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# Coexistence of Analytic and Distributional Solutions for Linear Differential Equations, I

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Necessary and sufficient conditions are discovered for the simultaneous existence of solutions to linear ordinary differential equations in the form of rational functions and finite linear combinations of the Dirac delta function and its derivatives. The results are also used in the study of polynomial solutions to some important classical equations. © 1990 Academic Press, Inc.

## 1. INTRODUCTION

This paper continues the study of interaction between analytic and distributional solutions of ordinary differential equations (ODE) and functional differential equations (FDE) initiated by J. Wiener [1, 2], S. M. Shah, and J. Wiener [3, 4], and K. L. Cooke and J. Wiener [5]. Recently, there has been considerable interest in problems concerning the existence of solutions to linear ODE and FDE in various spaces of generalized functions, due to increasing applications of the distributions theory in many important areas of theoretical and mathematical physics. It is well known that normal linear homogeneous systems of ODE with infinitely smooth coefficients have no generalized-function solutions other than the classical ones. However, distributional solutions may appear in the case of equations whose coefficients have singularities. The number $m$ is called the order of the distribution

$$x = \sum_{k=0}^{m} x_k \delta^{(k)}(t), \qquad x_m \neq 0, \tag{1}$$



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where $\delta^{(k)}$ denotes the $k$th derivative of the Dirac $\delta$ measure. The theory of such solutions to linear ODE and FDE has been developed in [6, 7]. In [7], for the first time an existence criterion of solutions (1) to any linear ODE was established. Solutions of FDE in the form of infinite series

$$x = \sum_{i=0}^{\infty} x_i \, \delta^{(i)}(t) \tag{2}$$

have been explored in [2, 6–9]. Integral transformations create close connections between entire and generalized functions [10]. Therefore, a unified treatment may be used in the study of both distributional and entire solutions to some classes of linear ODE and, especially, FDE with linear transformations of the argument [2–5]. This approach is employed here to explain the observation of some authors, in particular L. L. Littlejohn and R. P. Kanwal [11], on striking similarities between distributional and analytic solutions of linear ODE and FDE. Theorems are proved on the existence of finite-order distributional, rational, and polynomial solutions of linear ODE, with applications to important classical equations. The variable $t$ is real in the case of distributional solutions and complex for analytic solutions.

## 2. Distributional, Rational, and Polynomial Solutions of Linear ODE

Littlejohn and Kanwal [11] investigated distributional solutions of the confluent hypergeometric differential equation and presented some interesting glimpses into the general hypergeometric equation as well. Thus, it is easy to verify that $\delta(t-1)$ satisfies the equation

$$t(1-t)\,x''(t) + (1-3t)\,x'(t) - x(t) = 0$$

and that $(1-t)^{-1}$ is its classical solution. These functions exhibit intriguing similarities: $(1-t)^{-1}$ has a pole of order 1 and the distributional solution $\delta(t-1)$ also is a simple pole. Furthermore, $\delta'(t-1) - \delta''(t-1)$ is a distributional solution of the equation

$$t(1-t)\,x''(t) + (1-5t)\,x'(t) - 4x(t) = 0,$$

and $(1+t)(1-t)^{-3}$ is its classical solution. Again, we find that both these solutions have a pole of order 3. The following theorem shows that these features are not incidental.

THEOREM 2.1. *If the equation*

$$\sum_{i=0}^{n} q_i(t)\, x^{(n-i)}(t) = 0 \tag{2.1}$$

*with polynomial coefficients $q_i(t)$ admits a rational solution*

$$x = \sum_{k=0}^{m} (-1)^k k!\, x_k t^{-k-1}, \qquad x_m \neq 0 \tag{2.2}$$

*then it also has a distributional solution* (1) *of order m. Conversely, if* (2.1) *admits a distributional solution* (1) *of order m, then there exists a polynomial $q(t)$ such that the equation*

$$\sum_{i=0}^{n} q_i(t)\, x^{(n-i)}(t) = q(t) \tag{2.3}$$

*has a solution* (2.2).

*Proof.* First assume that (1) is a solution of Eq. (2.1). Then the (generalized) Laplace transform $L[x] = F(p)$ of (1) satisfies the equation

$$\sum_{i=0}^{n} q_i(-d/dp)(p^{n-i}F) = 0. \tag{2.4}$$

This implies that (2.4) admits a polynomial solution

$$F(p) = \sum_{k=0}^{m} x_k p^k \tag{2.5}$$

since $L[\delta^{(k)}(t)] = p^k$ [12]. Setting $p > 0$ and applying the right-sided Laplace transformation to (2.4) yields the equation

$$\sum_{i=0}^{n} (-1)^{n-i} q_i(-s)\, y^{(n-i)}(s) = q(-s), \tag{2.6}$$

where $y(s) = L[F(p)]$ and $q(-s)$ is a polynomial whose coefficients include certain derivatives of $F(p)$ at $p = 0$. The substitutions $s = -t$ and $y(s) = x(t)$ reduce (2.6) to (2.3). Since $k!s^{-k-1}$ is the Laplace transform of $p^k$, we conclude that (2.2) is a solution of (2.3).

On the other hand, if (2.2) is a solution of (2.1), then the function $y(s) = x(-s)$ satisfies the homogeneous equation corresponding to (2.6). This means that Eq. (2.4) has a polynomial solution (2.5) which, in turn, proves that (1) is a solution of Eq. (2.1).

*Remark.* If $N$ is the highest degree of the polynomials $q_i(t)$, then the degree of $q(t)$ in (2.3) does not exceed $N - 1$.

EXAMPLE 1. It has been proved in [6, 13] that Bessel's equation

$$t^2 x'' + tx' + (t^2 - \nu^2)x = 0$$

has a distributional solution (1) of order $m$ iff

$$v^2 = (m+1)^2$$

and it is given by the formula

$$x = C \sum_{k=0}^{[m/2]} \binom{m-k}{k} 4^{-k}\, \delta^{(m-2k)}(t), \qquad c = \text{const}.$$

Indeed, the Laplace transform $F(p)$ of solution (1) satisfies the equation

$$(p^2F)'' - (pF)' + F'' - v^2F = 0$$

differentiating which $k$ times at $p = 0$ leads to the relations

$$[v^2 - (k+1)^2]\, F_k = F_{k+2}, \qquad k = 0, ..., m-2$$

$$(v^2 - m^2)\, F_{m-1} = 0, \qquad [v^2 - (m+1)^2]\, F_m = 0,$$

$$(F_k = F^{(k)}(0) = k!\, x_k),$$

the last of which has a non-zero solution $F_m$. Substituting it in the foregoing equations enables us to find all $F_k$ $(k < m)$.

There exist constants $c_0$ and $c_1$ such that the equation

$$t^2x'' + tx' + (t^2 - v^2)\, x = c_0 + c_1 t$$

has a solution

$$x = C \sum_{k=0}^{[m/2]} (-1)^k\, 4^{-k} \frac{(m-k)!}{k!} t^{-m-1+2k}.$$

Theorem 2.1 describes the relationship between distributional and rational solutions of linear ODE. The proof makes use of the fact that the existence of a distributional solution to Eq. (2.1) implies that (2.4) has a polynomial solution. This can be also used to establish a correlation between the existence of polynomial and distributional solutions to linear ODE. This direction is of considerable interest for many important equations of mathematical physics which admit polynomial or distributional solutions.

THEOREM 2.2. *If Eq.* (2.1) *with polynomial coefficients* $q_i(t)$ *of the highest degree* $N$ *admits a polynomial solution, then Eq.* (2.4) *has a finite-order distributional solution. Furthermore, there exists a polynomial* $q(t)$ *of degree not exceeding* $N-1$ *such that the equation*

$$\sum_{i=0}^{n} q_i(-d/dt)(t^{n-i}x) = q(t) \tag{2.7}$$

*has a rational solution* (2.2).

*Proof.* Assume that Eq. (2.1) written in terms of variables $p$ and $F$,

$$\sum_{i=0}^{n} q_i(p)\, F^{(n-i)}(p) = 0, \tag{2.1'}$$

has a polynomial solution (2.5), with $x_m \neq 0$. Since (2.5) is the Laplace transform of (1), then

$$\sum_{i=0}^{n} q_i(d/ds)[(-s)^{n-i}\, y(s)] = 0,$$

the Laplace-transformed equation of which is (2.1′), has a solution

$$y(s) = \sum_{k=0}^{m} x_k\, \delta^{(k)}(s).$$

The substitutions $s = -t$, $y(s) = x(t)$ and the formula $\delta^{(k)}(-t) = (-1)^k\, \delta^{(k)}(t)$ show that the equation

$$\sum_{i=0}^{n} q_i(-d/dt)(t^{n-i}x) = 0 \tag{2.4'}$$

admits a distributional solution

$$x = \sum_{k=0}^{m} (-1)^k\, x_k\, \delta^{(k)}(t), \qquad x_m \neq 0.$$

Equation (2.4′) coincides with (2.4) written in variables $t$ and $x$. Setting $p > 0$ and applying the right-sided Laplace transformation to (2.1′) yields Eq. (2.7), where the coefficients of the polynomial $q(t)$ depend on the values of $F(p)$ and its derivatives at $p = 0$. Since $L[p^k] = k!\, t^{-k-1}$, the rational function (2.2) is a solution of (2.7).

EXAMPLE 2. The equation

$$pF'(p) = F(p)$$

has a solution $F = p$ and is the Laplace-transformed relation of

$$tx'(t) = -2x(t).$$

Hence, the latter equation admits a distributional solution $x = \delta'(t)$. It has also a rational solution $x = t^{-2}$. The equation

$$tx'(t) = -x(t)$$

has a distributional solution $x=\delta(t)$ and a rational solution $x=t^{-1}$. Therefore, the equation

$$\left(t\frac{d}{dt}-1\right)\left(t\frac{d}{dt}+1\right)x=0,$$

that is,

$$t^2x''(t)+tx'(t)-x(t)=0$$

has three types of solutions: polynomial $x=t$, rational $x=t^{-1}$, and distributional $x=\delta(t)$.

It has been proved in [6, 7] that if the equation

$$\sum_{i=0}^{n} t^i q_i(t)\, x^{(i)}(t)=0 \tag{2.8}$$

with coefficients $q_i(t)\in C^m$ and $q_n(0)\neq 0$ has a solution (1) of order $m$, then

$$\sum_{i=0}^{n} (-1)^i\, q_i(0)(m+i)!=0. \tag{2.9}$$

Conversely, if $m$ is the smallest non-negative integer root of relation (2.9), there exists an $m$-order solution (1) of Eq. (2.8).

Assume now that the coefficients $q_i(t)$ in (2.8) are polynomials or holomorphic functions in the neighborhood of $t=0$ and denote

$$Q_i(t)=t^iq_i(t).$$

The differential equation

$$\sum_{i=0}^{n} Q_i(-d/dp)[p^iF(p)]=0 \tag{2.10}$$

is obtained by applying the Laplace transformation to (2.8). Hence, we can formulate the following theorem which finds useful applications in the theory of orthogonal polynomials.

THEOREM 2.3. *Assume that the coefficients $q_i(t)$ are polynomials or holomorphic functions in the neighborhood of $t=0$ and $q_n(0)\neq 0$. If Eq.* (2.10) *has a polynomial solution* (2.5) *of degree m, then relation* (2.9) *takes place. Conversely, if m is the smallest non-negative integer root of* (2.9), *there exists a polynomial solution to* (2.10) *of degree m.*

*Remark.* The condition $q_n(0)\neq 0$ is equivalent to $Q_n^{(n)}(0)\neq 0$.

THEOREM 2.4. *The equation*

$$t^2x'' + 2tx' - [t^2 + \nu(\nu+1)]\, x = 0 \tag{2.11}$$

*has an m-order distributional solution* (1) *if and only if*

$$\nu(\nu+1) = m(m+1). \tag{2.12}$$

*It is given by the formula*

$$x(t) = \sum_{k=0}^{[m/2]} \frac{(-1)^k\,(2m-2k)!}{2^m k!\,(m-k)!\,(m-2k)!}\,\delta^{(m-2k)}(t), \tag{2.13}$$

*whose coefficients coincide with the corresponding coefficients of the Legendre polynomial* $P_m(t)$.

*Proof.* Equation (2.9) corresponding to (2.11) is

$$(m+2)(m+1) - 2(m+1) - \nu(\nu+1) = 0$$

and coincides with (2.12). Hence, condition (2.12) is necessary and sufficient for the existence of an $m$-order solution to (2.11). Substituting (1) in this equation and taking into account the formulas

$$t\delta^{(k+1)}(t) = -(k+1)\,\delta^{(k)}(t),$$
$$t^2\delta^{(k+2)}(t) = (k+2)(k+1)\,\delta^{(k)}(t)$$

gives

$$\sum_{k=0}^{m} [k(k+1) - \nu(\nu+1)]\, x_k\, \delta^{(k)}(t) - \sum_{k=0}^{m-2} (k+1)(k+2)\, x_{k+2}\, \delta^{(k)}(t) = 0,$$

whence

$$[\nu(\nu+1) - k(k+1)]\, x_k = -(k+1)(k+2)\, x_{k+2}, \qquad k = 0, ..., m-2$$
$$[\nu(\nu+1) - m(m-1)]\, x_{m-1} = 0,$$
$$[\nu(\nu+1) - m(m+1)]\, x_m = 0.$$

If (2.12) holds true, we choose $x_m \neq 0$ and successively find all $x_k (k < m)$. We have $x_{m-1} = 0$ and

$$[m(m+1) - k(k+1)]\, F_k = -F_{k+2}, \qquad F_k = k!\, x_k$$

and multiplying these relations for $k = m-2, m-4, ..., m-2j$, get

$$2^j j!\,(2m-1)(2m-3)\cdots(2m-2j+1)\, F_{m-2j} = (-1)^j\, F_m.$$

From here,

$$F_{m-2j}=\frac{(-1)^j\, m!\,(2m-2j)!}{(2m)!\, j!\,(m-j)!}F_m,$$

or

$$x_{m-2j}=\frac{(-1)^j\,(m!)^2\,(2m-2j)!}{(2m)!\, j!\,(m-j)!\,(m-2j)!}x_m. \tag{2.14}$$

We can require that the coefficients $x_{m-2j}$ of the distributional solution $x(t)$ coincide with the corresponding coefficients of the Legendre polynomial $P_m(t)$. Indeed, applying the Laplace transformation to (2.11) produces the differential equation

$$(p^2-1)F''+2pF'-\nu(\nu+1)F=0$$

for the Legendre polynomials. The usual way of finding their coefficients is by means of Rodrigues' formula [14]

$$P_m(t)=\frac{1}{2^m m!}\frac{d^m}{dt^m}(t^2-1)^m.$$

It is easy to see that here the coefficient $x_m$ of $t^m$ is

$$x_m=\frac{(2m)!}{2^m(m!)^2}. \tag{2.15}$$

Therefore, equating $x_m$ in (2.14) to the value (2.15) yields the coefficients

$$x_{m-2j}=\frac{(-1)^j\,(2m-2j)!}{2^m j!\,(m-j)!\,(m-2j)!}$$

of the distributional solution (2.13), where $[m/2]$ denotes the largest integer $\leqslant m/2$. Furthermore, there exist constants $c_0$ and $c_1$, such that the equation

$$t^2x''+2tx'-[t^2+m(m+1)]\,x=c_0+c_1t$$

has a rational solution

$$x(t)=\sum_{k=0}^{[m/2]}\frac{(2m-2k)!}{2^m k!\,(m-k)!}t^{-m-1+2k}.$$

THEOREM 2.5. *The equation*

$$tx'+\left(\frac{t^2}{2}+\nu+1\right)x=0 \tag{2.16}$$

*has an m-order distributional solution* (1) *if and only if*

$$\nu = m. \tag{2.17}$$

*This solution is given by the formula*

$$x(t) = \sum_{k=0}^{[m/2]} \frac{(-1)^k\, 2^{m-2k} m!}{k!\,(m-2k)!}\, \delta^{(m-2k)}(t), \tag{2.18}$$

*whose coefficients coincide with the corresponding coefficients of the Hermite polynomial* $H_m(t)$.

*Proof.* In the case of (2.16), Eq. (2.9) takes the form (2.17), which implies that (2.17) is a necessary and sufficient condition for the existence of an $m$-order solution (1) to (2.16). Substituting (1) in (2.16) leads to the relations

$$2(\nu - k)\, F_k = -F_{k+2}, \qquad k = 0, ..., m-2$$

$$(\nu - m + 1)\, F_{m-1} = 0, \qquad (\nu - m)\, F_m = 0$$

for the coefficients $F_k = k!\, x_k$. If $\nu = m$, we choose $F_m \neq 0$ and find

$$x_{m-2k} = \frac{(-1)^k\, m!}{4^k k!\,(m-2k)!}\, x_m. \tag{2.19}$$

The coefficients $x_{m-2k}$ of the distributional solution $x(t)$ can be chosen to coincide with the corresponding coefficients of the Hermite polynomial $H_m(t)$ defined by the formula [14]

$$H_m(t) = (-1)^m\, e^{t^2} \frac{d^m}{dt^m}\, e^{-t^2}.$$

It is easy to see that the leading term of $H_m(t)$ is $(2t)^m$. Therefore, setting $x_m = 2^m$ in (2.19) yields the coefficients

$$x_{m-2k} = \frac{(-1)^k\, 2^{m-2k} m!}{k!\,(m-2k)!}$$

of (2.18). They are identical to the coefficients of $H_m(t)$ because applying the Laplace transformation to (2.16) yields the differential equation

$$F'' - 2pF' + 2\nu F = 0$$

for the Hermite polynomials. Furthermore, there exist constants $c_0$ and $c_1$ such that the equation

$$tx' + \left(\frac{t^2}{2} + m + 1\right) x = c_0 + c_1 t$$

has a rational solution

$$x(t) = 2^m m!\, t^{-m-1} \sum_{k=0}^{[m/2]} \left(\frac{t^2}{4}\right)^k \Big/ k!,$$

where the sum on the right is the Taylor sum of order $[m/2]$ of the function $e^{t^2/4}$.

THEOREM 2.6. *The equation*

$$(t^2+t)\,x' + [(1-\alpha)t + (\nu+1)]\,x = 0, \qquad \alpha > -1 \tag{2.20}$$

*has an m-order distributional solution* (1) *if and only if* (2.17) *holds true. This solution is given by the formula*

$$x(t) = \sum_{k=0}^{m} \frac{(-1)^k\,\Gamma(m+\alpha+1)}{k!\,(m-k)!\,\Gamma(k+\alpha+1)}\,\delta^{(k)}(t), \tag{2.21}$$

*where $\Gamma$ denotes the gamma function and the coefficients coincide with the corresponding coefficients of the Laguerre polynomial $L_m^\alpha(t)$.*

*Proof.* Equation (2.9) corresponding to (2.20) is

$$-(m+1)! + (\nu+1)\,m! = 0,$$

which shows that (2.17) is a criterion for the existence of an $m$-order solution (1) to (2.20). Substituting (1) in (2.20) gives

$$\sum_{k=0}^{m-1} (k+1+\alpha)(k+1)\,x_{k+1}\,\delta^{(k)}(t) + \sum_{k=0}^{m} (\nu-k)\,x_k\,\delta^{(k)}(t) = 0,$$

whence

$$(\nu-m)\,x_m = 0, \; (\nu-k)\,x_k = -(k+1+\alpha)(k+1)\,x_{k+1}, \qquad 0 \leqslant k \leqslant m-1.$$

If $\nu = m$, we take an arbitrary $x_m \neq 0$ and find

$$x_k = \frac{(-1)^{m-k}\,m!\,\Gamma(m+\alpha+1)}{k!\,(m-k)!\,\Gamma(k+\alpha+1)}\,x_m. \tag{2.22}$$

We can normalize $x_m$ so that the coefficients of the distributional solution (1) coincide with the coefficients of the Laguerre polynomial $L_{(m)}^\alpha(t)$ defined by the formula [14]

$$(1-x)^{-\alpha-1}\,e^{-tx/(1-x)} = \sum_{m=0}^{\infty} L_m^\alpha(t)\,x^m, \qquad |x| < 1.$$

At $t=0$ we have the expansion

$$(1-x)^{-\alpha-1} = \sum_{m=0}^{\infty} L_m^\alpha(0)\, x^m$$

and differentiate it $m$ times at $x=0$. Consequently,

$$L_m^\alpha(0) = x_0 = \frac{\Gamma(m+\alpha+1)}{m!\,\Gamma(\alpha+1)}.$$

On the other hand, from (2.22) we have

$$x_0 = \frac{(-1)^m\,\Gamma(m+\alpha+1)}{\Gamma(\alpha+1)}\, x_m,$$

and putting $x_m = (-1)^m/m!$ in (2.22) delivers the coefficients of (2.21). They coincide with the coefficients of $L_m^\alpha(t)$ because applying the Laplace transformation to (2.20) produces the differential equation

$$pF'' + (\alpha+1-p)\, F' + vF = 0 \tag{2.23}$$

for the Laguerre polynomials. Furthermore, there exist constants $c_0$ and $c_1$ such that the equation

$$(t^2+t)\, x' + [(1-\alpha)t + (m+1)]\, x = c_0 + c_1 t$$

has a rational solution

$$x(t) = \sum_{k=0}^{m} \frac{\Gamma(m+\alpha+1)}{(m-k)!\,\Gamma(k+\alpha+1)}\, t^{-k-1},$$

which is the $m$th Taylor sum of the function $t^{-m-1}(1+t)^{m+\alpha}$, for $|t|<1$. In the case $\alpha=0$, Laguerre's equation

$$pF'' - (p-1)\, F' + mF = 0$$

is the Laplace-transformed relation of

$$(t^2+t)\, x' + (t+m+1)x = 0,$$

which admits a distributional solution

$$x(t) = \sum_{k=0}^{m} \frac{(-1)^k\, m!}{(k!)^2\,(m-k)!}\, \delta^{(k)}(t)$$

and a rational solution

$$x(t) = \sum_{k=0}^{m} \binom{m}{k}\, t^{-k-1} = t^{-m-1}(1+t)^m.$$

Equation (2.20) has two singular points, $t=0$ and $t=-1$. For $t=-1$, Eq. (2.9) becomes $m+1+\alpha+\nu=0$ or $\alpha=-\nu-m-1$. Let $\nu=n$ be a non-negative integer, then the equation

$$(t^2+t)\,x'+[(m+n+2)t+(n+1)]\,x=0 \tag{2.24}$$

has two finite-order distributional solutions: one of order $n$ with support $t=0$, and the other of order $m$ concentrated on $t=-1$. Substituting in (2.24) the unknown solution

$$x_n(t)=\sum_{k=0}^{n} x_k\,\delta^{(k)}(t),$$

it is easy to find

$$x_n(t)=\sum_{k=0}^{n}\binom{m+n-k}{m}\delta^{(k)}(t)/k!.$$

For the special case $m=0$, we have

$$x_n(t)=\sum_{k=0}^{n}\delta^{(k)}(t)/k!. \tag{2.25}$$

This implies that Laguerre's equation corresponding to (2.23),

$$pF''-(p+m+n)\,F'+nF=0,$$

has a polynomial solution

$$F_n(p)=\sum_{k=0}^{n}\binom{m+n-k}{m}p^k/k!,$$

and the equation

$$pF''-(p+n)\,F'+nF=0 \tag{2.26}$$

has a polynomial solution

$$F_n(p)=\sum_{k=0}^{n}p^k/k!,$$

which is the $n$th partial sum of $e^p$. Now, we recant Eq. (2.24) as

$$[(t+1)^2-(t+1)]\,x'+[(m+n+2)(t+1)-(m+1)]\,x=0$$

and substitute in it the distributional solution

$$x(t)=\sum_{k=0}^{m}c_k\,\delta^{(k)}(t+1),$$

which leads to the relations

$$(m-m)c_m = 0, \qquad (m-k)c_k = (k+1)(k-m-n)c_{k+1} \qquad (0 \leqslant k \leqslant m-1).$$

Letting $c_m = 1/m!$ yields

$$x(t) = \sum_{k=0}^{m} (-1)^k \binom{m+n-k}{n} \delta^{(k)}(t+1)/k!,$$

and for the special case $m=0$, we have

$$x(t) = \delta(t+1). \tag{2.27}$$

This implies that Eq. (2.26) has a solution

$$F(p) = e^p.$$

Hence, we have shown that (2.26) has the two solutions $F(p) = e^p = \sum_{k=0}^{\infty} p^k/k!$ and its partial sum $F_n(p) = \sum_{k=0}^{n} p^k/k!$. This remarkable property of Eq. (2.26) has been first noted by J. P. Hoyt [15]. W. Leighton [16] has also discussed (2.26), and T. A. Newton [17] has extended these results to third-order equations with polynomial or analytic at $t=0$ coefficients. Further generalizations for broader classes of linear ODE having $\sum_{k=0}^{\infty} c_k t^k$ and $\sum_{k=0}^{n} c_k t^k$ as solutions have been made by D. Zeitlin [18]. On the other hand, expanding (2.27) in a series

$$\delta(t+1) = \sum_{k=0}^{\infty} \delta^{(k)}(t)/k! \tag{2.28}$$

shows that the equation

$$(t^2+t)\,x' + [(n+2)t + (n+1)]\,x = 0$$

admits a formal distributional solution (2.28) whose $n$th partial sum (2.25) is also a distributional solution.

THEOREM 2.7. *Equation* (2.8), *with* $q_i(t) \in C^m$ *and*

$$q_i(t) = (t-a)^i\, p_i(t), \qquad p_i(t) \in C^m$$

*has the distributional solutions*

$$x_m(t) = \sum_{k=0}^{m} (-a)^k\, \delta^{(k)}(t)/k! \tag{2.29}$$

*and*

$$x(t) = \delta(t-a) = \sum_{k=0}^{\infty} (-a)^k\, \delta^{(k)}(t)/k! \tag{2.30}$$

*if and only if*

$$\sum_{i=0}^{n} (-1)^i i! \sum_{j=0}^{k} \binom{m+i-j}{i} a^{m-j} q_{i,k-j} = 0 \qquad (k=0, ..., m) \quad (2.31)$$

*and*

$$\sum_{i=0}^{n} (-a)^i q_i^{(i)}(a) = 0, \qquad (2.32)$$

*where* $q_{i,k-j} = q_i^{(k-j)}(0)/(k-j)!$.

*Proof.* Equation (2.8) and the truncated equation

$$\sum_{i=0}^{n} t^i x^{(i)}(t) \sum_{k=0}^{m} q_{ik} t^k = 0 \qquad (2.33)$$

have the same set of $m$-order solutions (1). Let $F(p)$ be the Laplace transform of (1); then the existence of an $m$-order solution (1) to (2.33) is equivalent to the existence of a polynomial solution of degree $m$ to the equation

$$\sum_{i=0}^{n} (-1)^i \sum_{k=0}^{m} (-1)^k q_{ik} (p^i F)^{(i+k)} = 0, \qquad (2.34)$$

which is obtained by applying the Laplace transformation to (2.33). Differentiating (2.34) $j$ times at $p=0$ leads to the equation

$$\sum_{i=0}^{n} (-1)^i i! \sum_{k=0}^{m} (-1)^k \binom{i+j+k}{i} q_{ik} F_{k+j} = 0$$

for the coefficients $F_k = F^{(k)}(0) = (-a)^k$ of the distributional solution (2.34). Since $F_k = 0$ for $k > m$, we have

$$\sum_{i=0}^{n} (-1)^i i! \sum_{j=0}^{m-k} (-1)^j \binom{i+j+k}{i} q_{ij} F_{k+j} = 0 \qquad (k=0, ..., m)$$

and make the substitution $k+j \to m-j$ to obtain

$$\sum_{i=0}^{n} (-1)^i i! \sum_{j=0}^{m-k} (-1)^{m-k-j} \binom{m+i-j}{i} q_{i,m-k-j} F_{m-j} = 0.$$

Changing $m-k$ to $k$ and putting $F_{m-j} = (-a)^{m-j}$ yields (2.31). Furthermore, we substitute (2.30) in (2.8) and get

$$\sum_{i=0}^{n} t^i (t-a)^i p_i(t)\, \delta^{(i)}(t-a) = 0,$$

whence

$$\sum_{i=0}^{n} (-1)^i\, i!\, t^i p_i(t)\, \delta(t-a) = 0.$$

Expanding $t^i p_i(t)$ in the neighborhood of $t = a$,

$$t^i p_i(t) = a^i p_i(a) + [a^i p_i'(a) + i a^{i-1} p_i(a)](t-a) + \cdots$$

and observing that

$$i!\, p_i(a) = q_i^{(i)}(a),\ (t-a)^j\, \delta(t-a) = 0, \qquad j = 1, 2, ...$$

produces (2.32).

COROLLARY 1. *Assume that the coefficients $q_i(t)$ are polynomials and $q_i^{(j)}(a) = 0$, for $j < i$. Then differential equation* (2.10) *admits the solutions*

$$F_m(p) = \sum_{k=0}^{m} (-ap)^k/k!$$

*and*

$$F(p) = e^{-ap} = \sum_{k=0}^{\infty} (-ap)^k/k!$$

*if and only if hypotheses* (2.31) *and* (2.32) *hold true.*

COROLLARY 2. *Assume that in the equation*

$$t q_1(t)\, x'(t) + q_0(t) x(t) = 0 \tag{2.35}$$

*$q_0(t)$, $q_1(t) \in C^m$, and $q_1(a) = 0$. Equation* (2.35) *admits distributional solutions* (2.29) *and* (2.30) *if and only if*

$$q_{0k} = (m+1)\, q_{1k} - a^{-k} \sum_{j=0}^{k-1} a^j q_{1j} \qquad (k = 0, ..., m) \tag{2.36}$$

*and*

$$q_0(a) = a q_1'(a), \tag{2.37}$$

*where $q_{ij} = q_i^{(j)}(0)/j!$, $i = 0, 1$.*

EXAMPLE 3. Consider the differential equation

$$t(t+1)\, x'(t) + q_0(t)\, x(t) = 0, \tag{2.38}$$

where $q_0(t)$ is a polynomial. For (2.38), we have

$$a=-1, q_1(t)=t+1, q_{10}=1, q_{11}=1, q_{1j}=0, \qquad j\geqslant 2$$

and conditions (2.36) become

$$q_{00}=m+1, q_{01}=m+2, q_{0k}=0, \qquad 2\leqslant k\leqslant m.$$

Condition (2.37) is $q_0(-1)=-1$. Hence, (2.38) has distributional solutions (2.25) and (2.28) if and only if

$$q_0(t)=n+1+(n+2)t+r_n(t),$$

where $r_n(t)$ is any polynomial of degree greater than $n$ containing a factor $t+1$ (because $r_n(-1)=0$).

THEOREM 2.8. *The equation*

$$t^2x''+3tx'-(t^2+v^2-1)x=0 \tag{2.39}$$

*has an m-order distributional solution* (1) *if and only if* $v^2=m^2$. *This solution is given by the formula*

$$x(t)=\frac{m}{2}\sum_{k=0}^{[m/2]}(-1)^k\frac{(m-k-1)!}{k!\,(m-2k)!}2^{m-2k}\,\delta^{(m-2k)}(t) \tag{2.40}$$

*whose coefficients coincide with the corresponding coefficients of the Chebyshev polynomial* $T_m(t)=\cos(m\arccos t)$.

*Proof.* In the case of (2.39), Eq. (2.9) takes the form

$$(m+2)(m+1)-3(m+1)-(v^2-1)=0,$$

that is, $v^2=m^2$, which implies that this condition is necessary and sufficient for the existence of an $m$-order solution (1) to (2.39). Substituting (1) in (2.39) gives

$$\sum_{k=0}^{m}[(k+2)(k+1)-3(k+1)-v^2+1]\,x_k\,\delta^{(k)}(t)$$
$$-\sum_{k=0}^{m-2}(k+2)(k+1)\,x_{k+2}\,\delta^{(k)}(t)=0,$$

that is,

$$\sum_{k=0}^{m}(k^2-v^2)\,x_k\,\delta^{(k)}(t)-\sum_{k=0}^{m-2}(k+2)(k+1)\,x_{k+2}\,\delta^{(k)}(t)=0.$$

Therefore,

$$(m^2 - \nu^2)\, x_m = 0, \qquad [(m-1)^2 - \nu^2]\, x_{m-1} = 0,$$

$$(k^2 - \nu^2)\, x_k = (k+1)(k+2)\, x_{k+2}, \qquad k = 0, ..., m-2.$$

If $\nu^2 = m^2$, we choose $x_m \neq 0$ and from the relations

$$(m^2 - k^2)\, F_k = -F_{k+2}, \qquad k = m-2, m-4, ..., m-2j$$

$$F_k = k!\, x_k$$

find

$$x_{m-2j} = \frac{(-1)^j\, m(m-j-1)!}{4^j j!\, (m-2j)!}\, x_m. \tag{2.41}$$

The constant $x_m$ can be selected so that the coefficients $x_{m-2j}$ of the distributional solution $x(t)$ coincide with the corresponding coefficients of $T_m(t)$. Indeed, if $m$ is even, we take $2j = m$ and get

$$x_0 = 2^{1-m}(-1)^{m/2}\, x_m.$$

On the other hand,

$$x_0 = T_m(0) = (-1)^{m/2},$$

hence

$$x_m = 2^{m-1},$$

and substituting this value in (2.41) yields the coefficients of (2.40). They are identical to the coefficients of $T_m(t)$ because applying the Laplace transformation to (2.39) produces the differential equation

$$(p^2 - 1)\, F'' + pF' - m^2 F = 0$$

for the Chebyshev polynomials. In the case of odd $m$, the same result follows by employing the derivative $T'_m(0)$. Furthermore, there exist constants $c_0$ and $c_1$ such that the equation

$$t^2 x'' + 3tx' - (t^2 + m^2 - 1)x = c_0 + c_1 t$$

has a rational solution

$$x(t) = \frac{m}{2} \sum_{k=0}^{[m/2]} \frac{(m-k-1)!}{k!\, t} \left(\frac{2}{t}\right)^{m-2k}.$$

THEOREM 2.9. *Bessel's equation of imaginary argument*

$$t^2 x'' + tx' - [t^2 + (\nu+1)^2]\, x = 0 \tag{2.42}$$

*has an m-order distributional solution* (1) *if and only if* $(\nu+1)^2=(m+1)^2$ *This solution is given by the formula*

$$x(t)=\sum_{k=0}^{[m/2]}\frac{(-1)^k\,(m-k)!}{k!\,(m-2k)!}\,2^{m-2k}\,\delta^{(m-2k)}(t), \tag{2.43}$$

*whose coefficients coincide with the corresponding coefficients of the Chebyshev polynomials of the second kind* $U_m(t)$.

*Proof.* Substituting (1) in (2.42) yields the relations

$$[(m+1)^2-(\nu+1)^2]\,x_m=0, \qquad [m^2-(\nu+1)^2]\,x_{m-1}=0,$$

$$[(k+1)^2-(\nu+1)^2]\,x_k=(k+1)(k+2)\,x_{k+2}, \qquad k=0,...,m-2.$$

If $(\nu+1)^2=(m+1)^2$, we take $x_m\neq 0$ and find

$$x_{m-2k}=\frac{(-1)^k\,(m-k)!}{4^k k!\,(m-2k)!}\,x_m. \tag{2.44}$$

The differential equation

$$(p^2-1)\,F''+3pF'-\nu(\nu+2)F=0$$

for the polynomials $U_\nu(p)$ is the Laplace-transformed relation of (2.42). These polynomials are generated by the expansion [19].

$$\frac{1}{1-2tx+x^2}=\sum_{m=0}^{\infty}U_m(t)\,x^m,$$

hence, $U_m(0)=(-1)^{m/2}$, for even $m$. Putting $k=m/2$ in (2.44) gives

$$x_0=(-1)^{m/2}\,2^{-m}x_m=(-1)^{m/2},$$

whence $x_m=2^m$. This value, together with (2.44), completely determines the coefficients of solution (2.43). Also, there exist constants $c_0$ and $c_1$ such that the equation

$$t^2x''+tx'-[t^2+(m+1)^2]\,x=c_0+c_1 t$$

has a rational solution

$$x(t)=\sum_{k=0}^{[m/2]}\frac{(m-k)!}{k!\,t}\left(\frac{2}{t}\right)^{m-2k}.$$

THEOREM 2.10. *The equation*

$$t^2x''+2(1-\gamma)\,tx'-[t^2+2\gamma+\nu(\nu+2\gamma+1)]\,x=0 \tag{2.45}$$

*has an m-order distributional solution* (1) *if and only if*

$$\nu(\nu+2\gamma+1)=m(m+2\gamma+1). \tag{2.46}$$

*This solution is given by the formula*

$$x(t)=\sum_{k=0}^{[m/2]}\frac{(-1)^k\,2^{m-2k}\Gamma(m+\lambda-k)}{k!\,(m-2k)!\,\Gamma(\lambda)}\,\delta^{(m-2k)}(t), \tag{2.47}$$

*whose coefficients coincide with the corresponding coefficients of the Gegenbauer polynomial* $C_m^\lambda(t)$, *where* $\Gamma$ *denotes the gamma function and*

$$\lambda=\gamma+\tfrac{1}{2}\geqslant 0. \tag{2.48}$$

*Proof.* Substituting (1) in (2.45) gives

$$(m-\nu)(m+\nu+2\gamma+1)\,x_m=0, \qquad (m-1-\nu)(m+\nu+2\gamma)\,x_{m-1}=0,$$
$$(\nu-k)(k+\nu+2\gamma+1)\,F_k=-F_{k+2}, \qquad F_k=k!\,x_k, \qquad k=0,...,m-2.$$

If (2.46) holds true, we can take $x_m\neq 0$ and find all $x_k$ $(k<m)$ from the equations

$$2(2m+2\lambda-2)\,F_{m-2}=-F_m, \qquad 4(2m+2\lambda-4)\,F_{m-4}=-F_{m-2},$$
$$..., 2k(2m+2\lambda-2k)\,F_{m-2k}=-F_{m-2k+2},$$

whence

$$x_{m-2k}=\frac{(-1)^k\,m!\,\Gamma(m+\lambda-k)}{4^k k!\,(m-2k)!\,\Gamma(m+\lambda)}\,x_m,$$

where $\lambda$ is defined by (2.48). Again, we can normalize $x_m$ so that the coefficients $x_{m-2k}$ coincide with the coefficients of the Gegenbauer polynomials $C_m^\lambda(t)$ generated by the expansion [19]

$$(1-2tx+x^2)^{-\lambda}=\sum_{m=0}^{\infty}C_m^\lambda(t)\,x^m.$$

Differentiating with respect to $t$ gives

$$\sum_{m=0}^{\infty}\frac{dC_m^\lambda}{dt}\,x^m=2\lambda x(1-2tx+x^2)^{-\lambda-1}$$
$$=2\lambda\sum_{m=0}^{\infty}C_m^{\lambda+1}(t)\,x^{m+1}$$
$$=2\lambda\sum_{m=0}^{\infty}C_{m-1}^{\lambda+1}(t)\,x^m$$

and

$$\frac{dC_m^\lambda}{dt} = 2\lambda C_{m-1}^{\lambda+1}.$$

Repeatedly applying this procedure yields the equation

$$\frac{d^j C_m^\lambda}{dt^j} = 2^j \frac{\Gamma(\lambda+j)}{\Gamma(\lambda)} C_{m-j}^{\lambda+j},$$

and since $C_0^\lambda = 1$, we obtain

$$\frac{d^m C_m^\lambda}{dt^m} = 2^m \frac{\Gamma(m+\lambda)}{\Gamma(\lambda)}.$$

Therefore, we set

$$x_m = 2^m \frac{\Gamma(m+\lambda)}{m!\,\Gamma(\lambda)}$$

and write the distributional solution in form (2.47), whose coefficients really coincide with the coefficients of $C_m^\lambda(t)$, because applying the Laplace transformation to (2.45) leads to the differential equation

$$(p^2-1)F'' + 2(1+\gamma)pF' - \nu(\nu+2\gamma+1)F = 0 \tag{2.49}$$

for the Gegenbauer polynomials. It remains to note that (2.49) generalizes Chebyshev's equations of the first and second kind, which follow for $\gamma = -\frac{1}{2}$ and $\gamma = \frac{1}{2}$, respectively. If $\gamma = 0$, we get Legendre's equation.

Theorems 2.4–2.10 show, in particular, that the study of polynomial solutions to some important classes of linear ODE with several singular points can be reduced to a technically easier task (which is important also in its own right) of exploring distributional solutions (1) of Eq. (2.8) with the only singular point $t=0$. This approach will be extended later to the study of analytic solutions of linear ODE and FDE.

Theorem 2.1 opens an easy way of constructing examples of linear ODE having finite-order distributional solutions. Indeed, take a linear ODE with constant coefficients

$$(D-k_1)(D-k_2)\ldots(D-k_n)\, y(s) = 0,$$

where some of the numbers $k_i$ are negative integers and $D = d/ds$, and make the substitutions $e^s = t$, $y(s) = x(t)$. Then Euler's equation

$$\left(t\frac{d}{dt} - k_1\right)\left(t\frac{d}{dt} - k_2\right)\cdots\left(t\frac{d}{dt} - k_n\right) x(t) = 0$$

has a rational solution (2.2) and, hence, admits also a finite-order distributional solution.

It has been proved in Theorem 2.1 that the existence of a rational solution (2.2) to Eq. (2.1) implies the existence of a distributional solution (1). A question arises: when is the converse true? In other words, when is $q(t)=0$ identically in (2.3)?

THEOREM 2.11. *If Eq.* (2.1) *with polynomial coefficients* $q_i(t)$ *admits a distributional solution* (1) *of order* $m$ *and if*

$$\deg q_i(t) \leqslant n-i, \qquad i=0, ..., n \tag{2.50}$$

*then it also has a rational solution* (2.2).

*Proof.* The Laplace transform $F(p)$ of solution (1) to (2.1) satisfies Eq. (2.4). Setting $p>0$ and applying the right-sided Laplace transformation to (2.4) yields Eq. (2.6), with $q(s)=0$ identically. Indeed, by virtue of (2.50), the order of the differential operator $q_i(-d/dp)$ in (2.4) does not exceed $n-i$. Hence, the function $p^{n-i}F(p)$ and all its derivatives of orders up to and including $\deg q_i - 1$ equal zero at $p=0$.

EXAMPLE 4. The equation

$$t^2x''' + (at-6)\,x' + ax = 0, \qquad a = \text{const}$$

admits a distributional solution $x=\delta(t)$ and satisfies (2.50). Hence, it has also a rational solution $x=t^{-1}$. The equation

$$tx'' + (t+3)\,x' + x = 0$$

admits a distributional solution $x=\delta(t)+\delta'(t)$ and also a rational solution $x=t^{-1}-t^{-2}$.

*Remark.* The conditions of Theorem 2.11 are sufficient but not necessary for the existence of a rational solution to Eq. (2.1). Thus, it has been shown above that the equation

$$(t^2+t)\,x' + (t+m+1)x = 0$$

admits a rational solution (2.2) but does not satisfy (2.50).

The class of equations

$$tx^{(n)}(t) + \sum_{i=1}^{n} a_i(t)\,x^{(n-i)}(t) = 0, \qquad a_i \in C^{m+n-i} \tag{2.51}$$

has been studied in [7] and a theorem proved that (2.51) has a distributional solution of order $m$ if and only if $a_1(0)=m+n$ and there exists a non-zero solution to the system

$$(a_1(0)\,k!-(k+1)!)\,x_{k+1-n} + \sum_{j=2}^{m+n} x_{k+j-n} \sum_{i=1}^{\min(j,n)} (-1)^{j-i} \times a_i^{(j-i)}(0)(k+j-i)!=0 \qquad (k=0,\ldots,m+n-1).$$

Since the verification of this condition is often difficult, we consider the particular case

$$\sum_{i=0}^{n} (a_i t+b_i)\,x^{(n-i)}(t)=0, \tag{2.52}$$

which admits simple methods of study and yet remains sufficiently important.

THEOREM 2.12. *Equation* (2.52) *with constant coefficients* $a_i, b_i$, *and* $a_0=1$, $b_0=0$ *has a finite-order distributional solution if and only if all poles* $p_i$ *of the function*

$$R(p)=\sum_{i=0}^{n} (b_i p-(n-i)\,a_i)\,p^{n-i-1}\Big/\sum_{i=0}^{n} a_i p^{n-i}$$

*are distinct, all residues*

$$r_i=\operatorname*{res}_{p=p_i} R(p)$$

*are non-negative integers, and the residues of the complex conjugate poles are equal. The solution is given by the formula*

$$x=C\prod_{i=0}^{n} (d/dt-p_i)^{r_i}\,\delta(t), \qquad C=\text{const} \tag{2.53}$$

*and its order is*

$$m=\sum_{i=0}^{n} r_i.$$

*If* $a_n=0$, *there exists also a solution*

$$x=C\prod_{i=0}^{n} (d/dt-p_i)^{r_i}\,t^{-1}. \tag{2.54}$$

*Proof.* Equation (2.52) has a finite-order solution if and only if the equation

$$F'(p) = R(p)F(p) \tag{2.55}$$

obtained by applying the Laplace transformation to (2.52) admits a polynomial solution. Hence, the integral of $R(p)$ must contain only logarithms of non-negative integer powers of real linear factors or irreducible quadratic trinomials, which is ensured by the conditions of the theorem. Furthermore, (2.53) follows directly from

$$(\ln F)' = \sum_{i=1}^{n} r_i (p - p_i)^{-1}. \tag{2.56}$$

Applying the Laplace transformation to (2.55), with $p > 0$, and taking into account $a_n = 0$, we get the relation

$$\sum_{i=0}^{n} (-1)^{n-i} (a_i s - b_i)\, x^{(n-i)}(s) = 0$$

which can be reduced to (2.52) by the substitution $s = -t$. Since $L[\delta] = 1$ and $L[1] = s^{-1}$, expression (2.54) is a solution of (2.52).

*Remark.* If $a_n \neq 0$, there exists a constant $c_0$ such that (2.54) is a solution of the equation

$$\sum_{i=0}^{n} (a_i t + b_i)\, x^{(n-i)}(t) = c_0.$$

COROLLARY. *The confluent hypergeometric equation*

$$tx'' + (b - t)\, x' - ax = 0 \tag{2.57}$$

*has a finite-order solution iff a and b are positive integers and $b \geqslant a + 1$. The solution is given by the formula* [6]

$$x = C d^{a-1}/dt^{a-1} (d/dt - 1)^{b-a-1}\, \delta(t) \tag{2.58}$$

*and its order is $m = b - 2$. There exists also a solution*

$$x = C d^{a-1}/dt^{a-1} (d/dt - 1)^{b-a-1}\, t^{-1}.$$

EXAMPLE 5. The equation [20]

$$tx'' + ax' + btx = 0, \qquad b \neq 0$$

has a finite-order solution iff the coefficient $a$ is a positive even integer. This solution is given by the formula

$$x = C(d^2/dt^2 + b)^{(a-2)/2}\,\delta(t)$$

and its order is $m = a - 2$. There exists a constant $c_0$ such that

$$x = C(d^2/dt^2 + b)^{(a-2)/2}\,t^{-1}$$

is a solution to

$$tx'' + ax' + btx = c_0.$$

EXAMPLE 6. The equation [20]

$$tx''' + (a+b)\,x'' - tx' - ax = 0$$

has a distributional solution iff $a$ is a positive integer and $b$ is even positive. This solution is given by the formula

$$x = Cd^{a-1}/dt^{a-1}(d^2/dt^2 - 1)^{b/2-1}\,\delta(t)$$

and its order is $m = a + b - 3$. There exists also a solution

$$x = Cd^{a-1}/dt^{a-1}(d^2/dt^2 - 1)^{b/2-1}\,t^{-1}.$$

EXAMPLE 7. The equation [20]

$$tx''' - (t+a)\,x'' - (t-a-1)\,x' + (t-1)\,x = 0$$

has a distributional solution iff $a$ is a negative integer, $a \leqslant -3$. This solution is given by the formula

$$x = C(d^2/dt^2 - 1)^{-(a+3)/2}\,\delta(t)$$

and its order is $m = -a - 3$. There exists also a solution

$$x = C(d^2/dt^2 - 1)^{-(a+3)/2}\,t^{-1}.$$

THEOREM 2.13. *Equation* (2.52) *has a polynomial solution of degree m if and only if the following conditions hold true*:

(i) $a_i = 0,\ i = n, n-1, ..., n-N;\ a_{n-N-1} \neq 0$

(ii) $b_i = 0,\ i = n, n-1, ..., n-N+1$

(iii) $b_{n-N}/a_{n-N-1} = -(m-N)$,

*for an integer N such that* $0 \leqslant N \leqslant \min(m, n-1)$.

*Proof.* Equation (2.52) written in terms of the variables $p$ and $F$ is the Laplace-transformed relation of

$$\left(\sum_{i=0}^{n} (-1)^{n-i} a_i t^{n-i}\right) x'(t) + \left(\sum_{i=0}^{n} (-1)^{n-i} (a_i(n-i) + b_i t)\, t^{n-i-1}\right) x(t) = 0. \tag{2.59}$$

Hence, (2.52) has a polynomial solution of degree $m$ if and only if (2.59) admits an $m$-order distributional solution (1). If $a_n \neq 0$, then $t=0$ is a regular point of Eq. (2.59), and in this case (2.59) has no singular distributional solution. Therefore, the condition $a_n = 0$ is necessary for the existence of a solution (1) to (2.59). Assuming $a_n = 0$, we write (2.59) as

$$t\left(\sum_{i=0}^{n-1} (-1)^i a_i t^{n-i-1}\right) x'(t) + \left(\sum_{i=0}^{n-1} (-1)^i a_i(n-i)\, t^{n-i-1} + \sum_{i=0}^{n} (-1)^i b_i t^{n-i}\right) x(t) = 0,$$

which is of form (2.8). Now, assume that (2.59) admits a distributional solution (1) of order $m$. If $a_{n-1} \neq 0$, then according to (2.9), we have

$$(-1)^{n-1} a_{n-1} + (-1)^n b_n - (-1)^{n-1} (m+1)\, a_{n-1} = 0,$$

that is,

$$b_n = -m a_{n-1}.$$

If $a_{n-1} = 0$ and $a_{n-2} \neq 0$, then (2.59) takes the form

$$t^2\left(\sum_{i=0}^{n-2} (-1)^i a_i t^{n-i-2}\right) x'(t) + t\left(\sum_{i=0}^{n-2} (-1)^i a_i(n-i)\, t^{n-i-2}\right) x(t) + \left(\sum_{i=0}^{n} (-1)^i b_i t^{n-i}\right) x(t) = 0,$$

and since $t^2x'$ and $tx$ are both distributions of order $m-1$, we conclude that $b_n = 0$. Therefore,

$$t^2\left(\sum_{i=0}^{n-2} (-1)^i a_i t^{n-i-2}\right) x'(t) + t\left(\sum_{i=0}^{n-2} (-1)^i a_i(n-i)\, t^{n-i-2} + \sum_{i=0}^{n-1} (-1)^i b_i t^{n-i-1}\right) x(t) = 0,$$

and the substitution $y = tx$ changes this equation to

$$t\left(\sum_{i=0}^{n-2} (-1)^i a_i t^{n-i-2}\right) y'(t) + \left(\sum_{i=0}^{n-2} (-1)^i a_i (n-i-1)\, t^{n-i-2} + \sum_{i=0}^{n-1} (-1)^i b_i t^{n-i-1}\right) y(t) = 0,$$

which is of form (2.8). Since $x(t)$ is an $m$-order distribution, $y(t)$ is of order $m-1$. Hence, by virtue of (2.9), we have

$$(-1)^{n-2} a_{n-2} + (-1)^{n-1} b_{n-1} - (-1)^{n-2} m a_{n-2} = 0,$$

that is,

$$b_{n-1} = -(m-1)\, a_{n-2}.$$

Assuming

$$a_i = 0, \qquad i = n, n-1, ..., n-N; \qquad a_{n-N-1} \neq 0$$

implies

$$b_i = 0, \qquad i = n, n-1, ..., n-N+1$$

and

$$t^{N+1}\left(\sum_{i=0}^{n-N-1} (-1)^i a_i t^{n-i-N-1}\right) x'(t) + t^N \left(\sum_{i=0}^{n-N-1} (-1)^i a_i (n-i)\, t^{n-i-N-1} + \sum_{i=0}^{n-N} (-1)^i b_i t^{n-i-N}\right) x(t) = 0.$$

The substitution $y = t^N x$ reduces this equation to

$$t\left(\sum_{i=0}^{n-N-1} (-1)^i a_i t^{n-i-N-1}\right) y'(t) + \left(\sum_{i=0}^{n-N-1} (-1)^i a_i (n-i-N)\, t^{n-i-N-1} + \sum_{i=0}^{n-N} (-1)^i b_i t^{n-i-N}\right) y(t) = 0,$$

and since $y(t)$ is a distribution of order $m-N$, we obtain, by virtue of (2.9),

$$(-1)^{n-N-1} a_{n-N-1} + (-1)^{n-N} b_{n-N} - (-1)^{n-N-1} (m-N+1)\, a_{n-N-1} = 0$$

or

$$b_{n-N} = -(m-N)\, a_{n-N-1}. \tag{2.60}$$

This proves the conditions of the theorem are necessary. They are also sufficient because hypothesis (iii) ensures that Eq. (2.60), which represents relation (2.9) for (2.52), has a unique (hence, the smallest) non-negative integer solution $m$.

THEOREM 2.14. *The differential equation*

$$(1-p^2)\, F''(p) + [\beta - \alpha - (\alpha+\beta+2)\, p]\, F'(p) + \gamma F(p) = 0, \tag{2.61}$$

*where $\gamma$ is a parameter, has a polynomial solution if and only if $\gamma$ is of the form*

$$\gamma = m(m+\alpha+\beta+1), \qquad m = 0, 1, 2, \ldots . \tag{2.62}$$

*Proof.* Equation (2.61) is the Laplace-transformed relation of

$$t^2x'' - (\alpha+\beta-2)\, tx' - [t^2 + (\alpha-\beta)\, t + (\alpha+\beta+\gamma)]\, x = 0.$$

This equation is of type (2.8), and it admits a finite-order distributional solution (1) if and only if the corresponding Eq. (2.9),

$$(m+2)(m+1) - (2-\alpha-\beta)(m+1) - (\alpha+\beta+\gamma) = 0,$$

that is,

$$m(m+\alpha+\beta+1) - \gamma = 0,$$

has a non-zero integer root $m$. This is so if $\gamma$ is chosen according to (2.62). A classical proof of this theorem, without the use of distributions, may be found in [19]. Equation (2.61), with $\gamma$ defined by (2.62), is the differential equation for the Jacobi polynomials.

THEOREM 2.15. *The hypergeometric equation*

$$p(1-p)\, F''(p) + [\gamma - (\alpha+\beta+1)\, p]\, F'(p) - \alpha\beta F(p) = 0 \tag{2.63}$$

*has a polynomial solution of degree $m$ if and only if*

$$\alpha = -m \qquad or \qquad \beta = -m.$$

*This solution is given by the formula*

$$F(\alpha, \beta, \gamma; p) = \sum_{k=0}^{m} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k k!}\, p^k, \tag{2.64}$$

*where*

$$(\lambda)_0 = 1, \qquad (\lambda)_k = \Gamma(\lambda+k)/\Gamma(\lambda) = \lambda(\lambda+1)\cdots(\lambda+k-1).$$

*Proof.* Equation (2.63) is the Laplace-transformed relation of

$$t^2x'' - [t + (\alpha+\beta-3)]\, tx' + [(\gamma-2)t + (\alpha\beta+1-\alpha-\beta)]\, x = 0. \tag{2.65}$$

Substituting (1) in (2.65) gives

$$\sum_{k=0}^{m-1} [(k+\alpha)(k+\beta)\, x_k - (k+1)(k+\gamma)\, x_{k+1}]\, \delta^{(k)}(t) + (m+\alpha)(m+\beta)\, x_m\, \delta^{(m)}(t) = 0,$$

whence

$$(m+\alpha)(m+\beta)\, x_m = 0,$$
$$(k+\alpha)(k+\beta)\, x_k - (k+1)(k+\gamma)\, x_{k+1} = 0, \qquad k = 0, ..., m-1.$$

The condition $(m+\alpha)(m+\beta) = 0$ enables us to choose $x_m = (\alpha)_m(\beta)_m/(\gamma)_m m!$ and to find the coefficients of (2.64). The substitution $p = 1-2s$ in (2.61) changes the Jacobi equation to the hypergeometric equation

$$s(1-s)\, F''(s) + [\alpha+1-(\alpha+\beta+2)s]\, F'(s) + m(m+\alpha+\beta+1)F(s) = 0$$

and leads to the important formula [19] for the Jacobi polynomials

$$P_m^{(\alpha,\beta)}(t) = \binom{m+\alpha}{m} F\left(-m, m+\alpha+\beta+1, \alpha+1; \frac{1-t}{2}\right)$$

normalized by the condition

$$P_m^{(\alpha,\beta)}(1) = \binom{m+\alpha}{m}.$$

THEOREM 2.16. *The hypergeometric equation* (2.63) *has a distributional solution* (1) *of order m if and only if*

(i) $\gamma = m+2$ *and*

(ii) $\alpha$ *or* $\beta$ *is an integer* $j = 1, ..., m+1$.

*This solution written in terms of the variables t and x is given by the formula*

$$x(t) = \sum_{k=\alpha-1}^{m} \binom{m+1-\alpha}{m-k} \delta^{(k)}(t)/\Gamma(k+2-\beta), \tag{2.66}$$

*where α is either the single integer or the greater of the two integers satisfying* (ii).

*Proof.* Substituting (1) in Eq. (2.63) written in terms of $t$ and $x$ leads to the relations

$$(m+2-\gamma)\,x_m=0,$$
$$(k+1-\alpha)(k+1-\beta)\,x_k=(m+1-k)\,x_{k-1}, \qquad k=0,...,m;\ x_{-1}=0. \tag{2.67}$$

Since $x_m \neq 0$, hypothesis (i) is a necessary condition for the existence of an $m$-order solution (1) to (2.63). The equation for $x_0$ is

$$(1-\alpha)(1-\beta)\,x_0=0,$$

and if $\alpha\neq 1$, $\beta\neq 1$, then $x_0=0$. Furthermore, if (ii) is not satisfied, (2.67) implies that $x_k=0$, for all $k=0,...,m$. On the other hand, if $\alpha$ is an integer such that $1\leqslant\alpha\leqslant m+1$ and $\beta$ is not, then

$$x_k=0, \qquad k=0,...,\alpha-2$$

and

$$\begin{aligned}
1\cdot(\alpha+1-\beta)\,x_\alpha &= (m+1-\alpha)\,x_{\alpha-1},\\
2\cdot(\alpha+2-\beta)\,x_{\alpha+1} &= (m-\alpha)\,x_\alpha,\\
&\cdots\\
j\cdot(\alpha+j-\beta)\,x_{\alpha+j-1} &= (m-\alpha-j+2)\,x_{\alpha+j-2} \qquad (0\leqslant j\leqslant m+1-\alpha).
\end{aligned}$$

Multiplying these relations yields (2.66), with the exactness to a constant factor, and this result remains valid also when both $\alpha$ and $\beta$ are integers such that $1\leqslant\alpha$, $\beta\leqslant m+1$, $\alpha\geqslant\beta$.

COROLLARY. *Under conditions of Theorem* 2.16, *the hypergeometric equation has a rational solution*

$$x(t)=\sum_{k=\alpha-1}^{m}(-1)^k\binom{m+1-\alpha}{m-k}k!\,t^{-k-1}/\Gamma(k+2-\beta).$$

THEOREM 2.17. *The hypergeometric equation* (2.63) *written in terms of t and x has a distributional solution of order m with support* $t=1$ *if and only if*

(i) $\alpha+\beta-\gamma=m+1$ *and*

(ii) $\alpha$ *or* $\beta$ *is an integer* $j=1,...,m+1$.

*This solution is given by the formula*

$$x(t)=\sum_{k=\alpha-1}^{m}(-1)^k\binom{m+1-\alpha}{m-k}\delta^{(k)}(t-1)/\Gamma(k+2-\beta), \qquad (2.68)$$

*where $\alpha$ is either the single integer or the greater of the two integers satisfying* (ii).

*Proof.* The substitutions $p=1-t$, $F(p)=x(t)$ change (2.63) to the hypergeometric equation

$$t(1-t)\,x''(t)+[(\alpha+\beta-\gamma+1)-(\alpha+\beta+1)\,t]\,x'(t)-\alpha\beta x(t)=0,$$

and it remains to apply Theorem 2.16 to the latter equation. Formula (2.68) follows from (2.66) and from the equality

$$\delta^{(k)}(1-t)=(-1)^k\,\delta^{(k)}(t-1).$$

Under conditions (i) and (ii), there exists also a solution

$$x(t)=\sum_{k=\alpha-1}^{m}\binom{m+1-\alpha}{m-k}k!\,(t-1)^{-k-1}/\Gamma(k+2-\beta). \qquad (2.69)$$

EXAMPLE 8. For the equation [11]

$$t(1-t)\,x''+(1-3t)\,x'-x=0,$$

with $\alpha=\beta=\gamma=1$, the condition $\alpha+\beta-\gamma=m+1$ implies $m=0$. Hence, there exist a distributional solution $x(t)=\delta(t-1)$ and a rational solution $x(t)=(t-1)^{-1}$. For the equation [11]

$$t(1-t)\,x''+(1-5t)\,x'-4x=0,$$

with $\alpha=\beta=2$, $\gamma=1$, the same condition implies $m=2$. Formula (2.68) yields the distributional solution $x(t)=\delta'(t-1)-\delta''(t-1)$, and (2.69) provides the rational solution

$$x(t)=(t-1)^{-2}+2(t-1)^{-3}=(t+1)(t-1)^{-3}.$$

It has been indicated above that applying the Laplace transformation to first- and second-order equations of type (2.8), whose coefficients are such that (2.9) admits a non-negative integer root, generates the most important linear ODE for orthogonal polynomials. New classes of linear ODE with polynomial solutions may be produced by applying the Laplace transformation to higher-order equations (2.8), in particular, to

$$\sum_{i=0}^{n} t^i(a_i t+b_i)\,x^{(i)}(t)=0$$

and

$$\sum_{i=0}^{n} t^i(a_i t^2 + b_i)\, x^{(i)}(t) = 0.$$

This approach is based on Theorem 2.3. Of course, the parameters $a_i$, $b_i$ should be selected in such a way that Eq. (2.9) has a non-negative integer root. This condition is very easy to satisfy. Indeed, it suffices to consider the non-negative integer $m$ and all coefficients of (2.9), except one, as given numbers, and to find this unknown coefficient. For instance, recently W. N. Everitt and L. L. Littlejohn [21] have studied the Legendre-type polynomials that satisfy the fourth-order equation

$$(p^2-1)^2 F^{(4)} + 8p(p^2-1) F^{(3)} + (4\alpha+12)(p^2-1) F'' + 8\alpha F' + \beta F = 0, \tag{2.70}$$

which is the Laplace-transformed relation of

$$t^4 x^{(4)} + 8t^3 x^{(3)} + 2[(2\alpha+6) - t^2]\, t^2 x'' + 8(\alpha - t^2)\, tx' + (\beta - t^2)\, x = 0.$$

Simplifying the corresponding equation (2.9)

$$(m+4)! - 8(m+3)! + (4\alpha+12)(m+2)! - 8\alpha(m+1)! + \beta m! = 0$$

yields the criterion

$$\beta = -m(m+1)(m^2+m+4\alpha-2)$$

for the existence of a polynomial solution of degree $m$ to (2.70).

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## References

1. J. Wiener, Generalized-function and analytic solutions of functional differential equations, *in* "Dynamical Systems II" (A. R. Bednarek and L. Cesari, Eds.), pp. 655–661, Academic Press, New York, 1982.
2. J. Wiener, Distributional and entire solutions of linear functional differential equations, *Internat. J. Math. Math. Sci.* **5** (1982), 729–736.
3. S. M. Shah and J. Wiener, Distributional and entire solutions of ordinary differential and functional differential equations, *Internat. J. Math. Math. Sci.* **6** (1983), 243–270.
4. S. M. Shah and J. Wiener, Existence theorems for analytic solutions of differential and functional differential equations, *J. Ramanujan Math. Soc.* **1** (1986), 85–97.

5. K. L. COOKE AND J. WIENER, Distributional and analytic solutions of functional differential equations, *J. Math. Anal. Appl.* **98** (1983), 111–129.
6. J. WIENER, Generalized-function solutions of linear systems, *J. Differential Equations* **38** (1980), 301–315.
7. J. WIENER, Generalized-function solutions of differential and functional differential equations, *J. Math. Anal. Appl.* **88** (1982), 170–182.
8. J. WIENER, Generalized-function solutions of differential equations with a countable number of argument deviations, *Differencial'nye Uravneniya* **2** (1978), 355–358.
9. J. WIENER, Existence of solutions of differential equations with deviating argument in the space of generalized functions, *Sibirsk. Mat. Zh.* **6** (1976), 1403–1405.
10. I. M. GEL'FAND AND G. E. SHILOV, "Generalized Functions," Vol. 2, "Spaces of Fundamental and Generalized Functions," Academic Press, New York, 1968.
11. L. L. LITTLEJOHN AND R. P. KANWAL, Distributional solutions of the hypergeometric differential equation, *J. Math. Anal. Appl.* **122** (1987), 325–345.
12. A. H. ZEMANIAN, "Generalized Integral Transformations," Interscience, New York, 1969.
13. J. WIENER, Solutions of linear systems in generalized functions, *Differencial'nye Uravneniya* **6** (1975), 1128–1130.
14. N. N. LEBEDEV, "Special Functions and Their Applications," Dover, New York, 1972.
15. J. P. HOYT, Remarks on a second order differential equation, *Amer. Math. Monthly* **74** (1967), 436.
16. W. LEIGHTON, On the differential equation $xy'' - (x+n)\, y' + ny = 0$, *Amer. Math. Monthly* **76** (1969), 52–53.
17. T. A. NEWTON, On using a differential equation to generate polynomials, *Amer. Math. Monthly* **81** (1974), 592–601.
18. D. ZEITLIN, On a class of ordinary linear differential equations having $\sum_{k=0}^{\infty} C_k x^k$ and $\sum_{k=0}^{n} C_k x^k$ as solutions, *Amer. Math. Monthly* **84** (1977), 716–720.
19. G. SZEGÖ, "Orthogonal Polynomials," 4th ed. AMS Colloquium Publ. XXIII, Providence, RI, 1978.
20. E. KAMKE, "Gewöhnliche Differentialgleichungen," Akad. Verlagsgesellschaft, Geest & Portig K.-G., Leipzig, 1959.
21. W. N. EVERITT AND L. L. LITTLEJOHN, Differential operators and the Legendre type polynomials, *Differential Integral Equations* **1** (1988), 97–116.