{h}^*_\Theta$. But $\text{gr}\,(\mathscr{X}(c'_\mu)) - p^{-1}\tau(-2\tilde{\mu})p$ is an element of $\mathscr{C}[[z^{-\tilde{\alpha}_i} \mid \alpha_i \in \pi^*]]\tau(-2\tilde{\mu})$. The only element of this set which acts as zero on $z^\lambda p$ is zero. This forces the expression in (5.5) to be identically equal to zero. $\square$

6. Computing graded radial components

In this section, we compute the graded image of radial components using information about rank one quantum symmetric pairs from Section 4. In particular, for each i such that $\alpha_i \in \pi^*$, we associate a semisimple Lie subalgebra $\mathfrak{g}_i$ of $\mathfrak{g}$ such that $\mathfrak{g}_i, \mathfrak{g}_i^\theta$ is an irreducible symmetric pair with rank one restricted root system as follows. Recall that ω_j denotes the fundamental weight corresponding to the root $\alpha_j \in \pi$. For all i such that $\alpha_i \in \pi^*$, set $\pi_i = \{\alpha_j \mid (\omega_j, \Theta(-\alpha_i)) \neq 0 \text{ or } (\omega_j, \Theta(-\alpha_{\mathrm{p}(i)})) \neq 0\}$. Let $\mathfrak{g}_i \subseteq \mathfrak{g}$ be the semisimple Lie subalgebra generated by the root vectors e_j and f_j with $\alpha_j \in \pi_i$. It follows that π_i is the set of simple roots associated to $\mathfrak{g}_i$. Moreover, the choice of π_i ensures that θ restricts to an involution of $\mathfrak{g}_i$ which we also refer to as θ. Recall the definition of the restricted root system Σ associated to $\mathfrak{g}, \mathfrak{g}^\theta$ given in (1.1). Set $\Sigma_i = \{\pm\tilde{\alpha}_i\}$ and note that Σ_i is precisely the set of restricted roots associated to the symmetric pair $\mathfrak{g}_i, \mathfrak{g}_i^\theta$.

Let Δ_i denote the root system associated to $\mathfrak{g}_i$ and set U_i equal to the subalgebra of U generated by $x_i, y_i, t_i^{\pm 1}$ for $\alpha_i \in \pi_i$. Note that U_i can be identified with the quantized enveloping algebra of $\mathfrak{g}_i$. Set $\Delta_i^+ = \Delta^+ \cap \Delta_i$. The coideal subalgebra $B \cap U_i$ of U_i can be thought of as a (standard) quantum analog of $U(\mathfrak{g}_i^\theta)$ inside of U_i. In particular, results in the previous sections of this paper apply to the quantum symmetric pair $U_i, B \cap U_i$. A similar comment can be made in reference to the subalgebra B' of U.

For most standard subsets of U, we use the subscript i to denote the intersection of this subset with U_i. For example, we write U_i^+ for $U^+ \cap U_i$. The exception to this rule is $B \cap U_i$ since B_i has already been defined as something different in (1.2).

Set $\mathfrak{h}^*_{\Theta i} = \{\lambda \in \mathbf{Q}\alpha_1 + \cdots + \mathbf{Q}\alpha_n \mid (\lambda, \eta) = 0 \text{ for all } \eta \in Q(\pi_i) \text{ such that } \Theta(\eta) = \eta\}$. Note that $\mathfrak{h}^*_\Theta$ is a subset of $\mathfrak{h}^*_{\Theta i}$. Given $\lambda \in \mathfrak{h}^*_{\Theta i}$, let v_λ be a T weight vector of weight λ and give $\mathscr{C}v_\lambda$ the structure of a trivial $\mathscr{M}_i U_i^+$ module. Write $\bar{M}_i(\lambda)$ for the (left) $\text{gr}\, U_i T$ module induced from the $\text{gr}\, T\mathscr{M}_i U_i^+$ module $\mathscr{C}v_\lambda$.

Lemma 6.1. *Fix $\lambda \in \mathfrak{h}^*_\Theta$ and let w be a weight vector in $\bar{M}(\lambda)$ of weight γ. Assume further that $sw = 0$ for all $s \in \mathrm{gr}\,(\mathscr{M}_i T_{\Theta i} U_i^+)_+$. Then $\gamma \in \mathfrak{h}^*_{\Theta i}$ and the map $uv_\gamma \mapsto uw$ is a* gr $U_i T$ *module isomorphism from* gr $U_i T v_\gamma$ *onto* gr $U_i T w$.

Proof. Since $sw = 0$ for all $s \in \mathscr{C}[T_{\Theta i}]_+$, it follows that $\tau(\eta)w = q^{(\eta,\gamma)}w = w$ for all $\tau(\eta) \in T_{\Theta i}$. Now $\tau(\eta) \in T_{\Theta i}$ if and only if $\eta \in Q(\pi_i)$ and $\Theta(\eta) = \eta$. Hence $\gamma \in \mathfrak{h}^*_{\Theta i}$.

Set $I = UU_+^+ + \sum_{\tau(\beta)\in T} U(\tau(\beta) - q^{(\gamma,\beta)})$. Note that $\bar{M}_i(\gamma)$ is isomorphic to the left gr $U_i T$ module $(\mathrm{gr}\, U_i T)/(\mathrm{gr}\,(I \cap U_i T))$. Since w is annihilated by $\mathrm{gr}\,(I \cap U_i T)$, it follows that the map $uv_\gamma \mapsto uw$ is a gr $U_i T$ module map.

By our assumptions on w, we have that $(\mathrm{gr}\, U_i T)w = N_i^- w$. Now $\bar{M}(\lambda)$ is a free N^- module. Hence the subspace $(\mathrm{gr}\, U_i T)w$ is a cyclic free N_i^- module. The lemma now follows from the fact that $\bar{M}_i(\gamma)$ is also a cyclic free N_i^- module. □

Let $\hat{M}_i(\lambda)$ be the completion of $\bar{M}_i(\lambda)$ consisting of possibly infinite sums of weight vectors in $\bar{M}_i(\lambda)$. Similarly, let $\hat{N}_i^-$ be the subspace of $\hat{N}^-$ consisting of possibly infinite sums of weight vectors in N_i^-. Given $\lambda \in \mathfrak{h}^*_{\Theta i}$, let $\zeta_{\lambda i}$ denote the $B' \cap U_i$ invariant vector of $\hat{M}_i(\lambda)$. By Lemma 5.6 (for the rank one symmetric pair $\mathfrak{g}_i, \mathfrak{g}_i^\theta$) there is an element b_i in $\hat{N}_i^-$ such that $\zeta_{\lambda i} = b_i v_\lambda$.

Lemma 6.2. *There exists $w = \sum_\gamma w_\gamma \in \hat{N}^-$ such that each w_γ is a weight vector of weight γ, $swv_\lambda = 0$ for all $s \in \mathrm{gr}\,(\mathscr{M}_i T_{\Theta i} U_i^+)_+$, and $b_i w v_\lambda = \zeta_\lambda$ for all $\lambda \in \mathfrak{h}^*_\Theta$.*

Proof. Fix $\lambda \in \mathfrak{h}^*_\Theta$. Let $m = \sum_\gamma m_\gamma$ be an element in $\hat{N}^-$ such that each $m_\gamma \in N_\gamma^-$ and $(B' \cap U_i)_+ m v_\lambda = 0$. It follows that $s m_\gamma v_\lambda = 0$ for all $s \in \mathscr{C}[T_{\Theta i}]_+$ and all γ. Thus m_γ nonzero implies that $\gamma \in \mathfrak{h}^*_{\Theta i}$.

Let β be the highest weight such that $m_\beta \neq 0$. The same argument as in Lemma 5.6 shows that $x_j m_\beta v_\lambda = 0$ for all $\alpha_j \in \pi_i$. By the previous paragraph, β is a weight in $\mathfrak{h}^*_{\Theta i}$. Hence $s m_\beta v_\lambda = 0$ for all $s \in \mathrm{gr}\,(T_{\Theta i} U_i^+)_+$.

Note that each $y_j t_j$ with $\alpha_j \notin \pi_\Theta$ is a highest weight vector for the action of ad $\mathscr{M}^+$. By [11, Section 4], ad y_k acts ad nilpotently on $y_j t_j$ whenever $k \neq j$. Hence [11, Theorem 5.9] ensures that each $y_j t_j$ such that $\alpha_j \notin \pi_\Theta$ generates a locally finite ad $\mathscr{M}$ module. By the definition of N^-, it follows that N^- is a locally finite module with respect to the action of ad $\mathscr{M}$. Hence m_β generates a finite dimensional ad $\mathscr{M}_i$ module. The fact that $(\beta, \alpha_j) = 0$ for all $\alpha_j \in \pi_i \cap \pi_\Theta$ further implies that m_β generates a one-dimensional trivial ad $\mathscr{M}_i$ module. It follows that $m_\beta v_\lambda$ is annihilated by any element in $\mathrm{gr}\,(\mathscr{M}_i T_{\Theta i} U_i^+)_+$. Thus by Lemma 6.1, $b_i m_\beta v_\lambda$ is a $B' \cap U_i$ invariant vector. Moreover, rescaling if necessary, we may assume by Lemma 5.6 that $b_i m_\beta v_\lambda = m_\beta v_\lambda +$ terms of weight lower than $\beta + \lambda$. Set $m' = m - b_i m_\beta$ and note that $m' v_\lambda$ is a $B' \cap U_i$ invariant vector. Moreover, when m' is written as a sum of weight vectors, the highest weight of a nonzero summand is strictly less than β.

Now assume that we are in the special case where $m v_\lambda = \zeta_\lambda$. (In particular, m is equal to b of Lemma 5.6 and is independent of the choice of λ.) Set $w_\beta = m_\beta$. By induction, we can find a sequence of weight vectors $\{w_\gamma\}$ in N^- such that each w_γ

generates a trivial gr $(\mathcal{M}_i T_{\Theta i} U_i^+)$ module and

$$m - b_i \sum_{0 \geqslant \gamma \geqslant \gamma'} w_\gamma \in \sum_{\alpha < \gamma'} \hat{N}^-$$

for all γ'. Thus by the definition of $\hat{N}^-$, we obtain $m = b_i \sum_\gamma w_\gamma$. $\square$

Let $G^-_{\pi \setminus \pi_i}$ denote the subalgebra of G^- generated by $(\mathrm{ad}\, U_i^-)\mathscr{C}[y_i t_i \mid \alpha_i \notin \pi_i]$. By [20, Section 6], we see that multiplication induces an isomorphism of vector spaces

$$G^- \cong G_i^- \otimes G^-_{\pi \setminus \pi_i}. \tag{6.1}$$

Furthermore, $G^-_{\pi \setminus \pi_i}$ is generated by elements of the form $(\mathrm{ad}\, y) y_j t_j$ where $\alpha_j \notin \pi_i$, $y \in G_i^-$, and the weight of $(\mathrm{ad}\, y) y_j t_j$ is a root in Δ. It follows that the weight of $(\mathrm{ad}\, y) y_j t_j$ cannot be an element of Δ_i. Hence the weights of vectors in $G^-_{\pi \setminus \pi_i}$ are elements of

$$\sum_{\gamma \in \Delta^+ \setminus \Delta_i^+} \mathbf{N}(-\gamma).$$

Suppose that $\beta \in \Delta$ and $\tilde{\beta} = \tilde{\alpha}_i$. Note that $2\tilde{\alpha}_i \in \sum_{\alpha \in \pi_i} \mathbf{N}\alpha_i$. Hence β must be a positive root. So both β and $-\Theta(\beta)$ are elements of $\sum_{\alpha \in \pi} \mathbf{N}\alpha$. Hence $2\tilde{\beta} \in \sum_{\alpha \in \pi_i} \mathbf{N}\alpha$ forces both β and $-\Theta(\beta)$ to be elements of $\sum_{\alpha \in \pi_i} \mathbf{N}\alpha$. In particular

$$\{\beta \in \Delta^+ \mid \tilde{\beta} = \tilde{\alpha}_i\} \subset \Delta_i^+. \tag{6.2}$$

By (6.2), if $\gamma \in \Delta^+ \setminus \Delta_i^+$, then $\tilde{\gamma} \notin \Sigma_i$. In particular, if β is a weight of an element of $G^-_{\pi \setminus \pi_i}$, then

$$\tilde{\beta} \in \sum_{\tilde{\gamma} \in \Sigma^+ \setminus \{\tilde{\alpha}_i\}} \mathbf{N}(-\tilde{\gamma}).$$

Lemma 6.3. *Let $\lambda \in \mathfrak{h}^*_\Theta$. Suppose w is a weight vector of weight β in N^- such that wv_λ generates a trivial* gr $(U_i^+ \mathcal{M}_i T_{\Theta i})$ *module and $\beta \in Q(\Sigma)$. Then the weight of w is contained in $\sum_{\tilde{\gamma} \in \Sigma^+ \setminus \{\tilde{\alpha}_i\}} \mathbf{N}(-\tilde{\gamma})$.*

Proof. Write $w = \sum_j w_{1j} w_{2j}$ where $w_{1j} \in G_i$ and $w_{2j} \in G^-_{\pi \setminus \pi_i}$. For each j, set γ_{1j} equal to the weight of w_{1j} and γ_{2j} equal to the weight of w_{2j}. We may further assume that $\{w_{2j}\}_j$ is a linearly independent set. Choose β' minimal in the set $\{\gamma_{2j}\}_j$ using the standard partial ordering on $Q(\pi)$. By the discussion preceding the lemma, we have that

$$\tilde{\beta}' \in \sum_{\tilde{\gamma} \in \Sigma^+ \setminus \{\tilde{\alpha}_i\}} \mathbf{N}(-\tilde{\gamma}).$$

Thus it is sufficient to show that $\beta = \beta'$.

Consider $\alpha_k \in \pi_i$. In particular, $x_k w v_\lambda = 0$. By Lemma 5.2, $\bar{M}(\lambda)$ is isomorphic to the submodule N^- of G^- as U^+ modules. Hence $x_k * w = \mathrm{gr}\,((\mathrm{ad}\,x_k)w) = 0$. On the other hand,

$$\mathrm{gr}\,((\mathrm{ad}\,x_k)w) \in \sum_{\gamma_{2j}=\beta'} \mathrm{gr}((\mathrm{ad}\,x_k)w_{1j})w_{2j} + G_i^- \sum_{\gamma \neq \beta'} (G^-_{\pi\setminus\pi_i})_\gamma.$$

Hence, by (6.1), $\mathrm{gr}\,(\mathrm{ad}\,x_k)w_{1j} = 0$ for all j such that $\gamma_{2j} = \beta'$. Fix j such that $\gamma_{2j} = \beta'$. It follows that w_{1j} is a highest weight vector with respect to the action of U_i^+ on G_i^-. By [12, Lemma 4.7(i)], w_{1j} is a scalar. Thus the weight of w agrees with the weight of w_{2j} which is just β'. □

Given $\lambda \in \mathfrak{h}^*_{\Theta i}$, let v^r_λ be a right T weight vector of weight λ and give $\mathscr{C}v^r_\lambda$ the structure of a one-dimensional trivial $\mathrm{gr}(\mathscr{M}_i G_i^-)$ module. Define $\bar{M}_i(\lambda)^r$ to be the right $\mathrm{gr}\,U_i T$ module induced from the one-dimensional $\mathrm{gr}\,(\mathscr{M}_i G_i^- T)$ module $\mathscr{C}v^r_\lambda$. Note that versions of Lemmas 6.1–6.3 hold for the right $\mathrm{gr}\,U_i$ modules $\bar{M}_i(\lambda)^r$. In particular, let b^r be chosen as in Lemma 5.6. Let b^r_i be also chosen as in Lemma 5.6 for the modules $\bar{M}_i(\lambda)^r$. As in Lemma 6.2, there exists $w^r \in \hat{N}^+$ such that $w^r b^r_i = b^r$. Moreover, $w^r = \sum_\gamma w^r_\gamma$ where each w_γ is annihilated by elements in $\mathrm{gr}\,(G_i^- \mathscr{M}_i T_{\Theta i})_+$. Furthermore, $w^r_\gamma \neq 0$ implies that γ is an element of $\sum_{\tilde{\beta} \in \Sigma^+ \setminus \{\tilde{\alpha}_i\}} \mathbf{N}\tilde{\beta}$.

Recall the graded version $\overline{\mathscr{P}}$ of the Harish-Chandra map defined in the last section. The vector space decomposition (3.5) extends in the obvious way to the corresponding graded algebras. Let $\overline{\mathscr{P}}_{\mathscr{A}}$ denote composition of $\overline{\mathscr{P}}$ with projection onto $\mathrm{gr}\,\mathscr{C}[\check{\mathscr{A}}]$ using a graded version of (3.5).

Given $\lambda \in \mathfrak{h}^*_\Theta$, assume that v^r_λ has been chosen so that $v^r_\lambda(v_\lambda) = 1$. It follows that $v^r_\lambda(\tau(\beta)v_\lambda) = q^{(\lambda,\beta)}$ for all $\tau(\beta) \in T$. In particular, $v^r_\lambda(t v_\lambda) = 1$ for all $t \in T_\Theta$. Hence $v^r_\lambda c\tau(\beta) d v_\lambda = q^{(\lambda-\gamma,\beta)}(\overline{\mathscr{P}}_{\mathscr{A}}(\mathrm{gr}\,cd))$ for all $c \in U^+_\gamma$, $\tau(\beta) \in T$, and $d \in G^-_{-\gamma}$. Thus

$$\bar{\Upsilon}(v^r_\lambda c \otimes d v_\lambda) = z^{\lambda-\gamma}(\overline{\mathscr{P}}_{\mathscr{A}}(\mathrm{gr}\,cd))$$

for all $c \in U^+_\gamma$ and $d \in G^-_{-\gamma}$. (Note that by the definition of $\overline{\mathscr{P}}_{\mathscr{A}}$ given in the previous paragraph, $\overline{\mathscr{P}}_{\mathscr{A}}(\mathrm{gr}\,cd)$ is just an element of $\mathrm{gr}\,\mathscr{C}[\check{\mathscr{A}}]$.)

Choose $p_i \in \mathscr{C}[[z^{-\tilde{\alpha}_i} \mid \alpha_i \in \pi^*]]$ as in Lemma 5.7 such that $\Upsilon(\zeta^r_{\lambda i} \otimes \zeta_{\lambda i}) = z^\lambda p_i$. Let $\mathscr{C}[[z^{-\tilde{\beta}} \mid \tilde{\beta} \in \Sigma^+ \setminus \tilde{\alpha}_i]]$ denote the subring of $\mathscr{C}[[z^{-\tilde{\alpha}_i} \mid \alpha_i \in \pi^*]]$ consisting of possibly infinite linear combinations of the $z^{-\nu}$ for $\nu \in \sum_{\beta \in \Sigma^+ \setminus \{\tilde{\alpha}_i\}} \mathbf{N}\beta$.

Lemma 6.4. *There exists $k_i \in \mathscr{C}[[z^{-\tilde{\beta}} \mid \tilde{\beta} \in \Sigma^+ \setminus \{\tilde{\alpha}_i\}]]$ such that $p = p_i k_i$. Furthermore, k_i has a nonzero constant term.*

Proof. By Lemma 5.7, p has a nonzero constant term. Hence if we can write $p = p_i k_i$, then both p_i and k_i have nonzero constant terms. Thus the second assertion follows from the first.

Fix $\lambda \in \mathfrak{h}^*_\Theta$. Using Lemma 5.6, choose $b \in \hat{N}^-$ such that $\zeta_\lambda = b v_\lambda$ and $b_i \in \hat{N}^-_i$ such that $\zeta_{\lambda i} = b_i v_\lambda$ Write

$$b_i = \sum_\delta b_{i\delta} \quad \text{and} \quad b^r_i = \sum_\delta b^r_{i\delta}$$

where each $b_{i\delta} \in (N_i^-)_{-\delta}$ and $b^r_{i\delta} \in (N_i^+)_{\delta}$. Note that

$$z^{\lambda} p_i = \sum_{\delta} \bar{\Upsilon}(v^r_{\lambda} b^r_{i\delta} \otimes b_{i\delta} v_{\lambda}) = \sum_{\delta} z^{\lambda-\delta}(\overline{\mathscr{P}}_{\mathscr{A}}(\mathrm{gr}\, b^r_{i\delta} b_{i\delta})).$$

Let $w = \sum_{\gamma} w_{\gamma}$ satisfy the conditions of Lemma 6.2. By Lemma 6.3, each γ with w_{γ} nonzero satisfies $-\gamma \in \sum_{\beta \in \Sigma^+ \setminus \{\tilde{\alpha}_i\}} \mathbf{N}\beta$. Choose γ so that $w_{\gamma} \neq 0$. Note that $U^+_{i+} w_{\gamma} v_{\lambda} = 0$. Using the identification of $\bar{M}(\lambda)$ with N^- as U^+ modules (Lemma 5.2), we obtain $\mathrm{gr}\,(\mathrm{ad}\, x_j) w_{\gamma} = 0$ for all $\alpha_j \in \pi_i$. Hence $\mathrm{gr}\, x_j w_{\gamma} = \mathrm{gr}\, t_j w_{\gamma} t_j^{-1} x_j$ for all $x_j \in U_i$. It follows that $\overline{\mathscr{P}}_{\mathscr{A}}(\mathrm{gr}\,(U U^+_{i+}) w_{\gamma}) = 0$. Similarly $\overline{\mathscr{P}}_{\mathscr{A}}(w^r_{\gamma}(\mathrm{gr}\, G^-_{i+} U)) = 0$. To make the next computation easier to read, we shorten $\overline{\mathscr{P}}_{\mathscr{A}}(\mathrm{gr}\, u)$ to $\overline{\mathscr{P}}_{\mathscr{A}}(u)$ for $u \in U$. Since $b_{i\delta} \in (N_i^-)_{-\delta}$ and $b^r_{i\delta} \in (N_i^+)_{\delta}$, it follows that

$$\begin{aligned}
\sum_{\delta} \overline{\mathscr{P}}_{\mathscr{A}}(w^r_{\gamma} b^r_{i\delta} \tau(\beta) b_{i\delta} w_{\gamma}) &= \sum_{\delta} \overline{\mathscr{P}}_{\mathscr{A}}(w^r_{\gamma} \overline{\mathscr{P}}_{\mathscr{A}}(b^r_{i\delta} \tau(\beta) b_{i\delta}) w_{\gamma}) \\
&= \sum_{\delta} q^{(-\delta,\beta)} \overline{\mathscr{P}}_{\mathscr{A}}(b^r_{i\delta} b_{i\delta}) \overline{\mathscr{P}}_{\mathscr{A}}(w^r_{\gamma} \tau(\beta) w_{\gamma}) \\
&= \overline{\mathscr{P}}_{\mathscr{A}}(w^r_{\gamma} \tau(\beta) w_{\gamma}) p_i \cdot \tau(\beta).
\end{aligned}$$

Now $\overline{\mathscr{P}}_{\mathscr{A}}(w^r_{\gamma} \tau(\beta) w_{\gamma})$ is equal to $a_{\gamma} z^{\gamma} \cdot \tau(\beta)$ for some scalar a_{γ} independent of β. Thus

$$\bar{\Upsilon}(v^r_{\lambda} b^r \otimes b v_{\lambda}) = z^{\lambda} \left(\sum_{\gamma} a_{\gamma} z^{\gamma} \right) p_i.$$

The lemma now follows from the fact that $-\gamma$ is in the $\mathbf{N}$ span of the set $\Sigma^+ \setminus \{\tilde{\alpha}_i\}$ whenever $a_{\gamma} \neq 0$. $\square$

Let $\tilde{s}_i$ be the reflection in W_{Θ} corresponding to the restricted root $\tilde{\alpha}_i$. Recall that $\tilde{s}_i$ restricts to a permutation on the set $\Sigma^+ \setminus \{\tilde{\alpha}_i\}$. Hence $\tilde{s}_i$ induces a linear map on $\mathscr{C}[[z^{-\tilde{\beta}} \mid \tilde{\beta} \in \Sigma^+ \setminus \{\tilde{\alpha}_i\}]]$ defined by

$$\tilde{s}_i \sum_{\tilde{\gamma}} z^{-\tilde{\gamma}} = \sum_{\tilde{\gamma}} z^{\tilde{s}_i(-\tilde{\gamma})}.$$

Lemma 6.5. *Choose $k_i \in \mathscr{C}[[z^{-\tilde{\beta}} \mid \tilde{\beta} \in \Sigma^+ \setminus \{\tilde{\alpha}_i\}]]$ such that $p = p_i k_i$. Then $\tilde{s}_i k_i = k_i$.*

Proof. Recall that ω_j denotes the fundamental weight corresponding to $\alpha_j \in \pi$. Given α_j and α_k in π^*, we know that $(\omega_j, \tilde{\alpha}_k)$ is a nonzero scalar multiple of δ_{jk}. Set $v_i = (\sum_{\{r \mid \alpha_r \in \pi^*\}} \omega_r) - \omega_i$. It follows that $(\tilde{\alpha}_j, v_i) \neq 0$ for all $j \neq i$ and $(\tilde{\alpha}_i, v_i) = 0$. Hence p_i commutes with $\tau(-2v_i)$. More generally,

$$z^{-\tilde{\beta}} \tau(-2v_i) = q^{(-2v_i, -\tilde{\beta})} \tau(-2v_i) z^{-\tilde{\beta}}$$

for all $\tilde{\beta} \in Q(\Sigma)$. Note that if $\tilde{\beta} \in \Sigma^+$ is not a scalar multiple of $\tilde{\alpha}_i$, then $(\tilde{\beta}, v_i) \neq 0$. Hence if $k \in \mathscr{C}[[z^{-\tilde{\beta}} \mid \tilde{\beta} \in \Sigma^+ \setminus \{\tilde{\alpha}_i\}]]$ and k commutes with $\tau(-2v_i)$ then k is a scalar.

Now by Corollary 3.3 and Theorems 3.4 and 5.8,

$$\mathcal{X}(c_{v_i}) = \sum_{w \in W_\Theta} w(k_i^{-1}\tau(-2\tilde{v}_i)k_i) + \sum_{\{\mu \in P^+(2\Sigma) \mid \mu < \tilde{v}_i\}} \sum_{w \in W_\Theta} w(f_\mu)\tau(-2w\mu)$$

up to a nonzero scalar for some $f_\mu \in \mathcal{C}(Q_\Sigma)$. Since $\mathcal{X}(c_{v_i})$ is $\tilde{s}_i$ invariant and $\tilde{s}_i\tau(-2v_i) = \tau(-2v_i)$, it follows that $\tilde{s}_i(k_i\tau(-2v_i)k_i^{-1}) = k_i\tau(-2v_i)k_i^{-1}$. Hence $\tilde{s}_ik_i = k_ik$ where k commutes with $\tau(-2v_i)$. Thus k must be a scalar. Since k_i has a nonzero constant term, and $\tilde{s}_i$ fixes constants, it follows that $k = 1$. □

Let a and x be an indeterminates and define

$$(x; a)_\infty = \prod_{i=0}^{\infty} (1 - xa^i). \tag{6.3}$$

Set $z_i = z^{2\tilde{\alpha}_i}$. Let ρ_i denote the half-sum of the positive roots in Δ_i. Note that ρ is just the sum of the fundamental weights corresponding to the simple roots in π. A similar statement can be made concerning ρ_i with respect to π_i. Hence $(\rho, \beta) = (\rho_i, \beta)$ for all $\beta \in Q(\pi_i)$.

For each $\tilde{\alpha} \in \Sigma^+$, we let $\mathcal{C}[[z^{-\tilde{\alpha}}]]$ denote the subring of $\mathcal{C}[[z^{-\tilde{\alpha}_i} \mid \alpha_i \in \pi^*]]$ consisting of elements of the form $\sum_{m \geqslant 0} a_m z^{-m\tilde{\alpha}}$ where the a_m are scalars. Recall that $z_i = z^{2\tilde{\alpha}_i}$. Using the rank one computations found in Section 4, we determine p_i.

Lemma 6.6. *Given $\alpha_i \in \pi^*$, we have*

$$p_i = \frac{(g_iz_i^{-1}; a_i)_\infty}{(z_i^{-1}; a_i)_\infty},$$

where $a_i = q^{(2\tilde{\alpha}_i, \tilde{\alpha}_i)}$ and $g_i = q^{2(\rho, \tilde{\alpha}_i)}$.

Proof. Let μ_i be chosen to satisfy the conditions of (4.7) with respect to $\mathfrak{g}_i$. By Theorems 4.7 and 5.8

$$p_i^{-1}\tilde{t}_i^{-1}p_i = \mathrm{gr}\,(\mathcal{X}(c'_{\mu_i})) = \tilde{t}_i^{-1}(1 - g_iz_i^{-1})(1 - z_i^{-1})^{-1}.$$

Note that $z_i^{-1}\tilde{t}_i^{-1} = a_i\tilde{t}_i^{-1}z_i^{-1}$. Hence,

$$\begin{aligned}\frac{(z_i^{-1}; a_i)_\infty}{(g_iz_i^{-1}; a_i)_\infty}\tilde{t}_i^{-1}\frac{(g_iz_i^{-1}; a_i)_\infty}{(z_i^{-1}; a_i)_\infty} &= \tilde{t}_i^{-1}\prod_{j=0}^{\infty}\frac{(1 - z_i^{-1}a_i^{j+1})}{(1 - g_iz_i^{-1}a_i^{j+1})}\frac{(1 - g_iz_i^{-1}a_i^j)}{(1 - z_i^{-1}a_i^j)} \\ &= \tilde{t}_i^{-1}(1 - g_iz_i^{-1})(1 - z_i^{-1})^{-1}.\end{aligned}$$

The lemma now follows from the fact that the only elements in $\mathcal{C}[[z^{-\tilde{\alpha}_i}]]$ which commute with $\tilde{t}_i$ are the scalars. □

Consider $\alpha \in \Delta$. Since Σ is reduced, $(\alpha, -\Theta(\alpha))$ equals 0 if $-\alpha \neq \Theta(\alpha)$ and equals (α, α) otherwise. Hence it is straightforward to check that $\tilde{\alpha} = \tilde{\beta}$ implies $(\alpha, \alpha) =$

(β, β) for all β in Δ. In particular, the length of a root whose restriction equals $\tilde{\alpha}$ is just a function of $\tilde{\alpha}$ and does not depend on the choice of root. Set

$$mult(\tilde{\alpha}) = |\{\beta \in \Delta | \tilde{\beta} = \tilde{\alpha}\}|.$$

We recall two well known facts about the *mult* function. First, $mult(\tilde{\alpha}_i) = 2(\rho, \tilde{\alpha}_i)/(\tilde{\alpha}_i, \tilde{\alpha}_i)$ for all $\alpha_i \in \pi^*$. Moreover, $mult(\tilde{\alpha}) = mult(w\tilde{\alpha})$ for all $w \in W_\Theta$ and $\alpha \in \Delta$.

Given $\tilde{\alpha} \in \Sigma^+$, set $a_{\tilde{\alpha}} = q^{(2\tilde{\alpha}, \tilde{\alpha})}$ and $g_{\tilde{\alpha}} = q^{mult(\tilde{\alpha})(\tilde{\alpha}, \tilde{\alpha})}$. Set

$$p_{\tilde{\alpha}} = \frac{(g_{\tilde{\alpha}} z^{-2\tilde{\alpha}}; a_{\tilde{\alpha}})_\infty}{(z^{-2\tilde{\alpha}}; a_{\tilde{\alpha}})_\infty}$$

for $\tilde{\alpha} \in \Sigma$. Note that $p_i = p_{\tilde{\alpha}_i}$. Furthermore, by the previous paragraph, $wp_{\tilde{\alpha}} = p_{\tilde{\beta}}$ whenever $w \in W_\Theta$ satisfies $w\tilde{\alpha} = \tilde{\beta}$. Now $p_{\tilde{\alpha}}$ is clearly an element of $\mathbf{C}((q))[[z^{-\tilde{\alpha}}]]$ where $\mathbf{C}((q))$ is the Laurent polynomial ring in the one variable q. However, by definition, $p_{\tilde{\alpha}_i}$ is actually an element of $\mathscr{C}[[z^{-2\tilde{\alpha}_i}]]$. Hence $p_{\tilde{\alpha}}$ is an element of $\mathscr{C}[[z^{-2\tilde{\alpha}}]]$ for each $\tilde{\alpha} \in \Sigma^+$.

Theorem 6.7. *Set*

$$p = \prod_{\tilde{\alpha} \in \Sigma^+} p_{\tilde{\alpha}}.$$

Then for each $\mu \in P^+(\pi)$,

$$\text{gr}\,(\mathscr{X}(c'_\mu)) = p^{-1}\tau(-2\tilde{\mu})p. \tag{6.4}$$

Proof. By Theorem 5.8, there exists an element p in $\mathscr{C}[[z^{-\tilde{\alpha}_i} \mid \alpha_i \in \pi^*]]$ such that $\text{gr}\,(\mathscr{X}(c'_\mu)) = p^{-1}\tau(-2\tilde{\mu})p$ for all $\mu \in P^+(\pi)$. Since each $p_{\tilde{\alpha}}$ is an element of $\mathscr{C}[[z^{-\tilde{\alpha}}]]$, it follows that $\prod_{\tilde{\alpha} \in \Sigma^+} p_{\tilde{\alpha}}$ is an element of $\mathscr{C}[[z^{-\tilde{\alpha}_i} \mid \alpha_i \in \pi^*]]$. Choose p' in $\mathscr{C}[[z^{-\tilde{\alpha}_i} \mid \alpha_i \in \pi^*]]$ so that $pp' = \prod_{\tilde{\alpha} \in \Sigma^+} p_{\tilde{\alpha}}$.

Set $k'_i = \prod_{\tilde{\alpha} \in \Sigma^+ \setminus \{\tilde{\alpha}_i\}} p_{\tilde{\alpha}}$. Note that $p_i k'_i = pp'$. Choose k_i as in Lemma 6.4. It follows that $k_i p' = k'_i$ for each i such that $\alpha_i \in \pi^*$. Write $k'_i = \sum_{\tilde{\gamma} \geqslant 0} d'_{\tilde{\gamma}} z^{-\tilde{\gamma}}$, $k_i = \sum_{\tilde{\gamma} \geqslant 0} d_{\tilde{\gamma}} z^{-\tilde{\gamma}}$, and $p' = \sum_{\tilde{\gamma} \geqslant 0} a_{\tilde{\gamma}} z^{-\tilde{\gamma}}$.

Note that by the definition of k'_i and Lemma 6.5,

$$\sum_{\tilde{\gamma} \geqslant 0} d'_{\tilde{\gamma}} z^{-\tilde{s}_i \tilde{\gamma}} = k'_i \quad \text{and} \quad \sum_{\tilde{\gamma} \geqslant 0} d_{\tilde{\gamma}} z^{-\tilde{s}_i \tilde{\gamma}} = k_i,$$

where equality holds in $\mathscr{C}[[z^{-\tilde{\alpha}_i} \mid \alpha_i \in \pi^*]]$. It follows that

$$\sum_{\tilde{\gamma} \geqslant 0} a_{\tilde{\gamma}} z^{-\tilde{s}_i \tilde{\gamma}} = p'$$

in $\mathscr{C}[[z^{-\tilde{\alpha}_i} \mid \alpha_i \in \pi^*]]$ for all simple reflections $\tilde{s}_i \in W_\Theta$. Hence

$$\sum_{\tilde{\gamma} \geqslant 0} a_{\tilde{\gamma}} z^{-w\tilde{\gamma}} = p' \tag{6.5}$$

in $\mathscr{C}[[z^{-\tilde{\alpha}_i} \mid \alpha_i \in \pi^*]]$ for all $w \in W_\Theta$. Suppose $\tilde{\gamma} > 0$ is such that $a_{\tilde{\gamma}}$ is nonzero. We can find $w \in W_\Theta$ such that $w\tilde{\gamma} < 0$. But then $z^{-w\tilde{\gamma}} \notin \mathscr{C}[[z^{-\tilde{\alpha}_i} \mid \alpha_i \in \pi^*]]$. This contradicts (6.5). Hence p' must be a scalar and so k_i is a scalar multiple of k_i'. $\square$

Let p be defined as in Theorem 6.7. It follows that, up to a nonzero scalar, p is the unique element of $\mathscr{C}[[z^{-\tilde{\alpha}_i} \mid \alpha_i \in \pi^*]]$ such that (6.4) holds for all $\mu \in P^+(\pi)$. Recall the definition of the projection map $\mathscr{P}_{\mathscr{A}}$ given at the end of Section 3. The next result extends Theorem 6.7 to other elements in $Z(\check{U})$ and, more generally, to elements of $\check{U}^B$.

Corollary 6.8. *For each* $c \in \check{U}^B$,

$$\mathrm{gr}\,(\mathscr{X}(c)) = p^{-1}\mathrm{gr}\,(\mathscr{P}_{\mathscr{A}}(c))p,$$

where

$$p = \prod_{\tilde{\alpha} \in \Sigma^+} p_{\tilde{\alpha}}.$$

Moreover, $c \notin B_+\check{U}$ *if and only if* $\mathscr{P}_{\mathscr{A}}(c) \neq 0$.

Proof. Let $c \in \check{U}^B$. The definition of the map $\mathscr{P}_{\mathscr{A}}$ ensures that $\mathscr{P}_{\mathscr{A}}(\check{U})$ is contained in $\mathscr{C}[\check{\mathscr{A}}]$. Thus $z^\lambda(\mathscr{P}_{\mathscr{A}}(c)) = 0$ for all $\lambda \in P^+(2\Sigma)$ if and only if $\mathscr{P}_{\mathscr{A}}(c) = 0$. Hence, by Theorem 3.6, $\mathscr{X}(c) = 0$ if and only if $\mathscr{P}_{\mathscr{A}}(c) = 0$.

Suppose $c \in B_+\check{U}$. Then $\mathscr{X}(c) = 0$. By the previous paragraph, $\mathscr{P}_{\mathscr{A}}(c) = 0$. Thus the lemma follows trivially in this case. Thus we may assume that $c \notin B_+\check{U}$.

Let $c' \in \check{\mathscr{A}}N^+$ such that $c - c' \in B_+\check{U}$. In particular, $\mathscr{X}(c) = \mathscr{X}(c')$. We can write $\mathrm{gr}\, c' = (\mathrm{gr}\, an) + (\mathrm{gr}\, n')$ where a is in $\mathscr{C}[\check{\mathscr{A}}]$, n is a weight vector in N^+ of weight γ, and $n' \in \sum_{\beta > \gamma} \check{\mathscr{A}}N^+_\beta$. Note that $a \neq 0$ since $c \notin B_+\check{U}$. Choose $\lambda \in P^+(2\Sigma)$ such that $z^\lambda(\mathrm{gr}\, a) \neq 0$. Since v^r_λ generates the gr U module $\bar{M}(\lambda)^r$, it follows that $v^r_\lambda n \neq 0$. Hence

$$\zeta^r_\lambda(\mathrm{gr}\, c') \in v^r_\lambda(\mathrm{gr}\, c') + \sum_{\beta > 0} v^r_\lambda N^+_\beta(\mathrm{gr}\, c') = z^\lambda(\mathrm{gr}\, a)v^r_\lambda n + \sum_{\beta > \gamma} v^r_\lambda N^+_\beta. \tag{6.6}$$

In particular, $\zeta^r_\lambda(\mathrm{gr}\, c')$ is nonzero.

Given $b \in B$, we have $c'b = cb + (c' - c)b \in bc + B_+\check{U}$. Thus $c'b \in B_+\check{U}$ for all $b \in B_+$. Hence

$$\zeta^r_\lambda(\mathrm{gr}\, c')B_+ = 0.$$

Therefore, by the discussion following Lemma 5.5, $\zeta^r_\lambda(\mathrm{gr}\, c')$ is a scalar multiple of ζ^r_λ. This forces n to be an element of $\mathscr{C}$. Without loss of generality, we may assume that $n = 1$. Thus (6.6) implies that $\zeta^r_\lambda(\mathrm{gr}\, c') = z^\lambda(\mathrm{gr}\, a)\zeta^r_\lambda$, which is a nonzero multiple of ζ^r_λ. Using the definition of the map $\mathscr{P}_{\mathscr{A}}$ (end of Section 3), we see that $\mathrm{gr}\, a = \mathrm{gr}\,(\mathscr{P}_{\mathscr{A}}(c')) = \mathrm{gr}\,(\mathscr{P}_A(c))$. Hence $\mathscr{P}_A(c)$ is nonzero, which completes the proof of the second assertion of the lemma.

Arguing as in Theorem 5.8, yields

$$\mathrm{gr}\,(\mathscr{X}(c')) = p^{-1}\mathrm{gr}\,(\mathscr{P}_A(c'))p.$$

Hence $\mathrm{gr}\,(\mathscr{X}(c)) = p^{-1}\mathrm{gr}\,(\mathscr{P}_A(c))p$. The first assertion now follows from Theorem 6.7. $\square$

An immediate consequence of Corollary 6.8 is that

$$\check{U}^B \cap (\check{\mathscr{A}}N_+^+ + (B\check{T}_\Theta)_+\check{U}) \subset (B\check{T}_\Theta)_+\check{U}.$$

A similar argument switching the right and left actions yields that

$$\check{U}^B \cap (\check{\mathscr{A}}N_+^+ + \check{U}(B\check{T}_\Theta)_+) \subset \check{U}(B\check{T}_\Theta)_+.$$

7. Minuscule and pseudominuscule weights

In this section, we find "small" elements in $\check{U}^B$ which correspond to minuscule or pseudominuscule weights of Σ and determine their radial components. For all but three types of irreducible symmetric pairs $\mathfrak{g}, \mathfrak{g}^\theta$, this small element is c'_μ plus a constant term where $\tilde{\mu}$ is either a minuscule or a pseudominuscule weight in Σ. Most of this section is devoted to finding a suitable element in $\check{U}^B$ in the remaining three problematic cases. This involves a separate construction using fine information about finite dimensional $\mathrm{ad}_r\, U$ submodules of $\check{U}$.

Recall that $\mathfrak{g}, \mathfrak{g}^\theta$ is an irreducible symmetric pair and Σ is reduced. It follows from the classification of irreducible symmetric pairs that Σ is an irreducible root system corresponding to a simple Lie algebra as classified in [7, Chapter III]. Before discussing elements of $\check{U}^B$, we briefly review facts concerning minuscule and pseudominuscule weights associated to root systems of simple Lie algebras. Since we will be applying this information to the restricted root system, all the results will be stated with respect to Σ.

A fundamental weight β is called minuscule with respect to the root system Σ if

$$0 \leqslant \frac{2(\beta,\alpha)}{(\alpha,\alpha)} \leqslant 1 \tag{7.1}$$

for all $\alpha \in \Sigma^+$. Since Σ is simple, it admits a minuscule weight if and only if it is not of type $E8, F4$, or $G2$. It is straightforward to check that the minuscule weights for Σ are exactly the smallest fundamental weights not contained in $Q^+(\Sigma)$. In particular, a minuscule weight β satisfies the following condition:

$$\text{There does not exist } \gamma \in P^+(\Sigma) \text{ such that } \beta - \gamma \in Q^+(\Sigma)\backslash\{0\}. \tag{7.2}$$

The longest root of Σ, when Σ is simply laced, and the longest short root of Σ, when Σ has two root lengths, is called a pseudominuscule weight.

A pseudominuscule weight β satisfies (7.1) for all $\alpha \in \Sigma^+ \backslash \beta$. Moreover, one checks easily that a pseudominuscule weight β satisfies the following condition similar to (7.2):

$$\text{The weight } \gamma \in P^+(\Sigma) \text{ satisfies } \beta - \gamma \in Q^+(\Sigma) \backslash \{0\} \text{ if and only if } \gamma = 0. \tag{7.3}$$

Recall the definition of the element p given in Theorem 6.7. Using the previous sections, we can compute the image of c_μ under $\mathscr{X}$ when μ satisfies one of the above conditions. In particular, we have the following.

Lemma 7.1. (i) *If $\mu \in P^+(\pi)$ is such that $\tilde{\mu}$ is minuscule in Σ then there exists a central element c in $Z(\check{U})$ with*

$$\mathscr{X}(c) = \sum_{w \in W_\Theta} w(p^{-1}\tau(-2\tilde{\mu})p).$$

(ii) *If $\mu \in P^+(\pi)$ is such that $\tilde{\mu}$ is pseudominuscule in Σ then there exists a central element c in $Z(\check{U})$ with*

$$\mathscr{X}(c) = \sum_{w \in W_\Theta} w((1 - \tau(2\tilde{\mu}))p^{-1}\tau(-2\tilde{\mu})p).$$

Proof. Suppose first that there exists $\mu \in P^+(\pi)$ such that $\tilde{\mu}$ is minuscule in Σ. Let $(\mathscr{C}(Q_\Sigma)\mathscr{A}_\geqslant)_+$ denote the subalgebra of $\mathscr{C}(Q_\Sigma)\mathscr{A}_\geqslant$ generated by $\mathscr{C}(Q_\Sigma)$ and $\mathscr{C}[\mathscr{A}_\geqslant]_+$. By Theorem 6.7 and Corollary 3.3,

$$\mathscr{X}(c'_\mu) \in p^{-1}\tau(-2\tilde{\mu})p + (\mathscr{C}(Q_\Sigma)\mathscr{A}_\geqslant)_+\tau(-2\tilde{\mu}).$$

Note that since $\tilde{\mu}$ satisfies (7.2), there does not exist $\tau(-2\tilde{\beta}) \in \mathscr{C}[\mathscr{A}_\geqslant]\tau(-2\tilde{\mu})$ such that $\tilde{\beta}$ is dominant. Hence by Theorem 3.4,

$$\mathscr{X}(c'_\mu) = \sum_{w \in W_\Theta} w(p^{-1}\tau(-2\tilde{\mu})p).$$

This proves (i).

Now assume that there exists $\mu \in P^+(\pi)$ such that $\tilde{\mu}$ is pseudominuscule in Σ. The same reasoning as in the previous paragraph, yields that

$$\mathscr{X}(c'_\mu) = \sum_{w \in W_\Theta} w(p^{-1}\tau(-2\tilde{\mu})p) + g$$

for some $g \in \mathscr{C}(Q_\Sigma)$. Now the zonal spherical functions φ_λ are eigenvectors for the action of $\mathscr{X}(c'_\mu)$. When $\lambda = 0$, φ_λ is just 1. Hence the action of $\mathscr{X}(c'_\mu)$ on 1 must be a scalar. Now the action of

$$\sum_{w \in W_\Theta} w(p^{-1}\tau(-2\tilde{\mu})p) - \sum_{w \in W_\Theta} w(\tau(2\tilde{\mu})p^{-1}\tau(-2\tilde{\mu})p)$$

on 1 is zero. Furthermore, $w(\tau(2\tilde{\mu})p^{-1}\tau(-2\tilde{\mu})p)$ is in $\mathscr{C}(Q_\Sigma)$ for each $w\in W_\Theta$. Hence $g+\sum_{w\in W_\Theta} w(\tau(2\tilde{\mu})p^{-1}\tau(-2\tilde{\mu})p)$ is a scalar, say g_0. It follows that $\mathscr{X}(c'_\mu - g_0)$ has the required form. $\square$

Assume for the moment that $\mathfrak{g},\mathfrak{g}^\theta$ is not of type EIV, EVII, or EIX. (See the Appendix, Section 9, for explicit descriptions of the types of irreducible symmetric pairs.) Suppose that β is a minuscule weight or pseudominuscule weight in $P^+(\Sigma)$. A straightforward computation shows that $P^+(\pi)$ contains a fundamental weight μ such that $\beta=\tilde{\mu}$. These values of μ and $\tilde{\mu}$ are given in the appendix.

Now assume that $\mathfrak{g},\mathfrak{g}^\theta$ is of type EIV. Then Σ is of type A3 with set of simple roots $\{\tilde{\alpha}_1,\tilde{\alpha}_6\}$. Recall that ω_i denotes the fundamental weight corresponding to the simple root α_i. For $i=1$ and $i=6$, let ω'_i denote the fundamental weight in the weight lattice of Σ corresponding to $\tilde{\alpha}_i$. Note that both ω'_1 and ω'_6 are minuscule. It is straightforward to check that neither ω'_1 nor ω'_6 is in the span of the set $\{\tilde{\omega}_i \mid 1\leqslant i\leqslant 6\}$. A similar computation shows that if $\mathfrak{g},\theta$ is of type EVII or EIX then the minuscule or pseudominuscule weights associated Σ are not the restriction of elements in $P^+(\pi)$. Thus, in these special cases, $Z(\check{U})$ does not contain elements whose radial components are of the form described in Lemma 7.1. The remainder of this section is devoted to finding elements in $\check{U}^B$ in the remaining cases which play the role of c_μ for μ minuscule or pseudominuscule.

We recall basic facts about the structure of $\check{U}$ as a U module with respect to the adjoint action (see [11,12], or [8, Section 7]). Here we use the right adjoint action instead of the left and will translate the results accordingly. For each $\eta\in P^+(\pi)$, observe that $\tau(2\eta)$ generates a finite dimensional $\mathrm{ad}_r\, U$ module. Let $F_r(\check{U})$ denote the locally finite part of $\check{U}$ with respect to the right adjoint action. One has that $F_r(\check{U})$ is a subalgebra of $\check{U}$. As an $(\mathrm{ad}_r\, U)$ module, $F_r(\check{U})$ is isomorphic to the direct sum of the $(\mathrm{ad}_r\, U)\tau(2\eta)$ as η runs over elements in $P^+(\pi)$.

Let G^+ be the subalgebra of U generated by the $x_jt_j^{-1}$ for $j=1,\dots,n$. By [10, Theorem 3.3], one can construct a basis of $(\mathrm{ad}_r\, U)\tau(2\eta)$ consisting of weight vectors contained in sets of the form

$$a_{-\beta}b_{\beta'}\tau(2\eta)+\sum_{\xi\in Q^+(\pi)}\ \sum_{\gamma\leqslant\beta-\xi}\ \sum_{\gamma'\leqslant\beta'-\xi} U^-_{-\gamma}G^+_{\gamma'}\tau(2\eta-2\xi), \tag{7.4}$$

where $a_{-\beta}\in U^-_{-\beta}$ and $b_{\beta'}\in G^+_{\beta'}$. Moreover, $0\leqslant\xi\leqslant\eta-w_0\eta$ and both β and β' are less than or equal to $\eta-w_0\eta$ [8, 7.1.20].

Lemma 7.2. *Let $\eta\in P^+(\pi)$ and suppose that $u\in(\mathrm{ad}_r\, U)\tau(2\eta)$. Then*

$$\mathscr{P}_{\mathscr{A}}(u)\in\tau(2\widetilde{w_0\eta})\mathscr{C}[\mathscr{A}_\geqslant]. \tag{7.5}$$

Proof. Suppose that $u\in U^-_{-\gamma}G^+_{\gamma'}\tau(2\eta-2\xi)$ where ξ and γ are elements of $Q^+(\pi)$ such that $0\leqslant\xi\leqslant\eta-w_0\eta$ and $0\leqslant\gamma\leqslant\eta-w_0\eta-\xi$. We argue that u satisfies (7.5). The lemma then follows by the linearity of $\mathscr{P}_{\mathscr{A}}$ and the description of a basis of $(\mathrm{ad}_r\, U)\tau(2\eta)$ using (7.4).

Note that (2.7) implies that $U_+^+ \check{U}^0 \subseteq \mathcal{M}_+^+ N^+ \check{U}^0 + N_+^+ \check{\mathcal{A}}$. Hence $\mathcal{P}_{\mathcal{A}}(U_+^+ \check{U}^0) = 0$. It further follows from (3.11) that

$$\check{U} U_+^+ \check{U}^0 \subseteq ((B\check{T}_\Theta)_+ \check{U} + N_+^+ \check{\mathcal{A}} + \mathscr{C}[\check{\mathcal{A}}]) U_+^+ \check{U}^0 \subseteq (B\check{T}_\Theta)_+ \check{U} + N_+^+ \check{\mathcal{A}}.$$

Hence $\mathcal{P}_{\mathcal{A}}(U^- G_+^+ \check{U}^0) = 0$. Thus, we may reduce to the case where $\gamma' = 0$ and so $u \in U_{-\gamma}^- \tau(2\eta - 2\xi)$. It follows that $u \in G_{-\gamma}^- \tau(2\eta - 2\xi - \gamma)$.

By Lemma 2.1 and (2.10), we have that

$$\mathcal{P}_{\mathcal{A}}(u) \in \tau(2\tilde{\eta} - 2\tilde{\xi} - \tilde{\gamma}) \mathscr{C}[\mathcal{A}_{\geqslant}].$$

To complete the proof of the lemma, we argue that $\tau(2\tilde{\eta} - 2\tilde{\xi} - \tilde{\gamma}) \in \tau(2\widetilde{w_0 \eta}) \mathscr{C}[\mathcal{A}_{\geqslant}]$. Our assumptions on ξ and γ ensure that $2\eta - 2\xi - \gamma \geqslant 2w_0\eta$. Furthermore, $2\tilde{\eta}$, $2\tilde{\xi}$, and $2w_0\tilde{\eta}$ are all elements of $Q(2\Sigma)$. Thus it is sufficient to show that $\tilde{\gamma} \in Q(2\Sigma)$ whenever $\mathcal{P}_{\mathcal{A}}(u) \neq 0$.

Note that $u\tau(-2\eta + 2\xi - \gamma)$ can be written as a sum of monomials $y_{i_1} t_{i_1} \cdots y_{i_m} t_{i_m}$ in the $y_i t_i$ of weight γ. Assume that $y_{i_1} \notin B_+$. Arguing as in the proof of Lemma 2.1, we obtain

$$y_{i_1} t_{i_1} \cdots y_{i_m} t_{i_m} \in -\tilde{\theta}(y_{i_1}) t_{i_1} y_{i_2} t_{i_2} \cdots y_{i_m} t_{i_m} + B_+ y_{i_2} t_{i_2} \cdots y_{i_m} t_{i_m}.$$

Thus by (2.5)

$$y_{i_1} t_{i_1} \cdots y_{i_m} t_{i_m} \in G^- \tilde{\theta}(y_{i_1}) t_{i_1} + G_{-\gamma + 2\tilde{\alpha}_i}^- T'_{\geqslant} T_\Theta + B_+ U.$$

It follows that $\mathcal{P}_{\mathcal{A}}(y_{i_1} t_{i_1} \cdots y_{i_m} t_{i_m}) \in \mathcal{P}_{\mathcal{A}}(G_{-\gamma+2\tilde{\alpha}_i}^- T'_{\geqslant} T_\Theta)$. Furthermore,

$$\mathcal{P}_{\mathcal{A}}(G_{-\gamma+2\tilde{\alpha}_i}^- T'_{\geqslant} T_\Theta) = \mathcal{P}_{\mathcal{A}}(G_{-\gamma+2\tilde{\alpha}_i}^-) \mathcal{P}_{\mathcal{A}}(T'_{\geqslant} T_\Theta).$$

By induction on γ, we obtain $\mathcal{P}_{\mathcal{A}}(G_{-\gamma}^-) = 0$ unless $\tilde{\gamma} \in Q^+(2\Sigma)$. Therefore $\mathcal{P}_{\mathcal{A}}(u) \neq 0$ implies $\tilde{\gamma} \in Q^+(2\Sigma)$. $\square$

Let ϕ be the Hopf algebra automorphism of U which fixes elements in $\mathcal{M}T$, $\phi(x_i) = q^{(-2\rho, \tilde{\alpha}_i)} x_i$ for all $1 \leqslant i \leqslant n$, and $\phi(y_i) = q^{(2\rho, \tilde{\alpha}_i)} y_i$ for all $1 \leqslant i \leqslant n$. Note that $\phi(B_i) = q^{(\rho, -\Theta(\alpha_i) - \alpha_i)} (q^{(2\rho, \alpha_i)} y_i t_i + q^{(2\rho, \Theta(\alpha_i))} \tilde{\theta}(y_i) t_i)$ for all $\alpha_i \notin \pi_\Theta$. The next lemma provides information about $(\mathrm{ad}_r\, B_+)\check{U}$ which will be applied later to elements of $\check{U}^B$.

Lemma 7.3. *Suppose* $u \in (\mathrm{ad}_r\, B_+)\check{U}$. *Then* $u \in (\phi(B)\check{T}_\Theta)_+ \check{U} + \check{U}(B\check{T}_\Theta)_+$.

Proof. Suppose $u \in (\mathrm{ad}_r\, (\mathcal{M}T_\Theta)_+)\check{U}$. Now $\mathcal{M}T_\Theta$ is a Hopf subalgebra of U. Hence $(\mathrm{ad}_r\, (\mathcal{M}T_\Theta)_+)\check{U} \subset (\mathcal{M}T_\Theta)_+ \check{U} + \check{U}(\mathcal{M}T_\Theta)_+$. The lemma now follows in this case since $\mathcal{M}T_\Theta$ is a subalgebra of both $\phi(B)$ and B. Thus, it is sufficient to show that $(\mathrm{ad}_r\, B_i)a \in (\phi(B)\check{T}_\Theta)_+ \check{U} + \check{U}(B\check{T}_\Theta)_+$ for all i such that $\alpha_i \notin \pi_\Theta$ and $a \in \check{U}$.

Fix i with $\alpha_i \notin \pi_\Theta$. By [20, Theorem 7.1], there exists a sequence $i_1, \ldots, i_s$, with $\alpha_{i_j} \in \pi_\Theta$ for $1 \leqslant j \leqslant s$, and a nonzero scalar g such that $\tilde{\theta}(y_i) = g(\mathrm{ad}_r\, x_{i_1}) \ldots (\mathrm{ad}_r\, x_{i_s})(t_{\mathrm{p}(i)}^{-1} x_{\mathrm{p}(i)})$. Using the fact that $(\mathrm{ad}_r\, x_j)a = -t_j^{-1} x_j a + t_j^{-1} a x_j$ for all j and for all $a \in \check{U}$, we obtain

$$\tilde{\theta}(y_i)t_i \in \mathscr{M}_+ U + g\tau(\Theta(\alpha_i))x_{\mathrm{p}(i)}x_{i_s}\cdots x_{i_1}t_i \tag{7.6}$$

and

$$\tilde{\theta}(y_i)t_i \in U\mathscr{M}_+ + (-1)^s g t_{i_1}^{-1}x_{i_1}\cdots t_{i_s}^{-1}x_{i_s}t_{\mathrm{p}(i)}^{-1}x_{\mathrm{p}(i)}t_i. \tag{7.7}$$

Note that $\tau(\Theta(-\alpha_i))t_i^{-1} \in T_\Theta$. Applying the antipode to (7.7) and using (7.6) to simplify yields

$$\begin{aligned}\sigma(\tilde{\theta}(y_i)t_i) &\in \mathscr{M}_+ U - g t_i^{-1}t_{\mathrm{p}(i)}^{-1}x_{\mathrm{p}(i)}t_{\mathrm{p}(i)}t_{i_s}^{-1}x_{i_s}t_{i_s}\cdots t_{i_1}^{-1}x_{i_1}t_{i_1}\\ &= (\mathscr{M}T_\Theta)_+ U - q^{(2\rho,\Theta(\alpha_i))}g\tau(\Theta(\alpha_i))x_{\mathrm{p}(i)}x_{i_s}\cdots x_{i_1}\\ &= (\mathscr{M}T_\Theta)_+ U - q^{(2\rho,\Theta(\alpha_i))}\tilde{\theta}(y_i).\end{aligned}$$

Hence

$$\begin{aligned}\sigma(B_i) &= \sigma(y_i t_i + \tilde{\theta}(y_i)t_i)\\ &= -q^{(2\rho,\alpha_i)}y_i - q^{(2\rho,\Theta(\alpha_i))}\tilde{\theta}(y_i) + (\mathscr{M}T_\Theta)_+ U\\ &= -q^{(\rho,\Theta(\alpha_i)+\alpha_i)}\phi(B_i)t_i^{-1} + (\mathscr{M}T_\Theta)_+ U.\end{aligned}$$

Recall that $\Delta(y_i t_i) = y_i t_i \otimes 1 + t_i \otimes y_i t_i$. It follows from [20, (7.14)] that

$$\Delta(\tilde{\theta}(y_i)t_i) \in \tilde{\theta}(y_i)t_i \otimes \tau(\Theta(\alpha_i))t_i + t_i \otimes \tilde{\theta}(y_i)t_i + U \otimes (\mathscr{M}T_\Theta)_+.$$

Note that $\tau(\Theta(\alpha_i))t_i \in T_\Theta$. Thus $\tau(\Theta(\alpha_i))t_i \in 1 + \mathscr{C}[T_\Theta]_+$. Combining the formulas for $\Delta(y_i t_i)$ and $\Delta(\tilde{\theta}(y_i))$ gives us

$$\Delta(B_i) \in B_i \otimes 1 + t_i \otimes B_i + U \otimes (\mathscr{M}T_\Theta)_+.$$

Hence $(\mathrm{ad}_r\, B_i)a \in -q^{(\rho,\Theta(\alpha_i)+\alpha_i)}\phi(B_i)t_i^{-1}a + t_i^{-1}aB_i + (\mathscr{M}T_\Theta)_+\check{U} + \check{U}(\mathscr{M}T_\Theta)_+$ for all $a \in \check{U}$. □

Recall the definition of the Hopf algebra automorphism χ following the proof of Theorem 4.5. In particular, $\chi(x_i) = q^{(\rho,\tilde{\alpha}_i)}x_i$ for all $1 \leqslant i \leqslant n$. Since $(\rho,\tilde{\alpha}_i) = (\tilde{\rho},\alpha_i)$ for all i, it follows that $\tau(-\tilde{\rho})x_i\tau(\tilde{\rho}) = \phi\chi(x_i)$ for $1 \leqslant i \leqslant n$. Similarly, $\tau(-\tilde{\rho})y_i\tau(\tilde{\rho}) = \phi\chi(y_i)$ for all $1 \leqslant i \leqslant n$. Thus $\tau(-\tilde{\rho})u\tau(\tilde{\rho}) = \phi\chi(u)$ for all $u \in U$.

Lemma 7.4. *Suppose that $a' \in \check{U}^B$ and $a \in \check{U}^0$ such that*

$$a' \in a + \phi(B)_+\check{U} + \check{U}(\mathscr{M}T_\Theta)_+ + (\mathrm{ad}_r\, B_+)F_r(\check{U}).$$

Then

$$(\varphi * \mathscr{X}(a'))(\tau(\tilde{\rho})) = \varphi(a\tau(\tilde{\rho})) \tag{7.8}$$

for all $\varphi \in \mathscr{C}[P(2\Sigma)]^{W_\Theta}$.

Proof. By [21, Corollary 5.4], $\{\varphi_\lambda \mid \lambda \in P^+(2\Sigma)\}$ is a basis for $\mathscr{C}[P(2\Sigma)]^{W_\Theta}$. Hence it is sufficient to verify (7.8) when $\varphi = \varphi_\lambda$ for $\lambda \in P^+(2\Sigma)$. We use here facts from Section 1 introduced before Theorem 1.1. In particular, ${}_{\phi\chi(B)}\mathscr{H}_{\phi(B)}$ is the subspace of $R_q[G]$ consisting of left $\phi\chi(B)$ and right $\phi(B)$ invariants. Moreover, the space ${}_{\phi\chi(B)}\mathscr{H}_{\phi(B)}$ contains a distinguished basis $\{g'_\lambda \mid \lambda \in P^+(2\Sigma)\}$ such that the set $\{\Upsilon(g'_\lambda) \mid \lambda \in P^+(2\Sigma)\}$ satisfies (1.3). For each $\lambda \in P^+(2\Sigma)$, $\Upsilon(g'_\lambda)$ is written as $\varphi^\lambda_{\phi(B),\phi\chi(B)}$. By [21, Theorem 6.3], $\varphi^\lambda_{\phi(B),\phi\chi(B)} = \varphi^\lambda_{B,B'} = \varphi_\lambda$ for all $\lambda \in P^+(2\Sigma)$.

The discussion preceding this lemma implies that

$$a'\tau(\tilde{\rho}) \in a\tau(\tilde{\rho}) + \phi(B)_+\check{U} + \check{U}\phi\chi(B)_+.$$

Hence

$$g'_\lambda(a'\tau(\tilde{\rho})) = \varphi_\lambda(a\tau(\tilde{\rho})).$$

Recall that $g'_\lambda \in L(\lambda)^{\phi\chi(B)} \otimes (L(\lambda)^*)^{\phi(B)}$. Thus arguing as in Section 3 (preceding Theorem 3.6),

$$g'_\lambda(a'\tau(\tilde{\rho})) = z^\lambda(\mathscr{P}_{\mathscr{A}}(a'))g'_\lambda(\tau(\tilde{\rho})). \tag{7.9}$$

The lemma now follows from Theorem 3.6 and the fact that $g'_\lambda(\tau(\tilde{\rho})) = \varphi_\lambda(\tau(\tilde{\rho}))$. $\square$

By [21, Theorem 3.1], $F_r(\check{U})$ can be written as a direct sum of finite dimensional simple $(\mathrm{ad}_r B)$ modules. Consider an element $a \in F_r(\check{U})$. Since the action of $\mathrm{ad}_r B$ on $F_r(\check{U})$ is locally finite, $(\mathrm{ad}_r B)a$ is a finite dimensional submodule of $F_r(\check{U})$. Now suppose that $a \notin (\mathrm{ad}_r B_+)\check{U}$. In particular, $(\mathrm{ad}_r B_+)a$ has codimension 1 in $(\mathrm{ad}_r B)a$. Since the action of $(\mathrm{ad}_r B)$ on $F_r(\check{U})$ is completely reducible, it follows that there exists a nonzero element $a' \in \check{U}^B$ such that

$$(\mathrm{ad}_r B)a = \mathscr{C}a' \oplus (\mathrm{ad}_r B_+)a.$$

We may further assume that a' has been chosen so that

$$a' \in a + (\mathrm{ad}_r B_+)a.$$

Now $(\mathrm{ad}_r B_+)\check{U} \cap \check{U}^B = 0$. Hence the choice of a' is unique. Thus we have a linear map $L : F_r(\check{U}) \to \check{U}^B$ such that $L(a)$ is the unique element of $\check{U}^B$ in the set $a + (\mathrm{ad}_r B_+)a$. (Note that L is very similar to the so-called Letzter map studied in [9].)

Recall that W_Θ acts on $\mathscr{C}[\check{\mathscr{A}}]$ (see the discussion following the proof of Corollary 3.3). Given $w \in W_\Theta$ and $u \in \mathscr{C}[\check{\mathscr{A}}]$ we write $w \cdot u$ for the action of w on u.

Lemma 7.5. *Suppose that d' is an element of $F_r(\check{U})$ such that $d' \in a + \phi(B)_+\check{U} + \check{U}(\mathscr{M}T_\Theta)_+$ where $a \in \mathscr{C}[\check{\mathscr{A}}]$ and*

$$\sum_{w \in W_\Theta} w \cdot (a\tau(\tilde{\rho})) \neq 0.$$

Then $\mathscr{P}_\mathscr{A}(L(d'))$ is nonzero.

Proof. It is straightforward to check that the bilinear pairing between $\mathscr{C}[P(2\Sigma)]^{W_\Theta}$ and $\mathscr{C}[\check{\mathscr{A}}]^{W_\Theta}$ given by $\langle f, k \rangle = f(k)$ for all $f \in \mathscr{C}[P(2\Sigma)]^{W_\Theta}$ and $k \in \mathscr{C}[\check{\mathscr{A}}]^{W_\Theta}$ is nondegenerate. Note that

$$\varphi(a\tau(\tilde{\rho})) = |W_\Theta|^{-1}\varphi\left(\sum_{w \in W_\Theta} w \cdot (a\tau(\tilde{\rho}))\right)$$

for all $\varphi \in \mathscr{C}[P(2\Sigma)]^{W_\Theta}$. Hence the assumptions on a ensure that there exists $\varphi \in \mathscr{C}[P(2\Sigma)]^{W_\Theta}$ such that $\varphi(a\tau(\tilde{\rho})) \neq 0$.

By the previous lemma and (7.9), $g_\lambda(L(d')\tau(\tilde{\rho})) = z^\lambda(\mathscr{P}_\mathscr{A}(L(d'))g_\lambda(\tau(\tilde{\rho})) = \varphi_\lambda(a\tau(\tilde{\rho}))$. Thus it is sufficient to find λ such that $\varphi_\lambda(a\tau(\tilde{\rho}))$ is nonzero. But the $\{\varphi_\lambda \mid \lambda \in P^+(2\Sigma)\}$ form a basis for $\mathscr{C}[P(2\Sigma)]^{W_\Theta}$. The result now follows using the nondegenerate pairing described in the first paragraph. □

We are now ready to associate an element of $\check{U}^B$ to a minuscule or pseudominuscule restricted weight.

Lemma 7.6. *There exists $a \in \check{U}^B$ such that*

$$\mathscr{P}_\mathscr{A}(a) \in \tau(-2\tilde{\eta}) + \tau(-2\tilde{\eta})\mathscr{C}[\mathscr{A}_\geqslant]_+, \tag{7.10}$$

where $\tilde{\eta}$ is a minuscule or pseudominuscule weight.

Proof. By the discussion preceding Theorem 3.6, $\mathscr{P}_\mathscr{A}$ restricted to $Z(\check{U})$ agrees with the composition of $\mathscr{P}$ followed by the projection onto $\mathscr{C}[\check{\mathscr{A}}]$ using (3.5). Suppose that $\eta \in P^+(\pi)$ such that $\tilde{\eta}$ is a minuscule or pseudominuscule restricted weight with respect to Σ. Then description (4.8) of the image of the central elements under $\mathscr{P}$ ensures that c_η satisfies (7.10). Thus we may reduce to the cases when $\mathfrak{g}, \mathfrak{g}^\theta$ is of type EIV, EVII, or EIX. We assume first that $\mathfrak{g}, \mathfrak{g}^\theta$ is of type EIV.

Recall that ω_i is the fundamental weight corresponding to the simple root α_i in π and ω'_i is the fundamental weight corresponding to the restricted root $\tilde{\alpha}_i$ in Σ. One checks that $\tilde{\omega}_6 = 2\omega'_6$, $\tilde{\omega}_1 = 2\omega'_1$, and $w_0\omega_6 = \omega_1$. Consider the $(\mathrm{ad}_r U)$ submodule $(\mathrm{ad}_r U)\tau(2\omega_6)$ of $F_r(\check{U})$. Note that the only restricted dominant integral weights less than $2\omega'_1$ is ω'_6.

By Lemma 7.2,

$$\mathscr{P}_\mathscr{A}(a) \in \tau(-4\omega'_1)\mathscr{C}[\mathscr{A}_\geqslant]_+$$

for any $a \in \check{U}^B \cap (\mathrm{ad}_r U)\tau(2\omega_6)$. On the other hand, Corollary 6.8 and Theorem 3.6 ensure that $\mathrm{gr}(\mathscr{P}_{\mathscr{A}}(a))$ is a linear combination of terms of the form $\tau(-2\gamma)$ where $\gamma \in P^+(\Sigma)$. Hence if $a \in \check{U}^B \cap (\mathrm{ad}_r U)\tau(2\omega_6)$, then

$$\mathscr{P}_{\mathscr{A}}(a) \in \tau(-2\gamma) + \tau(-2\gamma)\mathscr{C}[\mathscr{A}_{\geqslant}]_+,$$

where $\gamma = \omega_6'$, or $\gamma = 2\omega_1'$. Thus it is sufficient to find two elements a_1 and a_2 in $\check{U}^B \cap (\mathrm{ad}_r U)\tau(2\omega_6)$ such that $\mathscr{P}_{\mathscr{A}}(a_1)$ and $\mathscr{P}_{\mathscr{A}}(a_2)$ are linearly independent.

Note that $\tau(2\omega_6)$ is in $(\mathrm{ad}_r U)\tau(2\omega_6)$. It is straightforward to check that $4\omega_6' - 2\tilde{\alpha}_6 = 2\omega_1'$. Hence $\tau(2\omega_1') \in \tau(2\omega_6)\tau(-2\alpha_6) + \mathscr{C}[\check{T}_\Theta]_+$.

The action ad_r is a right action. In particular, $(\mathrm{ad}_r uv)w = (\mathrm{ad}_r v)(\mathrm{ad}_r u)w$ for all u, v, and $w \in \check{U}$. A direct computation yields that

$$\begin{aligned}(\mathrm{ad}_r y_6 x_6)\tau(2\omega_6) &- q^{-1}\tau(2\omega_6)\\ &= -q^{-2}(1-q^2)^2\tau(2\omega_6)t_6^{-1}y_6x_6 - q^{-1}\tau(2\omega_6)t_6^{-2}.\end{aligned}$$

Set $x = x_6x_5x_4x_2x_3x_4x_5x_6$. Arguing as in [L4, Lemma 5.1], one has, up to a nonzero scalar, that $(\mathrm{ad}_r x)\tau(2\omega_6)$ is an element of $\tilde{\theta}(y_6)x_6t_6^{-1}\tau(2\omega_6) + (\mathscr{M}T_\Theta)_+U + U(\mathscr{M}T_\Theta)_+$. Using this fact, it is straightforward to show that up to a nonzero scalar

$$(\mathrm{ad}_r x)\tau(2\omega_6) \in \tau(2\omega_6)t_6^{-1}y_6x_6 + \phi(B)_+\check{U} + \check{U}(\mathscr{M}T_\Theta)_+.$$

Thus a suitable linear combination of the terms $\tau(2\omega_6)$, $(\mathrm{ad}_r y_6x_6)\tau(2\omega_6)$ and $(\mathrm{ad}_r x)\tau(2\omega_6)$ yields an element $b \in (\mathrm{ad}_r U)\tau(2\omega_6)$ such that

$$b \in \tau(2\omega_1') + \phi(B)_+\check{U} + \check{U}(\mathscr{M}T_\Theta)_+.$$

Set $a_1 = L(\tau(2\omega_6))$ and $a_2 = L(b)$. Note that $2\omega_1' + \tilde{\rho}$ and $4\omega_6' + \tilde{\rho}$ are distinct dominant restricted weights and hence are in different orbits with respect to the action of W_Θ. It follows from Lemma 7.5 that $\mathscr{P}_{\mathscr{A}}(a_1)$, and $\mathscr{P}_{\mathscr{A}}(a_2)$ are linearly independent. This completes the proof of the lemma when $\mathfrak{g}, \mathfrak{g}^\theta$ is of type EIV for the restricted minuscule weight ω_1'. The same argument works for the restricted minuscule weight ω_6' using the diagram automorphism of E_6 which sends α_1 to α_6.

Now consider the case when $\mathfrak{g}, \mathfrak{g}^\theta$ is of type EVII. In this case, we use the $(\mathrm{ad}_r U)$ module $(\mathrm{ad}_r U)\tau(2\omega_7)$. Note that $\omega_7' = \tilde{\omega}_7$ and ω_1' are the only dominant integral restricted weights less than or equal to $\tilde{\omega}_7$. Furthermore, as above, there exists a linear combination b of $\tau(2\omega_7)$, $(\mathrm{ad}_r y_7y_6x_7x_6)\tau(2\omega_7)$, $(\mathrm{ad}_r y_7x_7)\tau(2\omega_7)$, $(\mathrm{ad}_r y_7xx_7)\tau(2\omega_7)$ and $(\mathrm{ad}_r x_7xx_7)\tau(2\omega_7)$ such that

$$b \in \tau(2\omega_1') + \mathscr{C}\tau(2(\tilde{\omega}_7 - \tilde{\alpha}_7)) + \phi(B)_+\check{U} + \check{U}(\mathscr{M}T_\Theta)_+.$$

One checks that $2\omega_1' + \tilde{\rho}, 2(\tilde{\omega}_7 - \tilde{\alpha}_7) + \tilde{\rho}$, and $\tau(2\tilde{\omega}_7) + \tilde{\rho}$ are in different W_Θ orbits. Hence Lemma 7.5 ensures that $\mathscr{P}_{\mathscr{A}}(L(\tau(2\omega_7)))$ and $\mathscr{P}(L(b))$ are linearly independent and the proof follows in this case. Similarly, if $\mathfrak{g}, \mathfrak{g}^\theta$ is of type EIX, one can find a linear combination b of elements in $(\mathrm{ad}_r U)\tau(2\omega_8)$ such that b is an element of

$$\tau(2\omega_1') + \mathscr{C}\tau(2(\tilde{\omega}_8 - \tilde{\alpha}_8)) + \mathscr{C}\tau(2(\tilde{\omega}_8 - \tilde{\alpha}_8 - \tilde{\alpha}_7)) + \phi(B)_+\check{U} + \check{U}(\mathscr{M}T_\Theta)_+.$$

In this case, there are three dominant integral weights, 1, ω_1', and ω_8', less than or equal to $\tilde{\omega}_8 = \omega_8'$. Now the W_Θ orbits of $\tilde{\rho}$, $2\omega_1' + \tilde{\rho}$, $2(\tilde{\omega}_8 - \tilde{\alpha}_8) + \tilde{\rho}, (\tilde{\omega}_8 - \tilde{\alpha}_8 - \tilde{\alpha}_7) + \tilde{\rho}$, and $2\tilde{\omega}_8 + \tilde{\rho}$ are distinct. This combined with Lemma 7.5 implies that the three elements $\mathscr{P}_\mathscr{A}(1), \mathscr{P}_\mathscr{A}(L(\tau(2\omega_8))$, and $\mathscr{P}_\mathscr{A}(L(b))$ are linearly independent. The desired element a in $\check{U}^B$ which satisfies (7.10) is then a suitable linear combination of 1, $L(\tau(2\omega_8)$, and $L(b)$. □

Using the proof of Lemma 7.1, we can compute the radial components of the "small" elements in $\check{U}^B$ described in Lemma 7.6.

Theorem 7.7. *Suppose that $\tilde{\mu}$ is a minuscule weight of Σ. Then there exists an element $c \in \check{U}^B$ such that*

$$\mathscr{X}(c) = \sum_{w \in W_\Theta} w(p^{-1}\tau(-2\tilde{\mu})p).$$

If Σ is of type F4, G2, *or* E8, *and $\tilde{\mu}$ is the pseudominuscule weight of Σ, then there exists an element c in $\check{U}^B$ such that*

$$\mathscr{X}(c) = \sum_{w \in W_\Theta} w[(1 - \tau(2\tilde{\mu}))p^{-1}\tau(-2\tilde{\mu})p].$$

Proof. If $\mathfrak{g}, \mathfrak{g}^\theta$ is not of type EIV, EVII, or EIX, then the result follows from Lemma 7.1 and the paragraph immediately following the lemma. More generally, let $\tilde{\mu}$ be a minuscule or pseudominuscule weight of Σ. Then Lemma 7.6 guarantees the existence of an element $c \in \check{U}^B$ such that $\mathrm{gr}\,\mathscr{P}_\mathscr{A}(c) = \tau(-2\tilde{\mu})$. Arguing as in the proof of Lemma 7.1, using Corollary 6.8 and Theorem 3.6 (instead of Theorem 6.7 and Corollary 3.3) we see that $\mathscr{X}(c)$ has the desired form. □

8. Macdonald polynomials

In this section we express the zonal spherical functions as Macdonald polynomials. Formally, Macdonald polynomials associated to a root system are Laurent polynomials which depend on an indeterminate a and a system of parameters $g = \{g_\alpha \mid \alpha \text{ is a root}\}$ such that $g_\alpha = g_{w\alpha}$ for all w in the corresponding Weyl group. Often, however, the g_α are taken to be powers of a. We will take this point of view here in reviewing basic facts and notations concerning these polynomials. To further simplify the presentation, we assume that a is a power of q and that the g_α are rational powers of a. In particular, each g_α is an element of $\mathscr{C}$.

Let $\check{\Sigma}$ denote the dual root system to Σ. For each $\lambda \in P^+(2\Sigma)$, set

$$m_\lambda = \sum_{w \in W_\Theta} z^{w\lambda}.$$

The set of Macdonald polynomials $\{P_\lambda(a,g) \mid \lambda \in P^+(2\Sigma)\}$ associated to $2\Sigma, \check{2}\Sigma$ is the unique basis of $\mathscr{C}[P(2\Sigma)]^{W_\Theta}$ which satisfy the following conditions:

(8.1) The polynomials are orthogonal with respect to the Macdonald inner product at a, g.
(8.2) There exists scalars $a_{\lambda,\mu}$ in $\mathscr{C}$ such that $P_\lambda(a,g) = m_\lambda + \sum_{\mu<\lambda} a_{\lambda,\mu} m_\mu$ for all $\lambda \in P^+(2\Sigma)$.

For more details, the reader is referred to [23] or [17].

Macdonald polynomials can also be characterized using certain difference operators associated to minuscule and pseudominuscule weights. In particular, set $a_{\tilde{\alpha}} = a^{2(\tilde{\alpha},\tilde{\alpha})}$ for $\tilde{\alpha} \in \Sigma$. Recall (6.3) the definition of $(x;a)_\infty$. Let

$$\Delta^+_{a,g} = \prod_{\tilde{\alpha} \in \Sigma^+} \frac{(z^{2\tilde{\alpha}}; a_{\tilde{\alpha}})_\infty}{(g_{\tilde{\alpha}} z^{2\tilde{\alpha}}; a_{\tilde{\alpha}})_\infty}.$$

Given $\beta \in P(2\Sigma)$, define an operator T_β on $\mathscr{C}[Q(2\Sigma)]$ by

$$T_\beta z^\alpha = q^{(\beta,\alpha)} z^\alpha.$$

Let $D_\beta(a,g)$ be the operator on $\mathscr{C}[Q(2\Sigma)]$ defined by

$$D_\beta(a,g)(f) = \sum_{w \in W_\Theta} w((\Delta^+_{a,g})^{-1}(T_\beta(\Delta^+_{a,g} f))).$$

Similarly, let $E_\beta(a,g)$ be the operator on $\mathbf{C}[a,g][z^\alpha \mid \alpha \in \Sigma]$ defined by

$$E_\beta(a,g)(f) = \sum_{w \in W_\Theta} w((\Delta^+_{a,g})^{-1}((T_\beta(\Delta^+_{a,g}))((T_\beta - 1)f))).$$

Set F_β equal to the difference operator D_β if β is a minuscule weight of 2Σ and equal to the difference operator E_β if β is pseudominuscule of 2Σ. The following result summarizes the relationship between Macdonald polynomials and these difference operators.

Theorem 8.1 (Macdonald [23]). *Let S be a subset of 2Σ consisting of*

(i) *one minuscule weight if 2Σ is not of type D_n,* E8, F4, *or* G2
(ii) *both minuscule weights if 2Σ is of type D_n*
(iii) *one pseudominuscule weight if Σ is of type* E8, F4, *or* G2.

Then the set $\{P_\lambda(a,g) \mid \lambda \in P^+(2\Sigma)\}$ is the unique basis of $\mathscr{C}[P(2\Sigma)]^W$ satisfying (8.2) *which consists of eigenvectors for the action of F_β as β ranges over S.*

In order to identify the zonal spherical functions with Macdonald polynomials, we show that the action of certain central elements (or more generally B invariant elements) on the right correspond to the left action of the difference operators described above.

Theorem 8.2. *Let* $\mathfrak{g}, \mathfrak{g}^\theta$ *be an irreducible symmetric pair. Let* $\{\varphi_\lambda \mid \lambda \in P^+(2\Sigma)\}$ *denote the unique* W_Θ *invariant zonal spherical family associated to* $\mathscr{B}$. *Then*

$$\varphi_\lambda = P_\lambda(a, g)$$

for all $\lambda \in P^+(2\Sigma)$. *Here* $a = q$ *and* $g_{\tilde{\alpha}} = q^{mult(\tilde{\alpha})(\tilde{\alpha},\tilde{\alpha})}$ *for each* $\tilde{\alpha} \in \Sigma$.

Proof. Let $\tilde{\mu}$ be a minuscule weight in Σ if Σ is not of type F4, EVIII, or G2 and pseudominuscule otherwise. Note that when $\tilde{\mu}$ is minuscule, so is $-w_0'\tilde{\mu}$ and similarly if $\tilde{\mu}$ is pseudominuscule then so is $-w_0'\tilde{\mu}$. Consider first the case when $\tilde{\mu}$ is minuscule. Applying Theorem 7.7 to the weight $w_0'\tilde{\mu}$ shows that there exists $c \in \check{U}^B$ such that

$$\mathscr{X}(c) = \sum_{w \in W_\Theta} w(p^{-1}\tau(2w_0'\tilde{\mu})p). \tag{8.3}$$

It is straightforward to see that $w_0'(p^{-1}) = \Delta^+_{a,g}$ using the definition of p given in Theorem 6.7. Hence we can rewrite (8.3) as

$$\mathscr{X}(c) = \sum_{w \in W_\Theta} w(\Delta^+_{a,g}\tau(2\tilde{\mu})(\Delta^+_{a,g})^{-1}).$$

It follows that the right action of $\mathscr{X}(c)$ agrees with the left action of $D_{2\tilde{\mu}}(a, g)$. Thus the basis $\{\varphi_\lambda \mid \lambda \in P^+(2\Sigma)\}$ is a basis of eigenvectors for the action of $D_{2\tilde{\mu}}(a, g)$ satisfying (8.2) when μ is minuscule. The theorem now follows from Theorem 8.1 when $\tilde{\mu}$ is minuscule. A similar argument using the operators $E_{2\tilde{\mu}}(a, g)$ works for $\tilde{\mu}$ pseudominuscule. $\square$

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Appendix A

We include in this appendix a complete list of all irreducible symmetric pairs with reduced restricted roots using Araki's classification [1]. For each case, we explicitly describe the involution Θ on Δ and the restricted roots Σ. (This information can also be read off the table in [1].) We further give the values of $g_{\tilde{\alpha}}$ and $a_{\tilde{\alpha}}$ for $\alpha \in \pi^*$ in each case. Moreover, if $\mathfrak{g}, \mathfrak{g}^\theta$ is not of type EIV,EVII, or EIX, then we explicitly give $\mu \in P^+(\pi)$ such that $\tilde{\mu}$ is the pseudominuscule or minuscule weight of Σ.

Note that in Case 1 below, when $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are simply laced, we have $a_{\tilde{\alpha}} = g_{\tilde{\alpha}}$ for all restricted roots $\tilde{\alpha}$. In particular, $a = g$ in these cases and the resulting Macdonald polynomials are just the Weyl character formulas.

Case I. $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ where both $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are simple and isomorphic to each other. Let $\pi_1 = \{\alpha_1, \dots, \alpha_n\}$ denote the simple roots corresponding to $\mathfrak{g}_1$ and $\pi_2 = \{\alpha_{n+1}, \dots, \alpha_{2n}\}$ denote the roots for $\mathfrak{g}_2$. We further assume that $\alpha_i \mapsto \alpha_{i+n}$ defines the isomorphism between the two root systems.

$\Theta(\alpha_i) = -\alpha_{i+n}$ for $1 \leqslant i \leqslant n$; $\Sigma = \{\tilde{\alpha}_1, \dots, \tilde{\alpha}_n\}$ and is of the same type as $\mathfrak{g}_1$; $\tilde{\mu}$ is minuscule (resp. pseudominuscule) when μ is a minuscule (resp. pseudominuscule) weight of $\mathfrak{g}_1$.

$a_{\tilde{\alpha}_i} = q^{(\alpha_i,\alpha_i)}$ and $g_{\tilde{\alpha}_i} = q^{(\alpha_i,\alpha_i)}$ for each i.

Case II. $\mathfrak{g}$ is simple

Type AI $\mathfrak{g}$ is of type A_n.

$\Theta(\alpha_i) = -\alpha_i$, for $1 \leqslant i \leqslant n$. $\Sigma = \{\alpha_1, \dots, \alpha_n\}$ is of type A_n.

minuscule weight: $\tilde{\mu} = \omega_1$ where $\mu = \omega_1$.

$a_{\tilde{\alpha}_i} = q^4$ and $g_{\tilde{\alpha}_i} = q^2$ for all $1 \leqslant i \leqslant n$.

Type AII $\mathfrak{g}$ is of type A_n, where $n = 2m+1$ is odd and $n \geqslant 3$.

$\Theta(\alpha_i) = \alpha_i$ for $i = 2j+1$, $0 \leqslant j \leqslant m$, $\Theta(\alpha_i) = -\alpha_{i-1} - \alpha_i - \alpha_{i+1}$ for $i = 2j$, $1 \leqslant j \leqslant m$. $\Sigma = \{\tilde{\alpha}_i \mid i = 2j+1, 0 \leqslant j \leqslant m\}$ is of type A_n.

minuscule weight: $\tilde{\mu} = \omega_1'$ is minuscule where $\mu = \omega_1$.

$a_{\tilde{\alpha}_i} = q^2$ and $g_{\tilde{\alpha}_i} = q^4$ for all i, $i = 2j, 1 \leqslant j \leqslant m$.

Type AIII *Case* 2. $\mathfrak{g}$ is of type A_n where $n = 2m+1$.

$\Theta(\alpha_i) = -\alpha_{n-i+1}$ for $1 \leqslant i \leqslant n$. $\Sigma = \{\tilde{\alpha}_i \mid 1 \leqslant i \leqslant m\}$ is of type C_m.

minuscule weight: $\tilde{\mu} = \omega_1'$ where $\mu = \omega_1$.

$a_{\tilde{\alpha}_i} = q^2$ for $1 \leqslant i \leqslant m-1$, $a_{\tilde{\alpha}_m} = q^4$, and $g_{\tilde{\alpha}_i} = q^2$ for $1 \leqslant i \leqslant m$.

Type BI $\mathfrak{g}$ is of type B_n, r is an integer such that $2 \leqslant r \leqslant n$.

$\Theta(\alpha_i) = \alpha_i$ for $r+1 \leqslant i \leqslant n$, $\Theta(\alpha_i) = -\alpha_i$ for $1 \leqslant i \leqslant r-1$, $\Theta(\alpha_r) = -\alpha_r - 2\alpha_{r+1} - 2\alpha_{r+2} - \cdots - 2\alpha_n$. $\Sigma = \{\tilde{\alpha}_1, \dots, \tilde{\alpha}_r\}$ is of type B_r.

minuscule weight: $\tilde{\mu} = \omega_r'$ where $\mu = \omega_n$.

$a_{\tilde{\alpha}_i} = q^4$ and $g_{\tilde{\alpha}_i} = q^2$ for $1 \leqslant i \leqslant r-1$, $a_{\tilde{\alpha}_r} = q^2$ and $g_{\tilde{\alpha}_r} = q^{2(n-r)+1}$.

Type BII $\mathfrak{g}$ is of type B_n.

$\Theta(\alpha_i) = \alpha_i$ for $2 \leqslant i \leqslant n$, $\Theta(\alpha_1) = -\alpha_1 - 2\alpha_2 - 2\alpha_3 - \cdots - 2\alpha_n$. $\Sigma = \{\tilde{\alpha}_1\}$ is of type A_1.

minuscule weight: $\tilde{\mu} = \omega_1'$ where $\mu = \omega_n$.

$a_{\tilde{\alpha}_1} = q^2$ and $g_{\tilde{\alpha}_1} = q^{2n-1}$.

Type CI $\mathfrak{g}$ is of type C_n.

$\Theta(\alpha_i) = -\alpha_i$ for $1 \leqslant i \leqslant n$. $\Sigma = \{\tilde{\alpha}_1, \dots, \tilde{\alpha}_n\}$ is of type C_n.

minuscule weight: $\tilde{\mu} = \omega_1'$ where $\mu = \omega_1$.

$a_{\tilde{\alpha}_i} = q^4$ and $g_{\tilde{\alpha}_i} = q^2$ for $1 \leqslant i \leqslant n-1$, $a_{\tilde{\alpha}_n} = q^8$, and $g_{\tilde{\alpha}_n} = q^4$.

Type CII *Case* 2. $\mathfrak{g}$ is of type C_n where $n \geqslant 3$ is even.

$\Theta(\alpha_i) = \alpha_i$ for $i = 2j-1, 1 \leqslant j \leqslant n/2$, $\Theta(\alpha_i) = -\alpha_{i-1} - \alpha_i - \alpha_{i+1}$ for $i = 2j, 1 \leqslant j \leqslant (n-2)/2$, $\Theta(\alpha_n) = -\alpha_n - 2\alpha_{n-1}$. $\Sigma = \{\tilde{\alpha}_i \mid i = 2j, 1 \leqslant j \leqslant n/2\}$ is of type $C_{n/2}$.

minuscule weight: $\tilde{\mu} = \omega'_2$ where $\mu = \omega_1$.
$a_{\tilde{\alpha}_i} = q^2$ and $g_{\tilde{\alpha}_i} = q^4$ for $i = 2j, 1 \leqslant j \leqslant (n-2)/2$, $a_{\tilde{\alpha}_n} = q^4$ and $g_{\tilde{\alpha}_n} = q^6$.
Type DI *Case* 1. $\mathfrak{g}$ is of type D_n, r is an integer such that $2 \leqslant r \leqslant n-2$
$\Theta(\alpha_i) = \alpha_i$ for $r+1 \leqslant i \leqslant n$, $\Theta(\alpha_i) = -\alpha_i$ for $1 \leqslant i \leqslant r-1$, $\Theta(\alpha_r) = -\alpha_r - 2\alpha_{r+1} - \cdots - 2\alpha_{n-2} - \alpha_{n-1} - \alpha_n$. $\Sigma = \{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_r\}$ is of type B_r.
minuscule weight: $\tilde{\mu} = \omega'_r$ where $\mu = \omega_n$.
$a_{\tilde{\alpha}_i} = q^4$ and $g_{\tilde{\alpha}_i} = q^2$ for $1 \leqslant i \leqslant r-1$, $a_{\tilde{\alpha}_r} = q^2$ and $g_{\tilde{\alpha}_r} = q^{2(n-r)}$.
Type DI *Case* 2. $\mathfrak{g}$ is of type D_n, $n \geqslant 4$.
$\Theta(\alpha_i) = -\alpha_i$ for $1 \leqslant i \leqslant n-2$, $\Theta(\alpha_{n-1}) = -\alpha_n, \Theta(\alpha_n) = -\alpha_{n-1}$. $\Sigma = \{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_{n-1}\}$ is of type B_{n-1}.
minuscule weight: $\tilde{\mu} = \omega'_{n-1}$ where $\mu = \omega_{n-1}$.
$a_{\tilde{\alpha}_i} = q^4$ for $1 \leqslant i \leqslant n-2$, $a_{\tilde{\alpha}_{n-1}} = q^2$, and $g_{\tilde{\alpha}_i} = q^2$ for $1 \leqslant i \leqslant n-1$.
Type DI *Case* 3. $\mathfrak{g}$ is of type D_n, $n \geqslant 4$.
$\Theta(\alpha_i) = -\alpha_i$ for $1 \leqslant i \leqslant n$. $\Sigma = \{\alpha_1, \ldots, \alpha_n\}$ is of type D_n.
minuscule weight: $\tilde{\mu}_i = \omega_{n-i}$ where $\mu_i = \omega_{n-i}$, for $i = 0$ and $i = 1$.
$a_{\tilde{\alpha}_i} = q^4$ and $g_{\tilde{\alpha}_i} = q^2$ for all $1 \leqslant i \leqslant n$.
Type DII $\mathfrak{g}$ is of type D_n, $n \geqslant 4$.
$\Theta(\alpha_i) = \alpha_i$ for $2 \leqslant i \leqslant n$, $\Theta(\alpha_1) = -\alpha_1 - 2\alpha_2 - \cdots - 2\alpha_{n-2} - \alpha_{n-1} - \alpha_n$. $\Sigma = \{\tilde{\alpha}_1\}$ is of type A_1.
minuscule weight: $\tilde{\mu} = \omega'_1$ where $\mu = \omega_n$.
$a_{\tilde{\alpha}_1} = q^2$ and $g_{\tilde{\alpha}_1} = q^{2(n-1)}$.
Type DIII *Case* 1. $\mathfrak{g}$ is of type D_n where n is even,
$\Theta(\alpha_i) = \alpha_i$ for $i = 2j-1, 1 \leqslant j \leqslant n/2$. $\Theta(\alpha_i) = -\alpha_{i-1} - \alpha_i - \alpha_{i+1}$ for $i = 2j$, $1 \leqslant j \leqslant (n-2)/2$, $\Theta(\alpha_n) = -\alpha_n$. $\Sigma = \{\tilde{\alpha}_2, \tilde{\alpha}_4, \ldots, \tilde{\alpha}_n\}$ is of type C_n.
minuscule weight: $\tilde{\mu} = \omega'_2$ where $\mu = \omega_1$.
$a_{\tilde{\alpha}_i} = q^2$ and $g_{\tilde{\alpha}_i} = q^4$ for $i = 2j, 1 \leqslant j \leqslant (n-2)/2$, $a_{\tilde{\alpha}_n} = q^4$, and $g_{\tilde{\alpha}_n} = q^2$.
Type E1 $\mathfrak{g}$ is of type E6.
$\Theta(\alpha_i) = -\alpha_i$ for all i. $\Sigma = \{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_6\}$.
minuscule weight: $\tilde{\mu} = \omega'_6$ where $\mu = \omega_6$.
$a_{\tilde{\alpha}_i} = q^4$ and $g_{\tilde{\alpha}_i} = q^2$ for all $1 \leqslant i \leqslant n$.
Type EII $\mathfrak{g}$ is of type E6.
$\Theta(\alpha_i) = -\alpha_{\mathrm{p}(i)}$ where $\mathrm{p}(1) = 6, \mathrm{p}(3) = 5, \mathrm{p}(4) = 4, \mathrm{p}(2) = 2, \mathrm{p}(5) = 3$, and $\mathrm{p}(6) = 1$. $\Sigma = \{\tilde{\alpha}_2, \tilde{\alpha}_4, \tilde{\alpha}_3, \tilde{\alpha}_1\}$ is of type F4.
pseudominuscule weight: $\tilde{\mu} = \omega'_1$ where $\mu = \omega_1$.
$a_{\tilde{\alpha}_i} = q^2$ for $i = 3, 1$, $a_{\tilde{\alpha}_i} = q^4$ for $i = 2, 4$, and $g_{\tilde{\alpha}_i} = q^2$, for $i = 1, 2, 3, 4$.
Type EIV $\mathfrak{g}$ is of type E6.
$\Theta(\alpha_i) = \alpha_i$ for $i = 2, 3, 4, 5$, $\Theta(\alpha_1) = -\alpha_1 - 2\alpha_3 - 2\alpha_4 - \alpha_5 - \alpha_2$, $\Theta(\alpha_6) = -\alpha_6 - 2\alpha_5 - 2\alpha_4 - \alpha_3 - \alpha_2$. $\Sigma = \{\tilde{\alpha}_1, \tilde{\alpha}_6\}$ is of type A3.
$a_{\tilde{\alpha}_i} = q^2$ and $g_{\tilde{\alpha}_i} = q^8$ for $i = 1, 6$.
Type EV $\mathfrak{g}$ is of type E7.
$\Theta(\alpha_i) = -\alpha_i$ for all i. $\Sigma = \{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_7\}$.
minuscule weight: $\tilde{\mu} = \omega'_7$ where $\mu = \omega_7$.
$a_{\tilde{\alpha}_i} = q^4$ and $g_{\tilde{\alpha}_i} = q^2$ for all $1 \leqslant i \leqslant 7$.
Type EVI $\mathfrak{g}$ is of type E7.

$\Theta(\alpha_i) = \alpha_i$ for $i = 2, 5, 7$, $\Theta(\alpha_6) = -\alpha_6 - \alpha_5 - \alpha_7$, $\Theta(\alpha_4) = -\alpha_2 - \alpha_5 - \alpha_4$, $\Theta(\alpha_i) = -\alpha_i$ for $i = 1, 3$. $\Sigma = \{\tilde{\alpha}_1, \tilde{\alpha}_3, \tilde{\alpha}_4, \tilde{\alpha}_6\}$ is of type F4.
pseudominuscule weight: $\tilde{\mu} = \omega_6'$ where $\mu = \omega_7$.
$a_{\tilde{\alpha}_i} = q^4$ and $g_{\tilde{\alpha}_i} = q^2$ for $i = 1, 3$, $a_{\tilde{\alpha}_i} = q^2$ and $g_{\tilde{\alpha}_i} = q^4$ for $i = 4, 6$.
Type EVII $\mathfrak{g}$ is of type E7.
$\Theta(\alpha_i) = \alpha_i$ for $i = 2, 3, 4, 5$, $\Theta(\alpha_1) = -\alpha_1 - 2\alpha_3 - 2\alpha_4 - \alpha_2 - \alpha_5$, $\Theta(\alpha_6) = -\alpha_6 - 2\alpha_4 - 2\alpha_5 - \alpha_2 - \alpha_3$, $\Theta(\alpha_7) = -\alpha_7$. $\Sigma = \{\tilde{\alpha}_1, \tilde{\alpha}_6, \tilde{\alpha}_7\}$ is of type C3.
$a_{\tilde{\alpha}_i} = q^2$ and $g_{\tilde{\alpha}_i} = q^8$ for $i = 1, 6$, $a_{\tilde{\alpha}_i} = q^4$ and $g_{\tilde{\alpha}_i} = q^2$ for $i = 7$.
Type EVIII $\mathfrak{g}$ is of type E8.
$\Theta(\alpha_i) = -\alpha_i$ for all i. $\Sigma = \{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_8\}$ is of type E8.
pseudominuscule weight: $\tilde{\mu} = \omega_8'$ where $\mu = \omega_8$.
$a_{\tilde{\alpha}_i} = q^4$ and $g_{\tilde{\alpha}_i} = q^2$ for all $1 \leqslant i \leqslant 8$.
Type EIX $\mathfrak{g}$ is of type E8.
$\Theta(\alpha_i) = \alpha_i$ for $i = 2, 3, 4, 5$, $\Theta(\alpha_1) = -\alpha_1 - 2\alpha_3 - 2\alpha_4 - \alpha_2 - \alpha_5$, $\Theta(\alpha_6) = -\alpha_6 - 2\alpha_4 - 2\alpha_5 - \alpha_2 - \alpha_3$, $\Theta(\alpha_i) = -\alpha_i$ for $i = 7, 8$. $\Sigma = \{\tilde{\alpha}_8, \tilde{\alpha}_7, \tilde{\alpha}_6, \tilde{\alpha}_1\}$ is of type F4. $a_{\tilde{\alpha}_i} = q^2$ and $g_{\tilde{\alpha}_i} = q^8$ for $i = 1, 6$, $a_{\tilde{\alpha}_i} = q^4$ and $g_{\tilde{\alpha}_i} = q^2$ for $i = 7, 8$.
Type FI $\mathfrak{g}$ is of type F4.
$\Theta(\alpha_i) = -\alpha_i$ for $i = 1, 2, 3, 4$. $\Sigma = \{\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3, \tilde{\alpha}_4\}$ is of type F4.
pseudominuscule weight: $\tilde{\mu} = \omega_4'$ where $\mu = \omega_4'$.
$a_{\tilde{\alpha}_i} = q^4$ and $g_{\tilde{\alpha}_i} = q^2$ for $i = 1, 2$, $a_{\tilde{\alpha}_i} = q^2$ and $g_{\tilde{\alpha}_i} = q$ for $i = 3, 4$.
Type G $\mathfrak{g}$ is of type G2.
$\Theta(\alpha_i) = -\alpha_i$ for $i = 1, 2$. $\Sigma = \{\tilde{\alpha}_1, \tilde{\alpha}_2\}$ is of type G2.
pseudominuscule weight: $\tilde{\mu} = \omega_1'$ where $\mu = \omega_1'$.
$a_{\tilde{\alpha}_i} = q^4$ and $g_{\tilde{\alpha}_i} = q^2$ for $i = 1$, $a_{\tilde{\alpha}_i} = q^{12}$ and $g_{\tilde{\alpha}_i} = q^6$ for $i = 2$.

Appendix B. Commonly used notation

Here is a list of notation defined in Section 1 (in the following order):
$\mathbf{C}$, $\mathbf{Q}$, $\mathbf{Z}$, $\mathbf{N}$, $\mathscr{C}$, Φ^+, $Q(\Phi)$, $P(\Phi)$, $Q^+(\Phi)$, $P^+(\pi)$, $\mathfrak{g}$, $\mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, Δ, $\pi = \{\alpha_1, \ldots, \alpha_n\}$, $(\ ,\ )$, ρ, θ, Θ, π_Θ, $\mathfrak{p}$, π^*, $\tilde{\alpha}$, Σ, W_Θ, U, $x_i, y_i, t_i^{\pm 1}$, U^-, U^+, U_+, S_+, T, U^0, τ, N_γ, T_Θ, $\mathscr{M}$, $\tilde{\theta}$, B_i, B_θ, $\mathbf{H}$, $\mathscr{B}$, $\mathscr{S}$, $L(\lambda)$, $R_q[G]$, Υ, ${}_{B'}\mathscr{H}_B$, ${}_{B'}\mathscr{H}_B(\lambda)$, $\varphi^\lambda_{B,B'}$, $\{\varphi_\lambda \mid \lambda \in P^+(2\Sigma)\}$.

Defined in Section 2:

T'	subgroup of T generated by $\{t_i \mid \alpha_i \in \pi^*\}$
G^-	subalgebra of U generated by $y_i t_i$, $1 \leqslant i \leqslant n$
$\mathscr{M}^-$	$\mathscr{M} \cap G^-$
$\mathscr{M}^+$	$\mathscr{M} \cap U^+$
N^-	subalgebra of G^- generated by $(\operatorname{ad} \mathscr{M}^-)\mathscr{C}[y_i t_i \mid \alpha_i \notin \pi_\Theta]$
N^+	subalgebra of U^+ generated by $(\operatorname{ad} \mathscr{M}^+)\mathscr{C}[x_i \mid \alpha_i \notin \pi_\Theta]$
$S_{\beta,r}$	the sum of weight spaces $S_{\beta'}$ with $\tilde{\beta}' = \tilde{\beta}$
$T'_{\geqslant}$	monoid generated by t_i^2 for $\alpha_i \in \pi^*$
$\mathscr{A}$	group generated by $\tau(2\tilde{\alpha})$ for $\alpha \in \pi^*$
$\mathscr{A}_{\geqslant}$	semigroup generated by $\tau(2\tilde{\alpha})$ for $\alpha \in \pi^*$

Defined in Section 3:

Q_Σ	$Q(\Sigma)$
$\mathscr{C}(Q_\Sigma)$	quotient ring of $\mathscr{C}[Q_\Sigma]$
$B' = B'_\theta$	algebra in $\mathscr{B}$ such that ${}_{B'}\mathscr{H}_B$ is W_Θ invariant
$U_\geqslant$	$U^+G^-\mathscr{A}_\geqslant$
$\check{U}$	simply connected quantized enveloping algebra
$\check{T}$	$\{\tau(\lambda) \mid \lambda \in P(\pi)\}$
$\check{\mathscr{A}}$	$\{\tau(\tilde{\mu}) \mid \mu \in P(\pi)\}$
$\check{U}^0$	$\mathscr{C}[\check{T}]$
$\check{T}_\Theta$	$\{\tau(\mu) \mid \tau(\mu) \in \check{T}$ and $\Theta(\mu) = \mu\}$
$T_\geqslant$	monoid generated by t_i^2 for $i = 1, \ldots, n$
g_λ	element in ${}_{B'}\mathscr{H}_B$ such that $\Upsilon(g_\lambda) = \varphi_\lambda$
$\mathscr{X}$	see Theorem 3.2 and Corollary 3.3
$Z(\check{U})$	the center of $\check{U}$
$\mathscr{P}$	the quantum Harish-Chandra projection of $\check{U}$ onto $\check{U}^0$
$\mathscr{C}((Q_\Sigma))$	formal Laurent series ring $\mathscr{C}((z^{-\tilde{\alpha}_i} \mid \alpha_i \in \pi^*))$
ω'_i	the fundamental weight in $P^+(\Sigma)$ corresponding to $\tilde{\alpha}_i$
ad_r	right adjoint action
$\check{U}^B$	$\mathrm{ad}_r B$ invariant elements in $\check{U}$
$\mathscr{P}_\mathscr{A}$	projection of $\check{U}$ onto $\mathscr{C}[\check{\mathscr{A}}]$ using (3.11)
ξ^*_λ	a particular choice of nonzero vector in $(L(\lambda)^*)^B$

Defined in Section 4:

c_μ	nonzero vector in $(\tau(-2\mu) + (\mathrm{ad}\ U_+)\tau(-2\mu)) \cap Z(\check{U})$
w_0	longest element in W
ω_i	fundamental weight in $P^+(\pi)$ corresponding to α_i
$\mathscr{P}'_\mathscr{A}$	$\mathscr{P}$ composed with projection of $\check{U}$ to $\mathscr{C}[\check{\mathscr{A}}]$ using (3.5)
w'_0	longest element in W_Θ
χ	Hopf algebra automorphism defined after proof of Theorem 4.5

Defined in Section 5:

$\mathfrak{h}^*_\Theta$	$\{\lambda \in \mathbf{Q}\alpha_1 + \cdots + \mathbf{Q}\alpha_n \mid \Theta(\lambda) = -\lambda\}$
$\bar{M}(\lambda)$ $(\bar{M}(\lambda)^r)$	graded version of generalized left (right) Verma module
$\hat{M}(\lambda)$ $(\hat{M}(\lambda)^r)$	completion of $\bar{M}(\lambda)$ $(\bar{M}(\lambda)^r)$
ζ_λ (ζ^r_λ)	nonzero vector in $\hat{M}(\lambda)^{B'}$ $((\hat{M}(\lambda)^r)^B)$
$\hat{N}^-(\hat{N}^+)$	completion of N^- (N^+)
$\bar{\Upsilon}$	map from $\hat{M}(\lambda)^r \otimes \hat{M}(\lambda)$ to $\mathscr{C}((z^{-\alpha} \mid \alpha \in \pi))$
$\overline{\mathscr{P}}$	projection of gr U onto gr U^0
c'_μ	multiple of c_μ satisfying (5.4)
p	defined in Theorem 5.8

Defined in Section 6:

π_i	$\{\alpha_j \mid (\omega_j, \Theta(-\alpha_i)) \neq 0$ or $(\omega_j, \Theta(-\alpha_{\mathrm{p}(i)})) \neq 0\}$
$\mathfrak{g}_i$	rank one semisimple Lie subalgebra of $\mathfrak{g}$ with simple roots π_i

Σ_i	$\{\pm\tilde{\alpha}_i\}$
Δ_i	root system associated to $\mathfrak{g}_i$
U_i	subalgebra of U generated by $x_i, y_i, t_i^{\pm 1}$ for $\alpha_i \in \pi_i$
$mult(\tilde{\alpha})$	number of elements in $\{\beta \in \Delta \mid \tilde{\beta} = \tilde{\alpha}\}$
$(x;a)_\infty$	defined in (6.3)
$a_{\tilde{\alpha}}$	$q^{(2\tilde{\alpha},\tilde{\alpha})}$
$g_{\tilde{\alpha}}$	$q^{mult(\tilde{\alpha})(\tilde{\alpha},\tilde{\alpha})}$

Defined in Section 7:

$F_r(\check{U})$	locally finite part of $\check{U}$ with respect to right adjoint action
L	map from $F_r(\check{U})$ to $\check{U}^B$ defined before Lemma 7.5

References

[1] S. Araki, On root systems and an infinitesimal classification of irreducible symmetric spaces, J. Math. Osaka City Univ. 13 (1) (1962) 1–34.

[2] M.S. Dijkhuizen, Some remarks on the construction of quantum symmetric spaces, in: Representations of Lie Groups, Lie Algebras and Their Quantum Analogues, Acta Applicandae Math. 44 (1–2) (1996) 59–80.

[3] M.S. Dijkhuizen, M. Noumi, A family of quantum projective spaces and related q-hypergeometric orthogonal polynomials, Trans. AMS 350 (8) (1998) 3269–3296.

[4] M.S. Dikhhuizen, J.V. Stokman, Some limit transitions between BC type orthogonal polynomials interpreted on quantum complex Grassmannians, Publ. Res. Inst. Math. Sci. 35 (1999) 451–500.

[5] S. Helgason, Fundamental solutions of invariant differential operators on symmetric spaces, Amer. J. Math. 86 (1964) 565–601.

[6] S. Helgason, Groups and Geometric Analysis, in: American Mathematical Society Mathematical Surveys and Monographs, Vol. 83, AMS, Providence, RI, 2000.

[7] J.E. Humphreys, Introduction to Lie Algebras and Representation Theory, Springer, New York, 1972.

[8] A. Joseph, Quantum Groups and Their Primitive Ideals, Springer, New York, 1995.

[9] A. Joseph, On a Harish-Chandra homomorphism, C. R. Acad. Sci. Paris Ser. I Math. 324 (7) (1997) 759–764.

[10] A. Joseph, A completion of the quantized enveloping algebra of a Kac–Moody algebra, J. Algebra 214 (1) (1999) 235–275.

[11] A. Joseph, G. Letzter, Local finiteness of the adjoint action for quantized enveloping algebras, J. Algebra 153 (1992) 289–318.

[12] A. Joseph, G. Letzter, Separation of variables for quantized enveloping algebras, Amer. J. Math. 116 (1994) 127–177.

[13] A. Joseph, G. Letzter, Evaluation of the quantum affine PRV determinants, Math. Res. Lett. 9 (2002) 307–322.

[14] T.H. Koornwinder, Askey–Wilson polynomials as zonal spherical functions on the SU(2) quantum group, SIAM J. Math. Anal. 24 (3) (1993) 795–813.

[15] T.H. Koornwinder, Askey–Wilson polynomials for root systems of type BC, in: Hypergeometric functions on domains of positivity, Jack polynomials and applications, Contemp. Math. 138 (1992) 189–204.

[16] M.S. Kébé, $\mathcal{O}$-algèbres quantiques, C. R. Acad. Sci. Sér. I, Math. 322 (1) (1996) 1–4.

[17] A.A. Kirillov Jr., Lectures on affine Hecke algebras and Macdonald's conjectures, Bull. AMS, New Ser. 34 (3) (1997) 251–292.

[18] G. Letzter, Subalgebras which appear in quantum Iwasawa decompositions, Canad. J. Math. 49 (6) (1997) 1206–1223.
[19] G. Letzter, Symmetric pairs for quantized enveloping algebras, J. Algebra 220 (2) (1999) 729–767.
[20] G. Letzter, Coideal subalgebras and quantum symmetric pairs, in: New Directions in Hopf Algebras, MSRI publications, Vol. 43, Cambridge University Press, Cambridge, 2002, pp. 117–166.
[21] G. Letzter, Quantum symmetric pairs and their zonal spherical functions, Transformation Groups 8 (3) (2003) 261–292.
[22] I.G. Macdonald, Spherical functions on a group of *p*-adic type, Publications of the Ramanujan Institute, Vol. 2, Ramanujan Institute, Center for Advanced Study in Mathematics, University of Madras, Madras, 1971.
[23] I.G. Macdonald, Orthogonal polynomials associated with root systems, Sém. Lotharingien Combin. 45 (2000/01) 40.
[24] M. Noumi, Macdonald's symmetric polynomials as zonal spherical functions on some quantum homogeneous spaces, Adv. Math. 123 (1) (1996) 16–77.
[25] M. Noumi, M.S. Dijkhuizen, T. Sugitani, Multivariable Askey–Wilson polynomials and quantum complex Grassmannians, Fields Inst. Commun. 14 (1997) 167–177.
[26] M. Noumi, T. Sugitani, Quantum symmetric spaces and related q-orthogonal polynomials, in: Group Theoretical Methods in Physics (ICGTMP), Toyonaka, Japan, 1994, World Science Publishing, River Edge, NJ, 1995, pp. 28–40.
[27] M. Rosso, Groupes Quantiques, Représentations linéaires et applications, Thèse, Paris, Vol. 7, 1990.
[28] T. Sugitani, Zonal spherical functions on quantum Grassmann manifolds, J. Math. Sci. Univ. Tokyo 6 (2) (1999) 335–369.
[29] N.R. Wallach, Real Reductive Groups, I, Pure and Applied Mathematics, Vol. 132, Boston, 1988.