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A New Class of Point Interactions in One Dimension*

PAUL R. CHERNOFF

Department of Mathematics, University of California at Berkeley, Berkeley, California 94720

AND

RHONDA J. HUGHES

Department of Mathematics, Bryn Mawr College, Bryn Mawr, Pennsylvania 19010-2899

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We present a class of self-adjoint extensions of the symmetric operator $-\Delta \mid C_0^\infty(\mathbb{R}^1 \setminus \{0\})$ which correspond formally to perturbations of the Laplacian by pseudopotentials involving δ^2. These operators, which provide new examples of generalized point interactions in the sense of Šeba, are defined by the boundary conditions $f(0^+) = e^{-z} f(0^-)$, $rf(0^+) + f'(0^+) = e^{z}[rf(0^-) + f'(0^-)]$, for $z \in \mathbb{C}$, $r \in \mathbb{R}$. We calculate their spectra, resolvents, and scattering matrices, and show that they can be realized as limits of Schrödinger operators with local short-range potentials.

1. INTRODUCTION

Hamiltonians involving generalized point interactions in one dimension were introduced by Šeba [Seb1], who showed that certain self-adjoint extensions of $-\Delta \mid C_0^\infty(\mathbb{R}^1 \setminus \{0\})$ could be realized as perturbations of the Laplacian by Fermi pseudopotentials. In two and three dimensions, the point interactions correspond to perturbations by δ-function potentials (cf. [AFH]), but the phenomena are much richer in one dimension. To understand this, note that the symmetric operator $A = -\Delta \mid C_0^\infty(\mathbb{R}^1 \setminus \{0\})$ has deficiency indices (2, 2), while in two and three dimensions the analogous operators have deficiency indices (1, 1). Consequently, in one dimension the operator A has a four-parameter family of self-adjoint extensions. In [Seb1], Šeba gives a comprehensive analysis of all the self-adjoint

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extensions of A in terms of the Cayley transform parametrization by 2×2 unitary matrices, and the associated boundary conditions.

Of particular interest are the boundary conditions which "link" the intervals $(-\infty, 0)$ and $(0, \infty)$. It is straightforward to show that the most general set of self-adjoint boundary conditions of this type is

$$f(0^+)=\alpha f(0^-)+\beta f'(0^-), \tag{1.1}$$

$$f'(0^+)=\gamma f(0^-)+\delta f'(0^-). \tag{1.2}$$

Here the coefficients, complex in general, must be of the form $\alpha=a\omega$, $\beta=b\omega$, $\gamma=c\omega$, $\delta=d\omega$, where ω is a complex number of modulus 1, and a, b, c, d are real with $ad-bc=1$ (cf. [GK]). The domain of the self-adjoint operator $L_{\alpha\beta\gamma\delta}$ corresponding to these boundary conditions consists of all functions f in $H^2(\mathbb{R}^1\setminus\{0\})$ which satisfy (1.1), (1.2); and we have $L_{\alpha\beta\gamma\delta}f=A^*f$; that is, $L_{\alpha\beta\gamma\delta}f(x)=-f''(x)$ for $x\neq 0$. In particular, $L_{\alpha\beta\gamma\delta}$ is a local operator; that is, if f vanishes on an open set U, then so does $L_{\alpha\beta\gamma\delta}f$. (This is a particular case of [AGHH, Lemma C.2].)

Some of these boundary conditions correspond to "point interactions," or perturbations of $-\Delta$ by distribution potentials. Thus the formal operators $-\Delta+c\delta$ and $-\Delta+b\delta'$ correspond to the respective boundary conditions $f(0^+)=f(0^-)$, $f'(0^+)-f'(0^-)=cf(0)$, and $f(0^+)-f(0^-)=bf'(0)$, $f'(0^+)=f'(0^-)$ (cf. [GHM], [AGHH]).

In [Seb1], Šeba finds a two-dimensional family of generalized point interactions formally given by perturbations of the Laplacian by one-dimensional "Fermi pseudopotentials"

$$-\Delta+c\delta(x)+(1-a)[\delta'(x)-\delta(x)\,d/dx]+b\delta'(x)\,d/dx.$$

The corresponding boundary conditions are

$$f(0^+)=af(0^-)+bf'(0^-),$$

$$f'(0^+)=cf(0^-)+(2-a)f'(0^-);$$

here a, b, and c are real parameters satisfying the relation $bc+(1-a)^2=0$.

Šeba asks whether other boundary conditions, for example, $f(0^+)=-f(0^-)$, $f'(0^+)=-f'(0^-)$ can be realized by pseudopotentials in a similar way, and whether it is possible to obtain the corresponding Hamiltonians as limits of Schrödinger operators with local short-range potentials.

In this paper we present the three dimensional class of self-adjoint extensions of A defined by boundary conditions of the form

$$\begin{gathered} f(0^+)=e^{-z}f(0^-), \\ rf(0^+)+f'(0^+)=e^{z}[rf(0^-)+f'(0^-)],\quad z\in\mathbb{C},\ r\in\mathbb{R}. \end{gathered} \tag{1.3}$$

We show that these extensions correspond to perturbations by pseudo-potentials formally involving, among other terms, δ^2. It is easy to see that the class of operators defined by the boundary conditions (1.3) is nearly disjoint from Šeba's class; the only common operator is the free Hamiltonian $-\Delta$.

Our approach exploits Segal's method of interpreting the formal operator $(1/i)(d/dx) + a\delta(x)$ in one dimension (cf. [Seg]) and enables us to answer the questions above, as well as to determine the resolvent, spectral, and scattering properties of the resulting operators.

In [Seg], Segal defines a self-adjoint operator on $L^2(\mathbb{R}^1)$ corresponding to the formal expression $(1/i)(d/dx) + V'$, where V is any real measurable function on $\mathbb{R}^1$, by $T_V = e^{-iV}(1/i)(d/dx)\, e^{iV}$. In particular, if V is the Heaviside function, then $V' = \delta$, the Dirac delta function. The operator T_V can be approximated in the strong resolvent sense by operators with smooth coefficients. A function $\phi \in L^2(\mathbb{R}^1)$ belongs to $\mathrm{Dom}(T_V)$ if and only if $e^{iV}\phi \in H^1(\mathbb{R}^1) = \mathrm{Dom}(d/dx)$.

In the same spirit, for $z \in \mathbb{C}$, we define $T_z = e^{-zH}(d/dx)\, e^{zH}$, where H is the Heaviside function; formally T_z appears to be $d/dx + z\delta$. However, noting that T_z is periodic in z with period $2\pi i$, we argue in Section 6 that the correct formal interpretation is $T_z = d/dx + 2\tau\delta$, where $\tau = (e^z - 1)/(e^z + 1) = \tanh(z/2)$. (If $z = \pm i\pi$, then τ is formally $\pm i\infty$. Note that τ never equals 1.) The operator T_z is closed with dense domain $\mathrm{Dom}(T_z) = \{\phi \in L^2(\mathbb{R}^1)\colon e^{zH}\phi \in H^1(\mathbb{R}^1)\}$. This means that $\phi \in H^1(\mathbb{R}^1 \setminus \{0\})$, and that ϕ satisfies the boundary condition $\phi(0^+) = e^{-z}\phi(0^-)$.

Because $T_z\phi = \phi'$ away from 0, the self-adjoint operator on $L^2(\mathbb{R}^1)$ defined by $B_z = T_z^* T_z$ is an extension of A. Since $T_z^* = -e^{\bar{z}H}(d/dx)\, e^{-\bar{z}H}$, B_z is formally

$$\left(-\frac{d}{dx} + 2\bar{\tau}\,\delta(x)\right)\left(\frac{d}{dx} + 2\tau\,\delta(x)\right)$$

$$= -\Delta - 4i \operatorname{Im} \tau\delta(x)\frac{d}{dx} - 2\tau\,\delta'(x) + 4|\tau|^2\,\delta(x)^2.$$

We get a more general family $L_{\{r,z\}}$ of extensions of A as follows. For $r \in \mathbb{R}$, $z \in \mathbb{C}$, define $Q_{\{r,z\}} = (r + T_z)^*\,(r + T_z)$, and $L_{\{r,z\}} = Q_{\{r,z\}} - r^2 I$, where I is the identity operator on $L^2(\mathbb{R}^1)$. Then $\mathrm{Dom}(L_{\{r,z\}}) = \mathrm{Dom}(Q_{\{r,z\}})$, and $\phi \in \mathrm{Dom}(Q_{\{r,z\}})$ if and only if $\phi \in \mathrm{Dom}(T_z)$ and $(rI + T_z)\,\phi \in \mathrm{Dom}(T_z^*)$. Thus $\phi(x) = e^{-zH(x)}\psi(x)$, where $\psi \in H^1(\mathbb{R}^1)$; in particular, $\phi(0^+)$ and $\phi(0^-)$ exist and satisfy the boundary condition

$$\phi(0^+) = e^{-z}\phi(0^-). \tag{1.4}$$

In addition, the function $(rI + T_z)\,\phi(x) = r\phi(x) + e^{-zH}\psi'(x)$ $(x \neq 0)$ must belong to $\mathrm{Dom}(T_z^*)$, so that

$$r\phi(x) + e^{-zH(x)}\psi'(x) = e^{zH(x)}\theta(x), \qquad \text{for some} \quad \theta \in H^1(\mathbb{R}^1);$$

this yields a second boundary condition

$$r\phi(0^+) + \phi'(0^+) = e^{z}[r\phi(0^-) + \phi'(0^-)]. \tag{1.5}$$

Note that, using (1.4), one sees that condition (1.5) is equivalent to

$$\phi'(0^+) - e^{z}\phi'(0^-) = r(e^{z} - e^{-z})\,\phi(0^-). \tag{1.6}$$

From the definitions it is clear that if $r = 0$ or z is purely imaginary, then $L_{\{r,z\}} = B_z$. Formally, $L_{\{r,z\}} = B_z + r(T_z + T_z^*)$, or

$$L_{\{r,z\}} = -\Delta - 4i\,\mathrm{Im}\,\tau\delta(x)\frac{d}{dx} - 2\tau\,\delta'(x)$$
$$+ 4|\tau|^2\,\delta(x)^2 + 4r\,\mathrm{Re}\,\tau\delta(x).$$

When $z = 0$, so that $\tau = 0$, the operator $L_{\{r,z\}}$ reduces to the free Laplacian $-\Delta$; moreover, for purely imaginary $z = i\gamma$, the operator $B_z = e^{-i\gamma H}(-d^2/dx^2)\,e^{i\gamma H}$ is unitarily equivalent to $-\Delta$.

The boundary conditions (1.4), (1.6) that define $L_{\{r,z\}}$ may be rewritten in the form

$$\phi(0^+) = \rho e^{i\varphi}\phi(0^-),$$
$$\phi'(0^+) = te^{i\varphi}\phi(0^-) + \rho^{-1}e^{i\varphi}\phi'(0^-),$$

where $e^{-z} = \rho e^{i\varphi}$ and $t = -r(\rho - \rho^{-1})$. These correspond to the general boundary conditions (1.1) and (1.2), with $\omega = e^{i\varphi}$, $a = \rho$, $b = 0$, $c = t$, and $d = \rho^{-1}$.

Finally, we note that the boundary conditions $f(0^+) = -f(0^-)$, $f'(0^+) = -f'(0^-)$ mentioned by Šeba correspond to the operator $B_{\pi i}$.

2. Cayley Transform Parametrization

In this section we will determine the 2×2 unitary matrix $\theta = (\theta_{ij})$ corresponding to the operator $L_{\{r,z\}}$ in the Cayley transform parametrization of self-adjoint extensions of $A = -\Delta \,|\, C_0^\infty(\mathbb{R}^1\setminus\{0\})$. Recall (cf. [RS, Theorem X.2]) that the extension A_θ corresponding to θ is the restriction of the adjoint A^* to the domain

$$\mathrm{Dom}(A_\theta) = \{f \in L^2(\mathbb{R}^1) \colon f = \phi + c_1 v_{\theta,1} + c_2 v_{\theta,2}\},$$

where $c_1, c_2 \in \mathbb{C}$, $\phi \in \mathrm{Dom}(\bar{A}) = H_0^2(\mathbb{R}^1 \setminus \{0\})$, and

$$v_{\theta,j}(x) = (h_+)_j - \theta_{1j}(h_-)_1 - \theta_{2j}(h_-)_2, \qquad j = 1, 2. \tag{2.1}$$

Here, with $\zeta = e^{\pi i/4}$,

$$(h_+)_1(x) = \frac{i\bar{\zeta}}{2} e^{i\zeta|x|}, \qquad (h_+)_2(x) = \frac{i}{2}\,\mathrm{sgn}(x)\, e^{i\zeta|x|},$$

$$(h_-)_1(x) = \frac{-i\zeta}{2} e^{-i\bar{\zeta}|x|}, \qquad (h_-)_2(x) = \frac{i}{2}\,\mathrm{sgn}(x)\, e^{-i\bar{\zeta}|x|}.$$

(These functions are bases for the deficiency subspaces of the operator A.)

Imposing the boundary condition (1.4), on $v_{\theta,1}$, we get

$$\bar{\zeta} + \zeta\theta_{11} + \theta_{21} = e^{z}(\bar{\zeta} + \zeta\theta_{11} - \theta_{21}),$$

or

$$\theta_{21} = \tau(\bar{\zeta} + \zeta\theta_{11}), \tag{2.2}$$

where, as in Section 1, $\tau = \tanh(z/2)$.

We note that, for $x \neq 0$,

$$v'_{\theta,1}(x) = -\frac{1}{2}\,\mathrm{sgn}(x)\, e^{i\zeta|x|} + \frac{\theta_{11}}{2}\,\mathrm{sgn}(x)\, e^{-i\bar{\zeta}|x|} - \frac{\zeta}{2}\theta_{21} e^{-i\bar{\zeta}|x|},$$

so that the second boundary condition (1.6) implies that

$$-1 + \theta_{11} - \bar{\zeta}\theta_{21} - e^{z}(1 - \theta_{11} - \bar{\zeta}\theta_{21}) = r(e^{z} - e^{-\bar{z}})(\bar{\zeta} + \zeta\theta_{11} + \theta_{21}).$$

Using (2.2) together with the formula $(e^{z} - e^{-\bar{z}})/(e^{z} + 1) = 2\,\mathrm{Re}\,\tau/(\bar{\tau} + 1)$, we obtain

$$\theta_{11} = \frac{1 + i|\tau|^2 + r\bar{\zeta}(1 + \tau)(e^{z} - e^{-\bar{z}})/(e^{z} + 1)}{1 + |\tau|^2 + r\zeta(1 + \tau)(e^{-\bar{z}} - e^{z})/(e^{z} + 1)}$$
$$= \frac{1 + i|\tau|^2 + 2r\bar{\zeta}\,\mathrm{Re}\,\tau}{1 + |\tau|^2 - 2r\zeta\,\mathrm{Re}\,\tau}.$$

Then

$$\theta_{21} = \tau\left(\bar{\zeta} + \zeta\left(\frac{1 + i|\tau|^2 + 2r\bar{\zeta}\,\mathrm{Re}\,\tau}{1 + |\tau|^2 - 2r\zeta\,\mathrm{Re}\,\tau}\right)\right)$$
$$= \frac{\sqrt{2}\,\tau}{1 + |\tau|^2 - 2r\zeta\,\mathrm{Re}\,\tau}.$$

Imposing the boundary condition (1.4) on $v_{\theta,2}$ yields

$$1 + \zeta\theta_{12} - \theta_{22} = e^{-z}[-1 + \zeta\theta_{12} + \theta_{22}],$$

whence we obtain the relation

$$\theta_{22} = 1 + \zeta\tau\theta_{12}. \tag{2.3}$$

Using orthogonality of the rows of θ together with (2.3), we see that

$$\theta_{12} = \frac{-\bar{\theta}_{21}}{\bar{\theta}_{11} + \zeta\tau\bar{\theta}_{21}} = \frac{-\sqrt{2}\,\bar{\tau}}{1 + |\tau|^2 + 2r\zeta \operatorname{Re} \tau}.$$

Finally, (2.3) yields the formula

$$\begin{aligned} \theta_{22} &= 1 + \zeta\tau\theta_{12} \\ &= 1 - \frac{\sqrt{2}\,\zeta|\tau|^2}{1 + |\tau|^2 + 2r\zeta \operatorname{Re} \tau} \\ &= \frac{1 - i|\tau|^2 + 2r\zeta \operatorname{Re} \tau}{1 + |\tau|^2 + 2r\zeta \operatorname{Re} \tau}. \end{aligned}$$

Thus the matrix corresponding to the Cayley transform parametrization of $L_{\{r,z\}}$ is

$$\theta(r, z) = \begin{pmatrix} \dfrac{1 + i|\tau|^2 + 2r\bar{\zeta} \operatorname{Re} \tau}{1 + |\tau|^2 - 2r\zeta \operatorname{Re} \tau} & \dfrac{-\sqrt{2}\,\bar{\tau}}{1 + |\tau|^2 + 2r\zeta \operatorname{Re} \tau} \\ \dfrac{\sqrt{2}\,\tau}{1 + |\tau|^2 - 2r\zeta \operatorname{Re} \tau} & \dfrac{1 - i|\tau|^2 + 2r\zeta \operatorname{Re} \tau}{1 + |\tau|^2 + 2r\zeta \operatorname{Re} \tau} \end{pmatrix}, \quad \tau = \tanh(z/2). \tag{2.4}$$

When $\tau = \pm i\infty$ we take limits in (2.4) to get

$$\theta(r, \pm i\pi) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Remark. In contrast to (2.4), the matrix corresponding to $-\Delta + c\delta$ is $\left(\begin{smallmatrix} e^{i\omega} & 0 \\ 0 & 1 \end{smallmatrix}\right)$, with $c = -\sqrt{2}(1 + \cot(\omega/2 - \pi/4))$. For $-\Delta + b\delta'$, the corresponding matrix is $\left(\begin{smallmatrix} 1 & 0 \\ 0 & e^{i\omega} \end{smallmatrix}\right)$, with $b = -\sqrt{2}\,(1 + \tan(\omega/2 - \pi/4))$.

3. Resolvent and Spectrum

In this section we compute the resolvent of the self-adjoint operator $L_{\alpha\beta\gamma\delta}$ corresponding to the boundary conditions (1.1), (1.2); the resolvent of the operator $L_{\{r,z\}}$ is a special case. When the parameters α, β, γ, δ are real, our resolvent formula follows from the Green's function calculation summarized in [Seb1]. We also discuss the spectrum, eigenfunctions, and resonances. Our main results are in the spirit of [AGHH], where similar properties of other solvable models are discussed. We use a perturbation theory method, namely Krein's formula [AGHH, Appendix A] together with the boundary conditions.

The resolvent of $L = L_{\alpha\beta\gamma\delta}$ is of the following form. For $\operatorname{Im} k > 0$, convenient orthogonal solutions of $(A^* - k^2)\phi = 0$ are $\psi_{1,k}(x) = e^{ik|x|}$ and $\psi_{2,k}(x) = \operatorname{sgn}(x)\, e^{ik|x|}$. There are coefficients $\lambda(k) = \{\lambda_{ij}\}_{i,j=1}^{2}$ so that

$$(L - k^2)^{-1} = \mathbf{G}_k + \sum_{l,m=1}^{2} \lambda_{l,m}(k) \langle \cdot, \overline{\psi_{m,k}} \rangle\, \psi_{l,k}, \tag{3.1}$$

where $\mathbf{G}_k = (-\Delta - k^2)^{-1}$.

Given $g \in L^2(\mathbb{R})$, set $f = (L - k^2)^{-1} g$. Then, using the familiar integral representation of $\mathbf{G}_k$, we get from (3.1) the formula

$$\begin{aligned} f(x) = {} & \frac{i}{2k} \int e^{ik|x-y|} g(y)\, dy + \lambda_{11} \int e^{ik(|x|+|y|)} g(y)\, dy \\ & + \lambda_{12} \int e^{ik(|x|+|y|)} \operatorname{sgn}(y)\, g(y)\, dy + \lambda_{21} \operatorname{sgn}(x) \int e^{ik(|x|+|y|)} g(y)\, dy \\ & + \lambda_{22} \operatorname{sgn}(x) \int e^{ik(|x|+|y|)} \operatorname{sgn}(y)\, g(y)\, dy. \end{aligned} \tag{3.2}$$

From this formula it follows that, for $x \neq 0$,

$$\begin{aligned} f'(x) = {} & \frac{1}{2} \int e^{ik|x-y|} \operatorname{sgn}(y - x)\, g(y)\, dy \\ & + ik\lambda_{11} \operatorname{sgn}(x) \int e^{ik(|x|+|y|)} g(y)\, dy \\ & + ik\lambda_{12} \operatorname{sgn}(x) \int e^{ik(|x|+|y|)} \operatorname{sgn}(y)\, g(y)\, dy \\ & + ik\lambda_{21} \int e^{ik(|x|+|y|)} g(y)\, dy \\ & + ik\lambda_{22} \int e^{ik(|x|+|y|)} \operatorname{sgn}(y)\, g(y)\, dy. \end{aligned} \tag{3.3}$$

By taking appropriate limits in formulas (3.2) and (3.3) we obtain the following expressions for $f(0^\pm)$ and $f'(0^\pm)$:

$$\begin{aligned} f(0^\pm) = {} & \left(\frac{i}{2k} + \lambda_{11} \pm \lambda_{21} \right) \int e^{ik|y|} g(y)\, dy \\ & + (\lambda_{12} \pm \lambda_{22}) \int e^{ik|y|} \operatorname{sgn}(y)\, g(y)\, dy \end{aligned} \tag{3.4}$$

and

$$f'(0^{\pm}) = (\pm ik\lambda_{11} + ik\lambda_{21}) \int e^{ik|y|} g(y)\, dy$$

$$+ \left(\frac{1}{2} \pm ik\lambda_{12} + ik\lambda_{22}\right) \int e^{ik|y|} \operatorname{sgn}(y)\, g(y)\, dy. \tag{3.5}$$

Next, impose boundary condition (1.1), using formulas (3.4) and (3.5) for $f(0^+)$, $f(0^-)$, and $f'(0^-)$. Because g is arbitrary and $\psi_{1,k}$ and $\psi_{2,k}$ are orthogonal, we get two equations:

$$\left(\frac{i}{2k} + \lambda_{11} + \lambda_{21}\right) = \alpha\left(\frac{i}{2k} + \lambda_{11} - \lambda_{21}\right) + ik\beta(-\lambda_{11} + \lambda_{21}) \tag{3.6}$$

and

$$(\lambda_{12} + \lambda_{22}) = \alpha(\lambda_{12} - \lambda_{22}) + \beta(\tfrac{1}{2} - ik\lambda_{12} + ik\lambda_{22}). \tag{3.7}$$

Similarly, from boundary condition (1.2), we get two more equations:

$$ik(\lambda_{11} + \lambda_{21}) = \gamma\left(\frac{i}{2k} + \lambda_{11} - \lambda_{21}\right) + ik\delta(-\lambda_{11} + \lambda_{21}) \tag{3.8}$$

and

$$(\tfrac{1}{2} + ik\lambda_{12} + ik\lambda_{22}) = \gamma(\lambda_{12} - \lambda_{22}) + \delta(\tfrac{1}{2} - ik\lambda_{12} + ik\lambda_{22}). \tag{3.9}$$

These four equations separate into two sets; (3.6), (3.8) give λ_{11} and λ_{21}, while (3.7), (3.9) give λ_{12} and λ_{22}. The result, in matrix form, is

$$\lambda(k) = \frac{1}{D_0(k)}\begin{pmatrix} i\gamma/2k + (\alpha + \delta - 1 - \Delta)/4 & -(\alpha - \delta + 1 - \Delta)/4 \\ -(\alpha - \delta - 1 + \Delta)/4 & i\beta k/2 - (\alpha + \delta - 1 - \Delta)/4 \end{pmatrix}, \tag{3.10}$$

where $\Delta = \alpha\delta - \beta\gamma$, and

$$D_0(k) = \beta k^2 + i(\alpha + \delta)\, k - \gamma. \tag{3.11}$$

Alternatively, if we write the parameters in the form $\alpha = a\omega$, $\beta = b\omega$, $\gamma = c\omega$, $\delta = d\omega$, where $|\omega| = 1$ and a, b, c, d are real with $ad - bc = 1$, then, after cancelling some factors of ω, we get

$$\lambda(k) = \frac{1}{D(k)}\begin{pmatrix} ic/2k + (a + d - \omega - \bar{\omega})/4 & -(a - d + \bar{\omega} - \omega)/4 \\ -(a - d + \omega - \bar{\omega})/4 & ibk/2 - (a + d - \omega - \bar{\omega})/4 \end{pmatrix}, \tag{3.12}$$

where

$$D(k) = bk^2 + i(a+d)\,k - c. \tag{3.13}$$

From the resolvent formula (3.1), we see that the eigenvalues E of the operator $L_{\alpha\beta\gamma\delta}$ correspond to the poles (if any) of $\lambda(k)$ such that $k = i\kappa$, $\kappa > 0$. (It is easy to see directly that 0 is not an eigenvalue of $L_{\alpha\beta\gamma\delta}$.) In other words, $E = -\kappa^2$, where $k = i\kappa$ is a zero of the polynomial $D(k)$ with $\kappa > 0$. Note that the substitution $k = i\kappa$ gives $-D(i\kappa) = b\kappa^2 + (a+d)\,\kappa + c$.

If $b = 0$ (i.e., $\beta = 0$), then $D(k)$ has one zero, namely

$$k_0 = i\kappa_0 = \frac{-ic}{a+d}. \tag{3.14}$$

Here the denominator $a+d$ does not vanish because $ad = 1$.

If $b \neq 0$ (i.e., $\beta \neq 0$), then $D(k)$ has two distinct zeros:

$$k_1 = i\kappa_1 = i[-(a+d) + \sqrt{(a-d)^2+4}]/2b \tag{3.15}$$

and

$$k_2 = i\kappa_2 = i[-(a+d) - \sqrt{(a-d)^2+4}]/2b; \tag{3.16}$$

here we have used the relation $ad - bc = 1$.

The following theorem summarizes our results.

THEOREM 3.1. *Let $\alpha = a\omega$, $\beta = b\omega$, $\gamma = c\omega$, $\delta = d\omega$ with $|\omega| = 1$, and a, b, c, d real, satisfying $ad - bc = 1$. Then the operator $L_{\alpha\beta\gamma\delta}$ has essential spectrum which is purely absolutely continuous with uniform multiplicity 2: $\sigma_{ess}(L_{\alpha\beta\gamma\delta}) = \sigma_{ac}(L_{\alpha\beta\gamma\delta}) = [0\ \infty)$. $L_{\alpha\beta\gamma\delta}$ has no positive eigenvalues, and at most two negative eigenvalues. For $\operatorname{Im} k > 0$, the resolvent of $L_{\alpha\beta\gamma\delta}$ is*

$$(L_{\alpha\beta\gamma\delta} - k^2)^{-1} = (-\Delta - k^2)^{-1} + \frac{1}{D(k)} \sum_{l,m=1}^{2} C_{l,m}(k) \langle \cdot, \overline{\psi_{m,k}} \rangle\, \psi_{l,k}. \tag{3.17}$$

Here $D(k) = bk^2 + i(a+d)\,k - c$, and the coefficients $C_{l,m}(k) = D(k)\,\lambda_{l,m}(k)$, where $\lambda(k)$ is given by Eq. (3.12).

The resolvent has integral kernel (Green's function)

$$\begin{aligned} K_{\alpha\beta\gamma\delta}(x, y; k) &= \frac{i}{2k} e^{ik|x-y|} \\ &\quad + \frac{1}{D(k)} \{[ic/2k + (a+d-\omega-\bar{\omega})/4] \\ &\quad - (a-d+\bar{\omega}-\omega)/4 \operatorname{sgn}(y) - (a-d+\omega-\bar{\omega})/4 \operatorname{sgn}(x) \\ &\quad + [ibk/2 - (a+d-\omega-\bar{\omega})/4] \operatorname{sgn}(x) \operatorname{sgn}(y)\}\, e^{ik(|x|+|y|)}. \end{aligned} \tag{3.18}$$

Proof. As everyone knows, in one dimension $-\Delta$ has spectrum $[0, \infty)$ which is purely absolutely continuous with uniform multiplicity 2. Since the resolvent of $L = L_{\alpha\beta\gamma\delta}$ is a finite-rank perturbation of the resolvent of $-\Delta$, the fact that its essential spectrum is $[0, \infty)$ follows from Weyl's theorem on perturbations by compact operators. The more refined conclusion that $\sigma_{ac}(L) = [0, \infty)$ with uniform multiplicity 2 is an application of [Ka, Theorem X.4.3].

We have already noted that the eigenvalues of L are of the form $E = -\kappa^2$, where $k = i\kappa$ is a zero of the quadratic polynomial $D(k)$, with $\kappa > 0$. ∎

Remark. From formula (3.18), it might appear that the Green's function has a simple pole at $k = 0$. However, if $c \neq 0$, it is easy to see that the residue vanishes, so in fact there is no pole at $k = 0$.

COROLLARY 3.1.1. (i) *If $b = 0$, let $\kappa_0 = -c/(a+d)$ and let*

$$\phi_0(x) = \begin{cases} e^{\kappa_0 x}, & x < 0 \\ \alpha e^{-\kappa_0 x}, & x > 0. \end{cases}$$

If $\kappa_0 > 0$, then $E_0 = -\kappa_0^2$ is a simple eigenvalue of $L_{\alpha\beta\gamma\delta}$ with eigenfunction ϕ_0. If $\kappa_0 < 0$, then $L_{\alpha\beta\gamma\delta}$ has a simple resonance (or "virtual state") at E_0, with resonance function ϕ_0.

(ii) *If $b \neq 0$, let $\kappa_j = -ik_j$, $j = 1, 2$, be given by* (3.15) *and* (3.16). *Let*

$$\phi_j(x) = \begin{cases} e^{\kappa_j x}, & x < 0 \\ (\alpha + \beta\kappa_j)\, e^{-\kappa_j x}, & x > 0. \end{cases}$$

If $\kappa_j > 0$, then $E_j = -\kappa_j^2$ is a simple eigenvalue of $L_{\alpha\beta\gamma\delta}$ with eigenfunction ϕ_j. If $\kappa_j < 0$, then there is a simple resonance at E_j with resonance function ϕ_j.

Proof. The Green's function (3.18), analytically continued to the entire k-plane, has a simple pole at $k = i\kappa_0$, if $b = 0$, or κ_j, $j = 1, 2$, if $b \neq 0$. If $\kappa_j > 0$, then $E_j = -\kappa_j^2$ is an eigenvalue of L. If $\kappa_j < 0$, then E_j corresponds to a resonance (also called a "virtual state" because E_j is real).

From the explicit formulas (3.10) or (3.12) for the matrix $\lambda(k)$, one can also show that the residue of the resolvent at $i\kappa_j$ is a rank-one operator P_j. Hence E_j is a simple eigenvalue (or simple resonance). The corresponding eigenfunction (or resonance function) can be found by inspecting the range of P_j. Alternatively, it is easy to compute the eigenfunctions directly by solving the equation $L\phi = -\phi'' = -\kappa^2\phi$ $(x \neq 0)$, and imposing the boundary conditions (1.1) and (1.2) at $x = 0$. ∎

Finally, we specialize the preceding general results to the case of the operator $L_{\{r,z\}}$. Here $\alpha = e^{-z}$, $\delta = e^{z}$, $\beta = 0$, $\gamma = -r(\alpha - \delta)$, and the determinant $\alpha\delta - \beta\gamma$ reduces to $\alpha\delta$. We will compute $\lambda(k)$ from the formulas (3.10) and (3.11) in terms of the parameters r and τ, where, as before,

$$\tau = (e^z - 1)/(e^z + 1) = \tanh(z/2),$$

so that

$$e^z = (1 + \tau)/(1 - \tau).$$

We find

$$\begin{aligned} \alpha + \delta &= 2(1 + |\tau|^2)/f(\tau), \\ \alpha - \delta = -4 \operatorname{Re} \tau/f(\tau), &\qquad \gamma = 4r \operatorname{Re} \tau/f(\tau), \end{aligned} \tag{3.19}$$

and

$$\alpha\delta = (1 - |\tau|^2 - 2i \operatorname{Im} \tau)/f(\tau),$$

where $f(\tau) = 1 - |\tau|^2 + 2i \operatorname{Im} \tau$.

A straightforward computation yields the formula

$$\lambda(k) = \frac{1}{g(k)} \begin{pmatrix} 2r \operatorname{Re} \tau/k - i|\tau|^2 & -i\bar{\tau} \\ -i\tau & i|\tau|^2 \end{pmatrix}, \tag{3.20}$$

where $g(k) = 2(1 + |\tau|^2)\, k + 4ir \operatorname{Re} \tau$.

Note that if $e^z = -1$, so that formally $\tau = \pm i\infty$, we can either compute $\lambda(k)$ directly or else take limits in (3.20) to get

$$\lambda(k) = \frac{1}{2k} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

THEOREM 3.2. *The resolvent of the operator* $L_{\{r,z\}}$ *has integral kernel*

$$\begin{aligned} K_{\{r,z\}}(x, y; k) = \frac{i}{2k} e^{ik|x-y|} - \frac{i}{g(k)} \{(|\tau|^2 + 2ir \operatorname{Re} \tau/k) \\ + \bar{\tau} \operatorname{sgn}(y) + \tau \operatorname{sgn}(x) - |\tau|^2 \operatorname{sgn}(x) \operatorname{sgn}(y)\}\, e^{ik(|x| + |y|)}, \end{aligned}$$

where $g(k) = 2(1 + |\tau|^2)\, k + 4ir \operatorname{Re} \tau$.

$L_{\{r,z\}}$ *has essential spectrum* $[0, \infty)$, *purely absolutely continuous with uniform multiplicity* 2, *together with at most one eigenvalue.*

Let $\kappa_0 = -(2r \operatorname{Re} \tau)/(1 + |\tau|^2)$ *and set* $E_0 = -\kappa_0^2$. *Define*

$$\phi(x) = [1 - \tau \operatorname{sgn}(x)]\, e^{-\kappa_0 |x|}.$$

If $r \operatorname{Re} \tau < 0$, *then* E_0 *is a simple eigenvalue of* $L_{\{r,z\}}$ *with corresponding eigenfunction* ϕ. *If* $r \operatorname{Re} \tau > 0$, *then* E_0 *is a simple resonance of* $L_{\{r,z\}}$ *with resonance function* ϕ. *If* $r \operatorname{Re} \tau = 0$, *then* $L_{\{r,z\}}$ *has no eigenvalues.*

Proof. The conclusions follow immediately from Theorem 3.1 and Corollary 3.1.1, together with formula (3.20). Concerning the eigenvalues or resonances, note that κ_0 and $r \operatorname{Re} \tau$ have opposite signs; hence $\kappa_0 > 0$ if $r \operatorname{Re} \tau < 0$, and then $E_0 = -\kappa_0^2$ is an eigenvalue. Case (i) of Corollary 3.1.1 shows that

$$\phi(x) = \begin{cases} (1+\tau)\, e^{\kappa_0 x}, & x < 0 \\ (1-\tau)\, e^{-\kappa_0 x}, & x > 0 \end{cases}$$

is the corresponding eigenfunction, since $(1-\tau)/(1+\tau) = e^{-z} = \alpha$. Similarly, if $r \operatorname{Re} \tau > 0$, there is a resonance at E_0 with resonance function ϕ. ∎

Remark. If $U_y f(x) = f(x+y)$ is the unitary translation operator, then

$$L_{\alpha\beta\gamma\delta;\, y} = U_y^{-1} L_{\alpha\beta\gamma\delta} U_y \tag{3.21}$$

is an extension of $-\Delta \mid C_0^\infty(\mathbb{R}^1 \setminus \{y\})$ with boundary conditions at y corresponding to (1.1) and (1.2). In particular, $L_{\{r,z\};\, y} = U_y^{-1} L_{\{r,z\}} U_y$ corresponds to our generalized point interaction shifted to y. By shifting the variables in the preceding formulas, one can immediately write down the resolvent, Green's kernel, and eigenfunctions for these operators. We omit the details.

4. Scattering Theory

In this section, we calculate the scattering matrix for the operators $L_{\alpha\beta\gamma\delta}$. In particular, we recover the known results for perturbations of $-\Delta$ by δ- and δ'-potentials found in [AGHH, I.3, I.4]. Finally we specialize our general results to the case of the operators $L_{\{r,z\}}$.

Because the resolvent of $L_{\alpha\beta\gamma\delta}$ is a finite-rank perturbation of the resolvent of $-\Delta$, general scattering theory (cf. [Ka, X.3, X.4]) implies that the wave operators exist and that the scattering operator S is unitary and intertwines $-\Delta$ with itself. Hence S can be represented as a direct integral of operators $S(k)$ on the generalized (or "continuum") eigenspaces $\mathcal{M}_k$ of $-\Delta$; for $k \geq 0$, the space $\mathcal{M}_k$ corresponds to the generalized eigenvalue k^2 in the spectrum, which, as we recalled in Section 3, is purely absolutely continuous with uniform multiplicity 2.

$\mathcal{M}_k$ is two dimensional, with a basis given by the functions

$$\phi_{r,k}(x) = e^{ikx} \qquad \text{and} \qquad \phi_{l,k}(x) = e^{-ikx};$$

in the formalism of time-independent scattering theory these represent waves with momentum k traveling to the right and left, respectively. With respect to this basis $S(k)$ has the matrix representation

$$S(k) = \begin{pmatrix} T_l(k) & R_r(k) \\ R_l(k) & T_r(k) \end{pmatrix};$$

here $R_l(k)$, $T_l(k)$ and $R_r(k)$, $T_r(k)$ are the "left-hand" and "right-hand" reflection and transmission coefficients, respectively.

We first consider waves incident from the left, and scattered by the point interaction at the origin. For $k \geqslant 0$, write

$$f_k(x) = \begin{cases} e^{ikx} + R_l(k)\, e^{-ikx}, & x < 0 \\ T_l(k)\, e^{ikx}, & x > 0. \end{cases}$$

This function must be a generalized eigenfunction of the operator $L_{\alpha\beta\gamma\delta}$, with eigenvalue k^2. Now $f_k(x)$ and its derivative have boundary values

$$f_k(0^+) = T_l(k), \qquad f'_k(0^+) = ikT_l(k),$$

and

$$f_k(0^-) = 1 + R_l(k), \qquad f'_k(0^-) = ik(1 - R_l(k))$$

whence, from the boundary conditions (1.1) and (1.2), we obtain the relations

$$T_l(k) = \alpha + ik\beta + (\alpha - ik\beta)\, R_l(k)$$

$$ikT_l(k) = \gamma + ik\delta + (\gamma - ik\delta)\, R_l(k).$$

Solving these equations, the left-hand coefficients are

$$T_l(k) = \frac{2ik(\alpha\delta - \beta\gamma)}{D_0(k)}, \qquad R_l(k) = \frac{\gamma + k^2\beta - ik(\alpha - \delta)}{D_0(k)}, \tag{4.1}$$

where $D_0(k) = k^2\beta + ik(\alpha + \delta) - \gamma$, as in Eq. (3.11). After cancelling some factors of ω, we obtain

$$T_l(k) \frac{2ik\omega}{D(k)}, \qquad R_l(k) = \frac{c + k^2 b - ik(a - d)}{D(k)}. \tag{4.2}$$

Here, as in (3.13), $D(k) = k^2 b + ik(a + d) - c$.

(Note: Using the fact that $ad - bc = 1$, it is straightforward to verify that $|T_l(k)|^2 + |R_l(k)|^2 = 1$. This is partial check that $S(k)$ is unitary.)

Next, we calculate the right-hand coefficients. For $k \geqslant 0$, let

$$f_k(x) = \begin{cases} T_r(k)\, e^{-ikx}, & x < 0 \\ e^{-ikx} + R_r(k)\, e^{ikx}, & x > 0. \end{cases}$$

Then $f_k(0^+) = 1 + R_r(k)$, $f'_k(0^+) = ik(-1 + R_r(k))$, $f_k(0^-) = T_r(k)$, and $f'_k(0^-) = -ikT_r(k)$. Using (1.1) and (1.2), we obtain

$$1 + R_r(k) = (\alpha - ik\beta)\, T_r(k)$$

and

$$ik(-1+R_r(k)) = (\gamma - ik\delta)\, T_r(k).$$

Accordingly,

$$T_r(k) = \frac{2ik}{D_0(k)}, \qquad R_r(k) = \frac{\gamma + k^2\beta + ik(\alpha - \delta)}{D_0(k)},$$

or, equivalently,

$$T_r(k) = \frac{2ik\bar{\omega}}{D(k)}, \qquad R_r(k) = \frac{c + k^2 b + ik(a-d)}{D(k)}. \tag{4.3}$$

From (4.2) and (4.3), we can immediately write down a formula for $S(k)$:

THEOREM 4.1. *The scattering matrix for $L_{\alpha\beta\gamma\delta}$ is given by*

$$S(k) = \frac{1}{D(k)} \begin{pmatrix} 2ik\omega & c + k^2 b + ik(a-d) \\ c + k^2 b - ik(a-d) & 2ik\bar{\omega} \end{pmatrix}, \tag{4.4}$$

where $D(k) = k^2 b + ik(a+d) - c$.

Remarks. (1) Note that the "on-shell" matrix $S(k)$ (k real) is unitary, as expected.

(2) From (4.4), we see that in general the analytically continued S-matrix has simple poles at the roots of the quadratic polynomial $D(k)$. Recall from (3.14)–(3.16) that if $b \neq 0$, then $D(k)$ has two distinct roots, and if $b=0$, then it has just one root, namely $k = -ic/(a+d)$. Assuming that $b \neq 0$, both roots are non-zero unless $c=0$; in which case a factor of k cancels and the matrix $S(k)$ reduces to

$$S(k) = \frac{1}{i(a+d)+kb} \begin{pmatrix} 2i\omega & kb + i(a-d) \\ kb - i(a-d) & 2i\bar{\omega} \end{pmatrix}.$$

Because $c=0$ we have $ad=1$, and therefore $a+d \neq 0$. So in this subcase we find that $S(k)$ has just one pole, at the non-zero point $k = -i(a+d)/b$.

Finally, if both b and c are 0, we find that

$$S(k) = \frac{1}{i(a+d)} \begin{pmatrix} 2i\omega & i(a-d) \\ -i(a-d) & 2i\bar{\omega} \end{pmatrix}$$

which is a constant matrix, independent of k.

Thus, we have the following corollary.

COROLLARY 4.1.1. *The poles of the analytically continued scattering matrix $S(k)$ are the same as those of the Green's kernel of $L_{\alpha\beta\gamma\delta}$, except that S never has a pole at $k=0$.*

That $S(k)$ never has a pole at $k=0$, seems to be a general phenomenon (cf. [AGHH, I.3.4 and I.4]), although it does not seem to have been proved in complete generality. (But observe that it follows from unitarity provided that $S(k)$ is continuous in k for real k.)

Finally, we turn to the particular case of the operator $L_{\{r,\tau\}}$. In this case, as noted in Section 3, the parameters are $\alpha = e^{\tau}$, $\delta = e^{\bar{\tau}}$, $\beta = 0$, and $\gamma = -r(\alpha - \delta)$. Using (3.19), we get the following formula for the scattering matrix from Theorem 4.1.

THEOREM 4.2. *The scattering matrix for the operator $L_{\{r,\tau\}}$ is given by*

$$S(k) = \frac{1}{h(k)} \begin{pmatrix} (1-\tau)(1+\bar{\tau})\, k & -2i \operatorname{Re} \tau(r - ik) \\ -2i \operatorname{Re} \tau(r+ik) & (1+\tau)(1-\bar{\tau})\, k \end{pmatrix},$$

where $h(k) = k(1+|\tau|^2) + 2ir \operatorname{Re} \tau$.

To conclude this section, we point out that completely similar results hold for interactions centered at any point $y \in \mathbb{R}$. Define $L_{\alpha\beta\gamma\delta;\,y}$ as in (3.21); then the corresponding scattering operator is

$$S_y = U_y^{-1} S_0 U_y,$$

where S_0 is the previously calculated scattering operator corresponding to $y=0$. In the momentum representation, U_y is given by the diagonal matrix

$$\tilde{U}_y(k) = \begin{pmatrix} e^{iky} & 0 \\ 0 & e^{-iky} \end{pmatrix},$$

so that

$$\begin{aligned} S_y(k) &= \tilde{U}_y(k)^{-1}\, S_0(k)\, \tilde{U}_y(k) \\ &= \frac{1}{D(k)} \begin{pmatrix} 2ik\omega & [c + k^2 b + ik(a-d)]\, e^{-2iky} \\ [c + k^2 b - ik(a-d)]\, e^{2iky} & 2ik\bar{\omega} \end{pmatrix} \end{aligned}$$

(cf. the corresponding calculations for the $\delta(x-y)$ and $\delta'(x-y)$ interactions in [AGHH, I.3.4 and I.4]).

5. APPROXIMATION BY OPERATORS WITH SMOOTH COEFFICIENTS

In [Seb1], Šeba asks whether the generalized point interaction Hamiltonians he studies can be approximated by Schrödinger operators with smooth (possibly nonlocal) potentials. It is well known that the δ-potential Hamiltonian can be represented as the norm resolvent limit of operators with local, scaled, short-range potentials (cf. [AGHK]), but this has been achieved only with nonlocal potentials in the case of the δ'-potential (cf. [Seb3]). In this section, we show that the family of point interactions $L_{\{r,z\}}$ can be approximated, in the strong resolvent sense, by Schrödinger operators with smooth local short-range potentials (Theorem 5.1).

We begin with the case of purely imaginary $z = i\gamma$, which is particularly tractable, for $L_{\{r,i\gamma\}} = B_{i\gamma}$ is unitarily equivalent to $-\Delta$. Indeed, let $\{h_n\}$ be any sequence of smooth functions such that $h_n \geqslant 0$, $\int_{-\infty}^{\infty} h_n(x)\,dx = 1$, and h_n is supported in $[0, 1/n]$ (so that $h_n \to \delta$ in the usual sense of approximation of distributions). Set $H_n(x) = \int_{-\infty}^{x} h_n(y)\,dy$; then $H_n(x) = 0$ for $x \leqslant 0$, $H_n(x) = 1$ for $x \geqslant 1/n$, $0 \leqslant H_n \leqslant 1$ everywhere, and $H_n(x) \to H(x)$. Consequently, $B_{i\gamma,n} = e^{-i\gamma H_n}(-\Delta)\,e^{i\gamma H_n}$ converges in the strong group sense, and hence the strong resolvent sense, to $B_{i\gamma}$. Moreover, the operator $B_{i\gamma,n}$ is obviously a perturbation of $-\Delta$ by a first-order differential operator with smooth coefficients supported in $[0, 1/n]$.

Note. This answers in the affirmative a question of Šeba regarding the operator $B_{\pi i}$ (cf. [Seb1]).

Before proceeding to the general case, let $\{H_n\}$ be a sequence of functions as defined above, and define

$$L_{\{r,z,n\}} = (r + T^*_{z,n})(r + T_{z,n}) - r^2 I,$$

where $T_{z,n} = e^{-zH_n}(d/dx)\,e^{zH_n}$. As above, the operator $L_{\{r,z,n\}}$ is also a perturbation of $-\Delta$ by a first-order differential operator with smooth coefficients supported in $[0, 1/n]$.

THEOREM 5.1. *Given $r \in \mathbb{R}$ and $z \in \mathbb{C}$, the sequence of operators $L_{\{r,z,n\}}$ converges to $L_{\{r,z\}}$ in the strong resolvent sense.*

Proof. First, consider the case $r \neq 0$. For $z \in \mathbb{C}$, T_z generates the uniformly bounded one-parameter group of operators on $L^2(\mathbb{R}^1)$ given by

$$\exp(tT_z) = e^{-zH} e^{tD} e^{zH},$$

where $e^{tD} = U_t$ is the translation group generated by $D = d/dx$, with $\mathrm{Dom}(D) = H^1(\mathbb{R}^1)$; explicitly, for $\phi \in L^2(\mathbb{R}^1)$, we have

$$(\exp(tT_z)\,\phi)(x) = e^{z[H(x+t) - H(x)]}\phi(x+t).$$

This immediately implies the norm estimate

$$\|\exp(tT_z)\,\phi\|_2 \leqslant M_z \|\phi\|_2, \qquad \text{where} \quad M_z = e^{|\mathrm{Re}\, z|}.$$

Moreover, if $\lambda \in \mathbb{C}$ is not purely imaginary, the usual integral formula for the resolvent implies that

$$\|(\lambda + T_z)^{-1}\| \leqslant \frac{M_z}{|\text{Re}\,\lambda|}; \tag{5.1}$$

the same estimates hold for the adjoint group $\exp(tT_z^*)$ generated by $T_z^* = -T_{-\bar{z}}$.

Next, since $H_n' = h_n$, we have $T_{z,n} = e^{-zH_n} D e^{zH_n} = d/dx + zh_n$ and $T_{z,n}^* = -d/dx + \bar{z}h_n$; these operators generate uniformly bounded one-parameter groups. Indeed, since $\exp(tT_{z,n}) = e^{-zH_n} e^{tD} e^{zH_n}$, we have the estimates

$$\|\exp(tT_{z,n})\| \leqslant M_z, \tag{5.2}$$

and

$$\|(\lambda + T_{z,n})^{-1}\| \leqslant \frac{M_z}{|\text{Re}\,\lambda|}. \tag{5.3}$$

The same estimates hold for the adjoints $T_{z,n}^*$. From the explicit formula for $\exp(tT_{z,n})$ and inequality (5.2), it is clear that for all $t \in \mathbb{R}$,

$$\operatorname*{s-lim}_{n \to \infty} \exp(tT_{z,n}) = \exp(tT_z), \tag{5.4}$$

and hence, for $\text{Re}\,\lambda \neq 0$,

$$\operatorname*{s-lim}_{n \to \infty} (\lambda + T_{z,n})^{-1} = (\lambda + T_z)^{-1}. \tag{5.5}$$

Similar results hold for the adjoints.

In particular, let $\lambda = r$. Then by (5.5), we have strong convergence of $(r + T_{z,n})^{-1}$ to $(r + T_z)^{-1}$, and of $(r + T_{z,n}^*)^{-1}$ to $(r + T_z^*)^{-1}$. Because the norms of all of these operators are bounded by $M_z/|r|$, independent of n, it follows that

$$\operatorname*{s-lim}_{n \to \infty} (r + T_{z,n})^{-1} (r + T_{z,n}^*)^{-1} = (r + T_z)^{-1} (r + T_z^*)^{-1}. \tag{5.6}$$

Since $L_{\{r,z\}} = (r + T_z^*)(r + T_z) - r^2 I$, we have

$$[r^2 + L_{\{r,z,n\}}]^{-1} = (r + T_{z,n})^{-1} (r + T_{z,n}^*)^{-1},$$

so Eq. (5.6) shows that

$$\operatorname*{s-lim}_{n \to \infty} [r^2 + L_{\{r,z,n\}}]^{-1} = [r^2 + L_{\{r,z\}}]^{-1}.$$

That is, we have strong convergence of resolvents for the specific real value $\lambda = r^2$, which for self-adjoint operators implies strong resolvent convergence for all non-real λ (cf. [Ka, Corollary VIII.1.4]).

Finally, suppose $r=0$. Then $L_{\{r,z\}}=B_z=T_z^*T_z$, and $L_{\{r,z,n\}}=B_{z,n}=T_{z,n}^*T_{z,n}$. For general $z\in\mathbb{C}$, we will prove strong graph convergence of $B_{z,n}$ to B_z; this is equivalent to strong resolvent convergence (cf. [RS, Theorem VIII.26]).

We employ a "localization" technique. Choose any $\phi\in\mathrm{Dom}(B_z)$, and decompose ϕ as $\psi+\eta$, where $\phi, \psi\in\mathrm{Dom}(B_z)$, ψ is supported in $[-1,1]$, and η vanishes in a neighborhood of 0. It then follows immediately that

$$B_{z,n}\eta\to B_z\eta=-\eta'', \qquad \text{as} \quad n\to\infty; \tag{5.7}$$

indeed, for n sufficiently large, $B_{z,n}\eta=-\eta''$.

Now define $\tilde{D}=d/dx$ on $L^2[-1,1]$, with $\mathrm{Dom}(\tilde{D})=\{f\in L^2[-1,1]: f$ absolutely continuous, $f'\in L^2$, and $f(1)=0\}$. Then $\tilde{D}^*=-\tilde{D}$, with domain prescribed by the adjoint boundary condition $f(-1)=0$. Both $\tilde{D}$ and its adjoint generate contraction semigroups on $L^2[-1,1]$.

Likewise, set $\tilde{T}_z=e^{-zH}\tilde{D}e^{zH}$ on $L^2[-1,1]$, and define $\tilde{B}_z=\tilde{T}_z^*\tilde{T}_z$. Given $\psi\in\mathrm{Dom}(B_z)$, with ψ supported in $[-1,1]$, let $\tilde{\psi}$ denote its restriction to $L^2[-1,1]$. Then clearly $\tilde{\psi}\in\mathrm{Dom}(\tilde{B}_z)$, and $\tilde{B}_z\tilde{\psi}=B_z\psi$ restricted to $[-1,1]$.

Define $\tilde{T}_{z,n}$ and $\tilde{B}_{z,n}$ on $L^2[-1,1]$ in the obvious way. An argument similar to that given above for the case $r\neq 0$ shows that $\tilde{B}_{z,n}\to\tilde{B}_z$ in the strong resolvent sense; for the operators $\tilde{T}_{z,n}$ and $\tilde{T}_{z,n}^*$ on $L^2[-1,1]$ have empty spectra (cf. [Ka, III.6.8]), and

$$(\tilde{B}_z)^{-1}=\tilde{T}_z^{-1}(\tilde{T}_z^*)^{-1}=\text{s-}\lim_{n\to\infty}\tilde{T}_{z,n}^{-1}(\tilde{T}_{z,n}^*)^{-1}=\text{s-}\lim_{n\to\infty}(\tilde{B}_{z,n})^{-1}.$$

Consequently, there exists a sequence of functions $\tilde{\psi}_n\in\mathrm{Dom}(\tilde{B}_{z,n})$ such that $\tilde{\psi}_n\to\tilde{\psi}$ and $\tilde{B}_{z,n}\tilde{\psi}_n\to\tilde{B}_z\tilde{\psi}$ in $L^2[-1,1]$.

Define $\psi_n=\tilde{\psi}_n$ on $[-1,1]$, and $\psi_n=0$ elsewhere. Then $\psi_n\in\mathrm{Dom}(B_{z,n})$, $\psi_n\to\psi$, and

$$B_{z,n}\psi_n=\tilde{B}_{z,n}\tilde{\psi}_n\to\tilde{B}_z\tilde{\psi}=B_z\psi \tag{5.8}$$

in $L^2(\mathbb{R}^1)$. Combining (5.7) and (5.8), we conclude that $B_{z,n}\to B_z$ in the sense of strong graph convergence. ∎

6. REMARKS

As we have noted in Section 1, Segal (cf. [Seg]) has suggested that, given a real L^∞ function V on $\mathbb{R}^1$, the formal expression $(1/i)(d/dx)+V'$ can be interpreted as the self-adjoint operator on $L^2(\mathbb{R}^1)$ given by

$$T_V=e^{-iV}\left(\frac{1}{i}\frac{d}{dx}\right)e^{iV}. \tag{6.1}$$

This is motivated by the elementary fact that if V is sufficiently well behaved—for example, absolutely continuous with bounded derivative—then $(1/i)(d/dx)+V'$ is a genuine operator which coincides with T_V.

For $V(x)=zH(x)$, where H is the Heaviside function and z is a real constant, the distributional derivative $V'(x)=z\,\delta(x)$; in this case, (6.1) specializes to the formula

$$\frac{1}{i}\frac{d}{dx}+z\,\delta=T_{iz}=e^{-izH}\left(\frac{1}{i}\frac{d}{dx}\right)e^{izH}. \tag{6.2}$$

However, Segal himself noted in [Seg] that the right side of (6.2) is periodic in z with period 2π. But the formal expression $(1/i)(d/dx)+z\delta$ has no such manifest periodicity. This suggests that (6.2) should be modified.

Indeed, there is a straightforward formal argument that leads to the interpretation

$$e^{-izH}\left(\frac{1}{i}\frac{d}{dx}\right)e^{izH}=\frac{1}{i}\frac{d}{dx}+2\tan\left(\frac{z}{2}\right)\delta. \tag{6.3}$$

This may be regarded as a "coupling constant renormalization" in the x-representation, analogous to the renormalization needed to interpret $(-\Delta+z\delta)$ in dimensions 2 and 3, and $(-\Delta+z\delta')$ in dimension 1. In particular, for $z=\pm\pi$, the coupling constant is formally infinite.

Somewhat more generally, if z is any complex number, one may argue for the interpretation

$$\frac{d}{dx}+\tau(z)\,\delta=e^{-zH}\frac{d}{dx}e^{zH}, \tag{6.4}$$

where $\tau(z)=2\tanh(z/2)$.

The justification is as follows: write $T_z=e^{-zH}(d/dx)\,e^{zH}$. Then T_z is a closed operator, similar to d/dx, with domain

$$\mathrm{Dom}(T_z)=e^{-zH}\,\mathrm{Dom}\left(\frac{d}{dx}\right)=e^{-zH}H^1(\mathbb{R}^1),$$

so that $f\in\mathrm{Dom}(T_z)$ if and only if $f=e^{-zH}g$, where $g\in H^1(\mathbb{R}^1)$. We then have

$$T_z f=e^{-zH}g'.$$

Now, outside any neighborhood of zero, f is absolutely continuous, and $f'=e^{-zH}g'$. Thus, if $(f')_0$ denotes the derivative of f away from zero, we have $(f')_0\in L^2(\mathbb{R}^1)$, and $T_z f=(f')_0$. Note that, since $f=e^{-zH}g$, with g absolutely continuous, f satisfies the following jump condition at zero:

$$f(0^+)=e^{-z}f(0^-). \tag{6.5}$$

This implies that the distributional derivative of f is given by

$$\frac{df}{dx} = f' = (f')_0 + [f(0^+) - f(0^-)]\,\delta(x)$$
$$= (f')_0 + [1 - e^z]\,f(0^+)\,\delta(x).$$

Thus

$$T_z f = (f')_0 = \left(\frac{d}{dx}\right) f + [e^z - 1]\,f(0^+)\,\delta(x). \tag{6.6}$$

Next, suppose we *define* $f(0) = (1/2)[f(0^+) + f(0^-)] = (1/2)(e^z + 1)\,f(0^+)$. Then

$$(e^z - 1)\,f(0^+) = 2\left(\frac{e^z - 1}{e^z + 1}\right) f(0) = 2\tanh\left(\frac{z}{2}\right) f(0).$$

So, we have from (6.6)

$$T_z f = \left(\frac{d}{dx}\right) f + 2\tanh\left(\frac{z}{2}\right) f(0)\,\delta. \tag{6.7}$$

If we agree to interpret $\delta(x)\,f(x)$ as $f(0)\,\delta(x)$, then (6.7) becomes

$$T_z f = \left[\frac{d}{dx} + 2\tanh\left(\frac{z}{2}\right)\delta\right] f, \tag{6.8}$$

which is precisely (6.4).

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