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Some Classes of Completely Regular Semigroups

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A semigroup is called completely regular if it is a union of groups. It has long been known [2, Theorem 4.6] that every completely regular semigroup S is a semilattice of completely simple semigroups; that is, S is a disjoint union $\bigcup\{S_\alpha: \alpha \in Y\}$ of completely simple semigroups S_α indexed by a semilattice Y, such that $S_\alpha S_\beta \subseteq S_{\alpha\beta}$ for all $\alpha, \beta \in Y$, where $\alpha\beta$ denotes the product (or meet) of α and β in Y. By the Rees theorem [2, Theorem 3.5], each S_α is isomorphic with a Rees matrix semigroup $\mathscr{M}(G_\alpha; I_\alpha, \Lambda_\alpha; P_\alpha)$, and so has known structure.

This result gives us what we might call the "gross structure" of S. Its "fine structure," just how the products $S_\alpha S_\beta$ are located in $S_{\alpha\beta}$, is quite another matter. A step in this direction was taken by Lallement [6, Theorem 2.10], who showed that the structure of S was determined by a system of mappings $\Phi_{\alpha\beta}: S_\alpha \to \Omega(S_\beta)$ (for $\alpha \geqslant \beta$ in Y), where $\Omega(T)$ denotes the translational hull of T.

A more informative expression for $\Omega(T)$ when T is completely simple is given in [8] and used in [9, Theorem 3] to amplify Lallement's theorem. While very useful in establishing various properties and classifications, as shown in [9], this result is very complicated. One of the purposes of the present paper is to give a simplified version of this theorem (Theorem 2), the simplification due to the (permissible) assumption that every sandwich matrix P_α is "normalized" [2, Sect. 3.2]. This is based on Theorem 1, which gives an improved expression for $\Omega(T)$, $T = \mathscr{M}(G; I, \Lambda; P)$, when we assume that P is normalized.

The rest of the paper gives two applications of Theorem 2, first to bands of groups (Theorems 3 and 4), and then to completely regular semigroups satisfying $\mathscr{D}$-covering (Theorem 5). Theorems 6.1 and 6.2 are specializations of these. Theorem 3 presents a description of bands of groups which is quite different from that of Leech [7, Theorem 1]. One finds a simpler construction (Theorem 4) for $\mathscr{D}$-split bands of groups.

A completely regular semigroup $S = \bigcup\{S_\alpha: \alpha \in Y\}$ is said to *satisfy $\mathscr{D}$-covering* [9, Sect. 8] if, whenever e and f are idempotent elements of S such that $e \in S_\alpha$ and $f \in S_\beta$ with $\alpha > \beta$, then $e > f$ in the usual partial ordering of idempotents

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($e \geqslant f$ if $ef = fe = f$). A band has this property if and only if it is almost commutative, and the structure of every such band was determined by Hall [4].

By an *orthogroup* we mean a completely regular semigroup S which is orthodox, i.e., the set E_S of idempotents of S is a subsemigroup of S. Beginning in 1960, the structure of orthogroups has been investigated by Fantham, Preston, Yamada, and others (see [1] for references). The structure of any orthogroup $S = \bigcup\{S_\alpha: \alpha \in Y\}$ satisfying $\mathscr{D}$-covering, and for which Y is totally ordered, was given by one of us [9, Theorem 11]. The same result was given previously by Fortunatov [3]. In Theorem 6.1, it is generalized to any orthogroup satisfying $\mathscr{D}$-covering.

Theorem 6.2 describes bands of groups satisfying $\mathscr{D}$-covering. These have arisen in the work of Kacman [5], and this is discussed briefly at the end of the paper.

Remark on notation. If X is a set, $\mathscr{T}(X)$ [$\mathscr{T}^*(X)$] will denote the semigroup of all right [left] transformations of X. If $x \in X$ and $\phi \in \mathscr{T}(X)$ [$\mathscr{T}^*(X)$], then the image of x under ϕ will be written $x\phi$ [ϕx], and the product $\phi_1\phi_2$ of two elements of $\mathscr{T}(X)$ is defined by $x(\phi_1\phi_2) = (x\phi_1)\phi_2$ [$(\phi_1\phi_2)x = \phi_1(\phi_2 x)$]. If $x_1 \in X$, then the constant transformation in $\mathscr{T}(X)$ [$\mathscr{T}^*(X)$] that sends every element of X into x_1 will be denoted by $\langle x_1\rangle$ [$\langle x_1\rangle^*$]. Thus $x\langle x_1\rangle = x_1$ and $\langle x_1\rangle^* x = x_1$ for every x in X. This differs from the notation in [9].

If X and Y are sets, Y^X will denote the set of all mappings $\phi: X \to Y$ when there is no question about composition of two such, and the image of x under ϕ may be written either $x\phi$ or ϕx.

1. An Expression for the Translational Hull of a Rees Matrix Semigroup with Normalized Sandwich Matrix

Let $S = \mathscr{M}(G;I,\Lambda;P)$ be a Rees $I \times \Lambda$ matrix semigroup over a group G, with $\Lambda \times I$ sandwich matrix $P = (P_{\lambda,i})$ over G. Recall [2, Lemma 3.6] that certain changes can be made in P without changing S (to within isomorphism), and we can thereby "normalize" P so that all the entries in any given row and any given column are the identity element e of G. Without loss of generality, we can assume that the index classes I and Λ have an element 1 in common. We shall say that P is *normalized* at (1, 1) if $p_{1,i} = e = p_{\lambda,1}$ for all i in I and λ in Λ.

The translational hull $\Omega(S)$ of S was described in [8, Sect. 1] as follows. If $\theta \in G^I$ and $\chi \in \mathscr{T}^*(I)$, the transformation $L(\chi, \theta)$ of S defined by

$$L(\theta, \chi)(a; i, \lambda) = ((\theta i)a; \chi i, \lambda) \tag{1.1}$$

for all $(a; i, \lambda)$ in S, is a left translation of S, and every left translation of S is uniquely expressible as one of these. Dually, the right translations of S are the transformations $R(\omega, \psi)$ defined by

$$(a; i, \lambda)(R(\omega, \psi) = (a(\lambda\omega); i, \lambda\psi), \tag{1.2}$$

where $\omega \in G^{\Lambda}$ and $\psi \in \mathscr{T}(\Lambda)$.

The product of two left or two right translations is given by

$$L(\theta, \chi) L(\theta', \chi') = L(\theta'', \chi\chi'), \tag{1.3}$$

where

$$\theta'' i = [\theta(\chi' i)](\theta' i) \qquad (\text{all } i \in I);$$

$$R(\omega, \psi) R(\omega', \psi') = R(\omega'', \psi\psi'), \tag{1.4}$$

where

$$\lambda\omega'' = (\lambda\omega)[(\lambda\psi)\omega'] \qquad (\text{all } \lambda \in \Lambda).$$

By definition, $\Omega(S)$ is the semigroup of all pairs $(L(\theta, \chi), R(\omega, \psi))$ of linked left and right translations (called "bitranslations"), and it is shown in [8, Sect. 1] that $L(\theta, \chi)$ and $R(\omega, \psi)$ are linked if and only if

$$p_{\lambda, \chi i}(\theta i) = (\lambda\omega)\, p_{\lambda\psi, i} \qquad (\text{all } i \in I, \lambda \in \Lambda). \tag{1.5}$$

The inner part $\Pi(S)$ of $\Omega(S)$ consists of the inner bitranslations $\pi_B = (\lambda_B, \rho_B)$ of S (where B is a fixed element of S). If $B = (b; j, \mu)$, then

$$\begin{aligned} \lambda_B &= L(\theta_B, \langle j \rangle^*), \qquad \text{where} \quad \theta_B i = b p_{\mu, i} \qquad (\text{all } i \in I); \\ \rho_B &= R(\omega_B, \langle \mu \rangle, \qquad \text{where} \quad \lambda\omega_B = p_{\lambda, j} b \qquad (\text{all } \lambda \in \Lambda). \end{aligned} \tag{1.6}$$

The natural action of $\Omega(S)$ on S is defined by

$$\begin{aligned} (L(\theta, \chi), R(\omega, \psi))A &= L(\theta, \chi)A, \\ A(L(\theta, \chi), R(\omega, \psi)) &= AR(\omega, \psi), \end{aligned} \tag{1.7}$$

for all $A \in S$.

We proceed now to give a convenient representation of $\Omega(S)$ when P is normalized.

THEOREM 1. *Let $S = \mathscr{M}(G; I, \Lambda; P)$ be a Rees matrix semigroup with normalized sandwich matrix P. Let $\Theta(S)$ be the set of all $(a; \chi, \psi)$ in $G \times \mathscr{T}^*(I) \times \mathscr{T}(\Lambda)$ satisfying the condition*

$$p_{\lambda, \chi i} a p_{1\psi, i} = p_{\lambda, \chi 1} a p_{\lambda\psi, i} \qquad (\textit{all } i \in I, \lambda \in \Lambda). \tag{1.8}$$

Define product in $\Theta(S)$ by

$$(a; \chi, \psi)(a'; \chi', \psi') = (a p_{1\psi, \chi' 1} a'; \chi\chi', \psi\psi'). \tag{1.9}$$

The mapping $(a; i, \lambda) \mapsto (a; \langle i \rangle^*, \langle \lambda \rangle)$ *is an isomorphism of* S *onto a subsemigroup* $\Theta_0(S)$ *of* $\Theta(S)$. *There exists an isomorphism of* $\Theta(S)$ *onto* $\Omega(S)$ *sending* $\Theta_0(S)$ *onto* $\Pi(S)$. *Using this isomorphism to transfer the natural action of* $\Omega(S)$ *on* S *over to* $\Theta(S)$, *we find*

$$\begin{aligned} (b; \chi, \psi)(a; i, \lambda) &= (bp_{1\psi, i}a; \chi i, \lambda), \\ (a; i, \lambda)(b; \chi, \psi) &= (ap_{\lambda, \chi 1}b; i, \lambda\psi), \end{aligned} \tag{1.10}$$

for all $(b; \chi, \psi)$ *in* $\Theta(S)$ *and all* $(a; i, \lambda)$ *in* S.

Proof. Define $\zeta\colon \Theta(S) \to \Omega(S)$ by

$$(a; \chi, \psi)\zeta = (L(\theta, \chi), R(\omega, \psi)), \tag{1.11}$$

where θ and ω are defined by

$$\begin{aligned} \theta i &= ap_{1\psi, i} \qquad (\text{all } i \in I), \\ \lambda\omega &= p_{\lambda, \chi 1}a \qquad (\text{all } \lambda \in \Lambda). \end{aligned} \tag{1.12}$$

Equation (1.5) is immediate from (1.8) and (1.12), so $(a; \chi, \psi)\zeta \in \Omega(S)$. The mapping ζ is evidently one-to-one. To show that it is onto, let $(L(\theta, \chi), R(\omega, \psi)) \in \Omega(S)$. Setting $i = 1 = \mu$ in (1.5), and using $p_{\lambda,1} = e = p_{1,i}$ (by hypothesis), we obtain $\theta 1 = 1\omega$. Let $a = \theta 1$ $(= 1\omega)$. Now setting $\lambda = 1$ in (1.5), we obtain $\theta i = ap_{1\psi, i}$. Setting $i = 1$ in (1.5), we get $\lambda\omega = p_{\lambda, \chi 1}a$. By (1.12), the given element of $\Omega(S)$ is the image of $(a; \chi, \psi)$ under ζ.

To show that ζ is a homomorphism (and hence an isomorphism), we have

$$\begin{aligned} (a; \chi, \psi)\zeta \cdot (a'; \chi', \psi')\zeta &= (L(\theta, \chi)L(\theta', \chi'), R(\omega, \psi)\, R(\omega', \psi')) \\ &= (L(\theta'', \chi\chi'), R(\omega'', \chi\chi')), \end{aligned}$$

where, by (1.3), (1.4), and (1.12),

$$\begin{aligned} \theta'' i &= ap_{1\psi, \chi' i}a'p_{1\psi', i} \qquad (\text{all } i \in I), \\ \lambda\omega'' &= p_{\lambda, \chi 1}ap_{\lambda\psi, \chi' 1}a' \qquad (\text{all } \lambda \in \Lambda). \end{aligned}$$

On the other hand, using (1.9),

$$\begin{aligned} [(a; \chi, \psi)(a'; \chi', \psi')]\zeta &= (ap_{1\psi, \chi' 1}a'; \chi\chi', \psi\psi')\zeta \\ &= (L(\theta_1, \chi\chi'), R(\omega_1, \psi\psi')), \end{aligned}$$

where, by (1.12),

$$\begin{aligned} \theta_1 i &= ap_{1\psi, \chi' 1}a'p_{1\psi\psi', i} \qquad (\text{all } i \in I), \\ \lambda\omega_1 &= p_{\lambda, \chi\chi' 1}ap_{1\psi, \lambda' 1}a' \qquad (\text{all } \lambda \in \Lambda). \end{aligned}$$

It suffices to prove $\omega_1 = \omega''$, the proof that $\theta_1 = \theta''$ being dual. But this is immediate from (1.8), replacing i by $\chi' 1$.

The first assertion in Theorem 1 is immediate from (1.9):

$$(a; \langle i\rangle^*, \langle\lambda\rangle)(b; \langle j\rangle^*, \langle\mu\rangle) = (ap_{\lambda_1}b; \langle i\rangle^*, \langle\mu\rangle).$$

That $\Theta_0(S)\zeta = \Pi(S)$ is evident from (1.12) and (1.6). For if $B = (b; j, \mu) \in S$, then

$$(b; \langle j\rangle^*, \langle\mu\rangle)\zeta = (L(\theta, \langle j\rangle^*), R(\omega, \langle\mu\rangle))$$

with $\theta i = bp_{\mu,i}$ and $\lambda\omega = p_{\lambda,j}b$, and hence equal to π_B by (1.6).

Again with θ given by (1.12), we have by (1.1),

$$\begin{aligned}(b; \chi, \psi)(a; i, \lambda) &= (b; \chi, \psi)\zeta \cdot (a; i, \lambda)\\ &= L(\theta, \chi)(a; i, \lambda)\\ &= ((\theta i)a; \chi i, \lambda)\\ &= (bp_{1\psi,i}a; \chi i, \lambda).\end{aligned}$$

This proves the first part of (1.10), and the second is dual.

2. A Normalized Structure Theorem for Completely Regular Semigroups

We begin by stating a variant of Lallement's structure theorem for completely regular semigroups [6, Theorem 2.19] (see also [9, Theorem 1]). It is obtained from Lallement's theorem by replacing each $\Omega(S_\alpha)$ by $\Theta(S_\alpha)$, and $\Phi_{\alpha,\beta}: S_\alpha \to \Omega(S_\beta)$ by $\Psi_{\alpha,\beta}: S_\alpha \to \Theta(S_\beta)$, where $\Psi_{\alpha,\beta} = \Phi_{\alpha,\beta}\zeta_\beta^{-1}$ and $\zeta_\beta: \Theta(S_\beta) \to \Omega(S_\beta)$ is the isomorphism established in Theorem 1.

THEOREM (Lallement). *With each element α of a semilattice Y associate a Rees matrix semigroup $S_\alpha = \mathscr{M}(G_\alpha; I_\alpha, \Lambda_\alpha; P_\alpha)$, with sandwich matrix P_α normalized at $(1_\alpha, 1_\alpha)$. Suppose that $S_\alpha \cap S_\beta = \varnothing$ if $\alpha \neq \beta$, and let $S = \bigcup\{S_\alpha: a \in Y\}$. With each pair $\alpha \geqslant \beta$ in Y, associate a mapping $\Psi_{\alpha,\beta}: S_\alpha \to \Theta(S_\beta)$ satisfying the following conditions.*

(A1) *For each $(a; i, \lambda)$ in S_α $(\alpha \in Y)$,*

$$(a; i, \lambda)\Psi_{\alpha,\alpha} = (a; \langle i\rangle^*, \langle\lambda\rangle).$$

(A2) *For arbitrary α, β in Y,*

$$(S_\alpha\Psi_{\alpha,\alpha\beta})(S_\beta\Psi_{\beta,\alpha\beta}) \subseteq \Theta_0(S_{\alpha\beta}).$$

If $A \in S_\alpha$ and $B \in S_\beta$, we define the product AB of A and B in S by

$$AB = [(A\Psi_{\alpha,\alpha\beta})(B\Psi_{\beta,\alpha\beta})]\, \Psi_{\alpha\beta,\alpha\beta}^{-1}\,. \tag{2.1}$$

(A3) *If* $\alpha\beta > \gamma$ *in* Y, $A \in S_\alpha$; *and* $B \in S_\beta$, *then*

$$(AB)\Psi_{\alpha\beta,\gamma} = (A\Psi_{\alpha,\gamma})(B\Psi_{\beta,\gamma}).$$

If (A1–A3) *hold, then with the product defined by* (2.1), S *is a completely regular semigroup. Conversely, every completely regular semigroup is isomorphic to one constructed in this way.*

We are now ready to state Theorem 2, which is a simplification of [9, Theorem 3], made possible by normalizing all the sandwich matrices.

THEOREM 2. *With each element* α *of a semilattice* Y *associate a Rees matrix semigroup* $S_\alpha = \mathscr{M}(G_\alpha; I_\alpha, \Lambda_\alpha; P_\alpha)$ *with sandwich matrix* $P_\alpha = (p^\alpha_{\lambda,i})$ *normalized at* $(1_\alpha, 1_\alpha)$. *Assume that* $S_\alpha \cap S_\beta = \varnothing$ *if* $\alpha \neq \beta$, *and let* $S = \bigcup\{S_\alpha : \alpha \in Y\}$. *With each pair* $\alpha \geqslant \beta$ *in* Y, *associate three mappings,*

$$\xi_{\alpha,\beta}: S_\alpha \to G_\beta, \qquad \chi_{\alpha,\beta}: S_\alpha \to \mathscr{T}^*(I_\beta), \qquad \psi_{\alpha,\beta}: S_\alpha \to \mathscr{T}(\Lambda_\beta),$$

satisfying the following conditions.

(B0) *If* $\alpha > \beta$ *in* Y, $A \in S_\alpha$, $i \in I_\beta$, *and* λ Λ_β , *then*

$$p^\beta_{\lambda,A1}(A\xi_{\alpha,\beta})\, p^\beta_{\lambda A,i} = p^\beta_{\lambda,Ai}(A\xi_{\alpha,\beta})\, p^\beta_{1A,i}\,,$$

where we have written Ai *for* $(A\chi_{\alpha,\beta})i$, λA *for* $\lambda(A\psi_{\alpha,\beta})$, *and* 1 *for* 1_β .

(B1) *If* $\alpha \in Y$ *and* $A = (a; i, \lambda) \in S_\alpha$, *then*

$$A\xi_{\alpha,\alpha} = a, \qquad A\chi_{\alpha,\alpha} = \langle i\rangle^*, \qquad A\psi_{\alpha,\alpha} = \langle\lambda\rangle.$$

(B2) *Let* α *and* β *be arbitrary elements of* Y, *and let* $A \in S_\alpha$, $B \in S_\beta$. *Then* $(A\chi_{\alpha,\alpha\beta})(B\chi_{\beta,\alpha\beta})$ *and* $(A\psi_{\alpha,\alpha\beta})(B\psi_{\beta,\alpha\beta})$ *are constant transformations of* $I_{\alpha\beta}$ *and* $\Lambda_{\alpha\beta}$, *respectively, say* $\langle k\rangle^*$ *and* $\langle\nu\rangle$. *Define the product* AB *of* A *and* B *in* S *by*

$$AB = ((A\xi_{\alpha,\alpha\beta})\, p^{\alpha\beta}_{1A,B1}(B\xi_{\beta,\alpha\beta});\, k, \nu). \tag{2.2}$$

(B3) *Let* α, β, γ *be elements of* Y *such that* $\alpha\beta > \gamma$, *and let* $A \in S_\alpha$, $B \in S_\beta$. *With* AB *defined by* (2.2),

$$(AB)\,\chi_{\alpha\beta,\gamma} = (A\chi_{\alpha,\gamma})(B\chi_{\beta,\gamma}),$$
$$(AB)\,\psi_{\alpha\beta,\gamma} = (A\psi_{\alpha,\gamma})(B\psi_{\beta,\gamma}),$$
$$(AB)\,\xi_{\alpha\beta,\gamma} = (A\xi_{\alpha,\gamma})\, p^\gamma_{1A,B1}\,(B\xi_{\beta,\gamma}).$$

If these conditions are satisfied, then, with product defined by (2.2), S *is a* *completely regular semigroup. Conversely, every completely regular semigroup is isomorphic to one constructed in this way.*

Since Theorem 2 is just a transcription of Lallement's theorem using Theorem 1, a formal proof is scarcely necessary. A mapping $\Psi_{\alpha,\beta}: S_\alpha \to \Theta_\beta(S_\beta)$ evidently determines a triple of mappings $(\xi_{\alpha,\beta}; \chi_{\alpha,\beta}, \psi_{\alpha,\beta})$. Noting (1.8) of Theorem 1, condition (B0) simply expresses the fact that

$$A\Psi_{\alpha,\beta} = (A\xi_{\alpha,\beta}; A\chi_{\alpha,\beta}, A\psi_{\alpha,\beta})$$

belongs to $\Theta_\beta(S_\beta)$, for every $A \in S_\alpha$. Conversely, every such triple satisfying (B0) determines a unique mapping $\Psi_{\alpha,\beta}$.

(B1) is visibly the same as (A1). If $\alpha\beta \geqslant \gamma$ in Y, $A \in S_\alpha$, and $B \in S_\beta$, then, by (1.9),

$$(A\Psi_{\alpha,\gamma})(B\Psi_{\beta,\gamma}) = ((A\xi_{\alpha,\gamma})\, p^{\gamma}_{1A,B1}(B\xi_{\beta,\gamma}); (A\chi_{\alpha,\gamma})(B\chi_{\beta,\gamma}), (A\psi_{\alpha,\gamma})(B\psi_{\beta,\gamma})).$$

When $\gamma = \alpha\beta$, (A2) requires this to lie in $\Theta_0(S_{\alpha\beta})$, hence to be of the form $(c; \langle k\rangle^*, \langle\nu\rangle)$, and this is exactly what (B2) requires. Since $(c; \langle k\rangle^*, \langle\nu\rangle)\Psi^{-1}_{\alpha\beta,\alpha\beta} = (c; k, \nu)$, definitions (2.1) and (2.2) of the product in S coincide. When $\gamma < \alpha\beta$,

$$(AB)\Psi_{\alpha\beta,\gamma} = ((AB)\xi_{\alpha\beta,\gamma}; (AB)\chi_{\alpha\beta,\gamma}, (AB)\psi_{\alpha\beta,\gamma}),$$

and (B3) is just the componentwise formulation of (A3).

If, as in (B0), we write Ai for $(A\chi_{\alpha,\beta})i$ when $\alpha > \beta$, we see from (B3) that $(AB)i = A(Bi)$ for all $i \in I_\gamma$. Similarly, $\lambda(AB) = (\lambda A)B$ for all $\lambda \in \Lambda_\gamma$. In particular, when $\alpha > \beta$, $\chi_{\alpha,\beta}$ defines a left action of S_α on I_β, and $\psi_{\alpha,\beta}$ a right action of S_α on Λ_β. We could state Theorem 2 in terms of these actions and the mappings $\xi_{\alpha,\beta}$ without introducing the letters $\chi_{\alpha,\beta}$ and $\psi_{\alpha,\beta}$.

When applying Theorem 2 to bands (idempotent semigroups) each group G_α is trivial, and the mappings $\xi_{\alpha,\beta}$ disappear altogether.

If the conditions of the theorem hold, then (B3) also holds for $\gamma = \alpha\beta$, by (B1), (B2), and the definition (2.2). The latter agrees with the given product in each component S_α by virtue of (B1). Of course (B1) may be regarded as a definition.

We remark also that, when $\alpha > \beta$, $A \in S_\alpha$, and $B = (b; j, \mu) \in S_\beta$,

$$AB = ((A\xi_{\alpha,\beta})\, p^{\beta}_{1A,j}b; Aj, \mu). \tag{2.3}$$

For if $l \in I_\beta$, (B2) implies that $A(Bl) = k$. But $Bl = j$ by (B1), so $k = Aj$. Similarly, if $\lambda \in \Lambda_\beta$, $(\lambda A)B = \nu$. Also by (B1), $B\xi_{\beta,\beta} = b$. Equation (2.2) thus becomes (2.3). Dually, we have

$$BA = (bp_{\mu,A1}(A\xi_{\alpha,\beta}); j, \mu A). \tag{2.4}$$

3. Bands of Groups

Let S be a completely regular semigroup, and let E_S be the set of idempotents of S. For each $a \in S$, denote by H_a the $\mathscr{H}$-class (maximal subgroup) of S containing a. S is called a ***band of groups*** if $H_a H_b \subseteq H_{ab}$ for all $a, b \in S$; that is, if $\mathscr{H}$ is a congruence on S. Evidently $S/\mathscr{H}$ is then a band. Since E_S is a transversal of $\mathscr{H}$, we can transfer the structure of $S/\mathscr{H}$ over to E_S by defining $e * f$ $(e, f \in E_S)$ to be the identity element of H_{ef}.

A Rees matrix semigroup $M = \mathscr{M}(G; I, \Lambda; P)$ is a band of groups, and $E_M(*)$ is isomorphic to the rectangular band on $I \times \Lambda$. Denoting the idempotent $(p_{\lambda,i}^{-1}; i, \lambda)$ of M by (i, λ), we have the rule $(i, \lambda) * (j, \mu) = (i, \mu)$. If $e = (i, \lambda)$, write p_e for $p_{\lambda,i}$, and write (a, e) for the element $(a; i, \lambda)$ of M. We then have the following rule for multiplication in M:

$$(a, e)(b, f) = (ap_{f*e}b, e * f). \tag{3.1}$$

Accordingly, we may write $\mathscr{M}(G, E(*), p)$ instead of $\mathscr{M}(G; I, \Lambda; P)$, indicating that we have a group G, a rectangular band $E(*)$, a mapping $p: E \to G$ (the image of $e \in E$ under p denoted by p_e), and product on the set $G \times E$ defined by (3.1).

Suppose that P is normalized at $(1, 1)$. Let u denote the idempotent $(1, 1)$ of M. We identify $H_u = H_{11}$ with G, thus u with the identity element of G. If $e = (i, \lambda)$, then $p_{1,i} = p_{e*u}$ and $p_{\lambda,1} = p_{u*e}$. Hence P is normalized at u if and only if

$$p_{e*u} = p_{u*e} = u \qquad (\text{all } e \in E). \tag{3.2}$$

Suppose now that S is a band of groups, and $S = \bigcup\{S_\alpha : \alpha \in Y\}$, its decomposition into completely simple subsemigroups S_α. We may assume that each $S_\alpha = \mathscr{M}(G_\alpha, E_\alpha(*), p^\alpha)$. Then the operation $*$ introduced above on $E_S = \bigcup\{E_\alpha : \alpha \in Y\}$ agrees with the operation $*$ on each component E_α.

We take the (reasonable) point of view in this section that the structure of $E_S(*)$ is known. (The structure of bands should be elucidated before we attack the more general problem!) Since $\mathscr{H}$ is a congruence, the actions of S_α on I_β and Λ_β are determined by those of E_α, and these in turn by the structure of $E_S(*)$. Consequently we can apply Theorem 2, discarding all reference to χ's and ψ's, to obtain the following.

Theorem 3. *Let $E(*)$ be a band, and $E = \bigcup\{E_\alpha : \alpha \in Y\}$ its decomposition into a semilattice Y of rectangular bands $E_\alpha(*)$. With each element α of Y associate a group G_α and a sandwich matrix $p^\alpha : E_\alpha \to G_\alpha$ normalized at an element u_α of E_α. Let S_α be the Cartesian product $G_\alpha \times E_\alpha$, and $S = \bigcup\{S_\alpha : \alpha \in Y\}$. With each pair $\alpha \geqslant \beta$ in Y associate a mapping $\xi_{\alpha,\beta} : S_\alpha \to G_\beta$ such that $\xi_{\alpha,\alpha} : S_\alpha \to G_\alpha$ is defined by $(a, e)\xi_{\alpha,\alpha} = a$, for all $(a, e) \in S_\alpha$.*

Define product in S as follows. If $\alpha, \beta \in Y$, $A = (a, e) \in S_\alpha$, *and* $B = (b, f) \in S_\beta$, *then let*

$$AB = ((A\xi_{\alpha,\alpha\beta})\, p^{\alpha\beta}_{f*u_{\alpha\beta}*e}(B\xi_{\beta,\alpha\beta}),\, e * f). \tag{3.3}$$

This makes each S_α *into the Rees matrix semigroup* $\mathcal{M}(G_\alpha, E_\alpha(*), p^\alpha)$. *Product on S defined by* (3.3) *is associative, and S becomes thereby a band of groups with* $E_S(*) \cong E(*)$, *if and only if the following conditions are satisfied.*

(D1) *If* $\alpha > \beta$ *in* Y, $A = (a, e) \in S_\alpha$, *and* $f \in E_\beta$, *then*

$$p^\beta_{e*f}(A\xi_{\alpha,\beta})\, p^\beta_{f*u_\beta*e} = p^\beta_{e*u_\beta*f}(A\xi_{\alpha,\beta})\, p^\beta_{f*e}\,.$$

(D2) *Let* $\alpha\beta > \gamma$ *in* Y, *and let* $A = (a, e) \in S_\alpha$ *and* $B = (b, f) \in S_\beta$. *Then, with AB defined by* (3.3),

$$(AB)\,\xi_{\alpha\beta,\gamma} = (A\xi_{\alpha,\gamma})\, p^\gamma_{f*u_\gamma*e}(B\xi_{\beta,\gamma}).$$

Conversely, every band of groups S, such that $E_S(*)$ *is isomorphic to the band* $E(*)$, *is isomorphic to one constructed in this way.*

No formal proof is necessary. Note, however, that in proving the direct part, we can infer from Theorem 2 only that the semigroup S so constructed is completely regular. But it is clear from (3.3) that $\mathcal{H}$ is a congruence on S; in fact, $H_eH_f \subseteq H_{e*f}$.

4. $\mathcal{D}$-Split Bands of Groups

In this section we simplify Theorem 3 considerably by assuming that $E(*)$ is $\mathcal{D}$-split.

Let ρ be a congruence on a semigroup S. If T is a subsemigroup of S which is also a transversal of ρ (i.e., contains exactly one element from each ρ-class), then we call T a *ρ-splitting subsemigroup* of S. In that case, $T \cong S/\rho$. We say that S is *ρ-split* if it contains a ρ-splitting subsemigroup.

If $S = \bigcup\{S_\alpha : \alpha \in Y\}$ is a completely regular semigroup, then Green's relation $\mathcal{D}$ is a congruence on S; the $\mathcal{D}$-classes of S are the completely simple components S_α, and $S/\mathcal{D} \cong Y$. Any $\mathcal{D}$-splitting subsemigroup T of S consists of idempotents, one from each S_α. If $T \cap S_\alpha = \{t_\alpha\}$, then $T = \{t_\alpha : \alpha \in Y\}$, and $t_\alpha t_\beta = t_{\alpha\beta}$ for all $\alpha, \beta \in Y$. T is also a $\mathcal{D}$-splitting subsemigroup of the band $E_S(*)$, since $t_\alpha t_\beta = t_\alpha * t_\beta$.

Conversely, if $U = \{u_\alpha : \alpha \in Y\}$ is a $\mathcal{D}$-splitting subsemigroup (hence subsemilattice) of $E_S(*)$, it is also one of S. For we have $u_\alpha u_\beta \in H_{u_\alpha * u_\beta} = H_{u_{\alpha\beta}}$, whence $u_\alpha u_{\alpha\beta} = u_{\alpha\beta}$. Similarly, $u_\beta u_\alpha \in H_{u_{\alpha\beta}}$, so $u_\beta u_{\alpha\beta} = u_{\alpha\beta}$. Hence $u_\alpha u_\beta = (u_\alpha u_\beta)u_{\alpha\beta} = u_\alpha(u_\beta u_{\alpha\beta}) = u_\alpha u_{\alpha\beta} = u_{\alpha\beta}$, so that U is also a subsemigroup of S. Thus

$$\mathcal{D}\text{-split (band of groups)} = (\mathcal{D}\text{-split band) of groups.}$$

THEOREM 4. *Let $E(*) = \bigcup\{E_\alpha(*)\colon \alpha \in Y\}$ be a $\mathcal{D}$-split band, and let $U = \{u_\alpha\colon \alpha \in Y\}$ be a $\mathcal{D}$-splitting subsemilattice of $E(*)$ with $u_\alpha \in E_\alpha$. With each element α of Y associate a group G_α and a sandwich matrix $p^\alpha\colon E_\alpha \to G_\alpha$ normalized at u_α. Let S_α be the Cartesian product $G_\alpha \times E_\alpha$, and $S = \bigcup\{S_\alpha\colon \alpha \in Y\}$. With each pair $\alpha \geqslant \beta$ in Y associate a homomorphism $\phi_{\alpha,\beta}\colon G_\alpha \to G_\beta$, defining $\phi_{\alpha,\alpha}$ to be the identity automorphism of G_α, such that the following conditions hold.*

(E1) *If $\alpha > \beta > \gamma$ in Y, then $\phi_{\alpha,\beta}\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$.*

(E2) *If $\alpha > \beta$ in Y and $e \in E_\alpha$, then $p_e{}^\alpha\phi_{\alpha,\beta} = p^\beta_{e*u_\beta*e}$.*

(E3) *If $\alpha > \beta$ in Y, $a \in G_\alpha$, $e \in E_\alpha$, and $f \in E_\beta$, then*

$$p^\beta_{e*f}(a\phi_{\alpha,\beta})\, p^\beta_{f*u_\beta*e} = p^\beta_{e*u_\beta*f}(a\phi_{\alpha,\beta})\, p^\beta_{f*e}.$$

Define product in S as follows. If $\alpha, \beta \in Y$, $(a, e) \in S_\alpha$, and $(b, f) \in S_\beta$, then let

$$(a, e)(b, f) = ((a\phi_{\alpha,\alpha\beta})\, p^{\alpha\beta}_{f*u_{\alpha\beta}*e}(b\phi_{\beta,\alpha\beta}),\ e * f). \tag{4.1}$$

Then S becomes a $\mathcal{D}$-split band of groups; conversely, every such is isomorphic to a semigroup constructed in this way.

Proof. We prove the converse first. Let S be a $\mathcal{D}$-split band of groups. Then (to within isomorphism) we can consider S to be constructed as in Theorem 3. Moreover, we normaize the p^α at the elements u_α of a $\mathcal{D}$-splitting subsemigroup U of $E_S(*)$ (hence of S, as noted above).

Note first that if $\alpha \geqslant \beta$ in Y, and $e \in E_\alpha$, then

$$p^\beta_{e*u_\beta} = p^\beta_{u_\beta*e} = u_\beta. \tag{4.2}$$

For, by (3.2), $p^\beta_{f*u_\beta} = u_\beta$ for every $f \in E_\beta$, and we can take $f = e * u_\beta$ to obtain first equation in (4.2); proof of the second is similar.

Observe next the following special case of condition (D2) of Theorem 3 replacing β by α and γ by β. If $\alpha > \beta$, and if $A = (a, e)$ and $B = (b, f)$ both belong to S_α, then

$$(AB)\, \xi_{\alpha,\beta} = (A\xi_{\alpha,\beta})\, p^\beta_{f*u_\beta*e}(B\xi_{\alpha,\beta}). \tag{4.3}$$

If either $e * u_\alpha = e$ or $u_\alpha * f = f$, this reduces to

$$(AB)\xi_{\alpha,\beta} = (A\xi_{\alpha,\beta})(B\xi_{\alpha,\beta}). \tag{4.4}$$

Taking the first case (proof for the second being similar), and using $u_\beta * u_\alpha = u_\beta$ and $u_\alpha * e * u_\alpha = u_\alpha$,

$$\begin{aligned} f * u_\beta * e &= f * u_\beta * e * u_\alpha = f * u_\beta * u_\alpha * e * u_\alpha \\ &= f * u_\beta * u_\alpha = f * u_\beta. \end{aligned}$$

By (4.2), the sandwich term in (4.3) is u_β.

One immediate consequence of (4.4) is that, for any $e \in E_\alpha$, $(e * u_\alpha)\xi_{\alpha,\beta}$ and $(u_\alpha * e)\xi_{\alpha,\beta}$ are idempotent elements of G_β. Hence

$$(e * u_\alpha)\xi_{\alpha,\beta} = (u_\alpha * e)\xi_{\alpha,\beta} = u_\beta . \tag{4.5}$$

Define $\phi_{\alpha,\beta}: G_\alpha \to G_\beta$ to be the restriction of $\xi_{\alpha,\beta}$ to G_α. By (4.4), $\phi_{\alpha,\beta}$ is a homomorphism.

If $(a, e) \in S_\alpha$, then $(a, e) = (e * u_\alpha)\, a(u_\alpha * e)$, where we identify a with (a, u_α). Two applications of (4.4), along with (4.5), yield

$$(a, e)\xi_{\alpha,\beta} = a\phi_{\alpha,\beta} . \tag{4.6}$$

Equation (4.1) is now immediate from (4.3), and (E3), follows from (D1).

By (3.1), or first principles, if $e \in E_\alpha$ then

$$\begin{aligned}(u_\alpha * e)(e * u_\alpha) &= (u_\alpha , u_\alpha * e)(u_\alpha , e * u_\alpha) \\ &= (p^\alpha_{e*u_\alpha*e} , u_\alpha * e * u_\alpha) \\ &= (p_e^\alpha, u_\alpha) = p_e^\alpha.\end{aligned}$$

Hence, by (4.3), and using (4.5),

$$p_e^\alpha\phi_{\alpha,\beta} = p_e^\alpha\xi_{\alpha,\beta} = p^\beta_{e*u_\alpha*u_\beta*u_\alpha ee} .$$

Since $u_\alpha * u_\beta * u_\alpha = u_\beta$, we obtain (E2).

Finally, to show (E1), let $\alpha > \beta > \gamma$ in Y. From (4.1) we find that, for any $a \in G_\alpha$, $au_\beta = (\alpha\phi_{\alpha,\beta} , u_\beta) = a\phi_{\alpha,\beta}$. Since $u_\beta u_\gamma = u_\gamma$, we then have

$$a\phi_{\alpha,\beta}\phi_{\beta,\gamma} = (au_\beta)\phi_{\beta,\gamma} = au_\beta u_\gamma = au_\gamma = a\phi_{\alpha,\gamma} .$$

Hence (E1) holds.

Turning now to the direct part of Theorem 4, let $S = \bigcup\{S_\alpha : \alpha \in Y\}$ be constructed as described in the theorem, with conditions (E1–E3) holding. We proceed to show that the hypotheses for the direct part of Theorem 3 hold. It will then follow that S is a band of groups. Since $S/\mathscr{H} \simeq E(*)$, and $E(*)$ is $\mathscr{D}$-split by hypothesis, it then follows that S is $\mathscr{D}$-split.

We define $\xi_{\alpha,\beta}: S_\alpha \to G_\beta$ $(\alpha > \beta)$ by (4.6). Equation (4.1) then implies (3.3), and condition (E3) implies (D1). All that remains is to prove (D2). Let $\alpha\beta > \gamma$ in Y, and let $A = (a, e) \in S_\alpha$ and $B = (b, f) \in S_\beta$. By (4.1), (4.6), and (E2),

$$\begin{aligned}(AB)\, \xi_{\alpha\beta,\gamma} &= [(a\phi_{\alpha,\alpha\beta})\, p^{\alpha\beta}_{f*u_{\alpha\beta}*e}(b\phi_{\beta,\alpha\beta})]\, \phi_{\alpha\beta,\gamma} \\ &= (a\phi_{\alpha,\alpha\beta}\phi_{\alpha\beta,\gamma})(p^{\alpha\beta}_{f*u_{\alpha\beta}*e})\, \phi_{\alpha\beta,\gamma}(b\phi_{\beta,\alpha\beta}\phi_{\alpha\beta,\gamma}) \\ &= (a\phi_{\alpha,\gamma})\, p_g^{\gamma}(b\phi_{\beta,\gamma}),\end{aligned}$$

where

$$\begin{aligned} g &= (f * u_{\alpha\beta} * e) * u_\gamma * (f * u_{\alpha\beta} * e) \\ &= f * u_{\alpha\beta} * e * u_{\alpha\beta} * u_\gamma * u_{\alpha\beta} * f * u_{\alpha\beta} * e \\ &= f * u_{\alpha\beta} * u_\gamma * u_{\alpha\beta} * e = f * u_\gamma * e . \end{aligned}$$

Again using (4.6), we arrive at (D2).

5. Completely Regular Semigroups Satisfying $\mathscr{D}$-Covering

Let $S = \bigcup\{S_\alpha : \alpha \in Y\}$ be a completely regular semigroup satisfying $\mathscr{D}$-covering, the latter meaning that if e and f are idempotents of S such that $e \in S_\alpha$ and $f \in S_\beta$ with $\alpha > \beta$, then $e > f$. We shall apply Theorem 2 to give a structure theorem for such semigroups (Theorem 5).

As in Section 2, we can assume $S_\alpha = \mathscr{M}(G_\alpha; I_\alpha, \Lambda_\alpha; P_\alpha)$, for each $\alpha \in Y$, with $P_\alpha = (p^\alpha_{\lambda,i})$ normalized at $(1_\alpha, 1_\alpha)$. We identify $a \in G_\alpha$ with $(a; 1_\alpha, 1_\alpha)$, and, in particular, S_α with G_α if S_α is a group (i.e., $|I_\alpha| = |\Lambda_\alpha| = 1$).

Lemma 5.1. *If $\alpha > \gamma$ in Y, then every idempotent element of S_α acts as a two-sided identity element on S_γ. If $\alpha \neq \alpha\beta \neq \beta$ in Y, then $S_{\alpha\beta} = G_{\alpha\beta}$, and $ef = e_{\alpha\beta}$ for any idempotents e of S_α and f of S_β.*

Proof. Let $e^2 = e \in S_\alpha$ and $c \in S_\gamma$. Let f be the identity element of H_c. Since S satisfies $\mathscr{D}$-covering, $e > f$, and hence $ec = efc = fc = c$; similarly, $ce = c$.

Now let $\alpha \neq \alpha\beta \neq \beta$ in Y, and let $e^2 = e \in S_\alpha$ and $f^2 = f \in S_\beta$. Since $\alpha > \alpha\beta$ and $\beta > \alpha\beta$, both e and f act as identities on $S_{\alpha\beta}$, and clearly the same must be true of ef. But $ef \in S_{\alpha\beta}$. Hence $S_{\alpha\beta}$ is a completely simple semigroup containing an identity element, hence is a group. By convention, $S_{\alpha\beta} = G_{\alpha\beta}$, and we have also shown that $ef = e_{\alpha\beta}$.

Lemma 5.2. *Let $\alpha > \beta$ in Y, and let $(a; i, \lambda) \in S_\alpha$, $(b; j, \mu) \in S_\beta$. Then*

$$(a; i, \lambda)(b; j, \mu) = a(b; j, \mu),$$
$$(b; j, \mu)(a; i, \lambda) = (b; j, \mu)a.$$

Proof. Since P_α is normalized, $(e_\alpha; i, 1_\alpha)$ and $(e_\alpha; 1_\alpha, \lambda)$ are idempotent. By Lemma 5.1, they act as identity elements on S_β, and hence

$$\begin{aligned} (a; i, \lambda)(b; j, \mu) &= (e_\alpha; i, 1_\alpha)(a; 1_\alpha, 1_\alpha)(e_\alpha; 1_\alpha, \lambda)(b; j, \mu) \\ &= (a; 1_\alpha, 1_\alpha)(b; j, \mu) = a(b; j, \mu). \end{aligned}$$

The proof of the second equation is similar.

THEOREM 5. *With each element* α *of a semilattice* Y *associate a Rees matrix semigroup* $S_\alpha = \mathcal{M}(G_\alpha; I_\alpha, \Lambda_\alpha; P_\alpha)$ *with* P_α *normalized at* $(1_\alpha, 1_\alpha)$, *and with* $S_\alpha \cap S_\beta = \varnothing$ *if* $\alpha \neq \beta$. *Assume that if* $\alpha \neq \alpha\beta \neq \beta$, *then* $S_{\alpha\beta}$ *is a group* (*which we identify with* $G_{\alpha\beta}$). *For each* $\alpha > \beta$ *in* Y, *assume that* G_α *acts by permutations on* I_β *from the left and on* Λ_β *from the right. With each pair* $\alpha > \beta$ *in* Y *associate a mapping* $\phi_{\alpha,\beta}: G_\alpha \to G_\beta$ *such that the following conditions are satisfied, for all* $\alpha > \beta$ *in* (C1–C4) *and all* $\alpha > \beta > \gamma$ *in* (C5–C6):

(C1) $(ab)\phi_{\alpha,\beta} = (a\phi_{\alpha,\beta})p^{\beta}_{1a,b1}(b\phi_{\alpha,\beta})$ (*all* $a, b \in G_\alpha$);

(C2) $p^{\beta}_{\lambda,a1}(a\phi_{\alpha,\beta})p^{\beta}_{\lambda a,i} = p^{\beta}_{\lambda,ai}(a\phi_{\alpha,\beta})p^{\beta}_{1a,i}$ (*all* $a \in G_\alpha$, $i \in I_\beta$, $\lambda \in \Lambda_\beta$);

(C3) $p^{\alpha}_{\lambda,i}\phi_{\alpha,\beta} = e_\beta$ (*all* $i \in I_\alpha$, $\lambda \in \Lambda_\alpha$);

(C4) $p^{\alpha}_{\lambda,i}j = j$ *and* $\mu p^{\alpha}_{\lambda,i} = \mu$ (*all* $i \in I_\alpha$, $j \in I_\beta$, $\lambda \in \Lambda_a$, $\mu \in \Lambda_\beta$);

(C5) $\phi_{\alpha,\beta}\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$;

(C6) $(a\phi_{\alpha,\beta})i = ai$ *and* $\lambda(a\phi_{\alpha,\beta}) = \lambda a$ (*all* $a \in G_\alpha$, $i \in I_\gamma$, $\lambda \in \Lambda_\gamma$).

Define product in $S = \bigcup\{S_\alpha : \alpha \in Y\}$ *as follows. If* $(a; i, \lambda) \in S_\alpha$ *and* $(b; j, \mu) \in S_\beta$, *let*

$$\begin{aligned}(a; i, \lambda)(b; j, \mu) &= ((a\phi_{\alpha,\beta})\, p^{\beta}_{1a,j}b;\ aj, \mu) && \text{if } \alpha > \beta,\\ &= (ap^{\alpha}_{\lambda,b1}(b\phi_{\beta,\alpha});\ i, \lambda b) && \text{if } \alpha < \beta,\\ &= (ap^{\alpha}_{\lambda,j}b;\ i, \mu) && \text{if } \alpha = \beta,\\ &= (a\phi_{\alpha,\alpha\beta})(b\phi_{\beta,\alpha\beta}) && \text{if } \alpha \neq \alpha\beta \neq \beta.\end{aligned} \tag{5.1}$$

Then S *becomes a completely regular semigroup satisfying* $\mathscr{D}$*-covering. Conversely, every such semigroup is isomorphic to one constructed in this way.*

Proof. We prove the converse first. Let S be a completely regular semigroup satisfying $\mathscr{D}$-covering. Being a completely regular semigroup, S has the structure described in Theorem 2. We use the language and notation of actions (of S_α on I_β and Λ_β when $\alpha > \beta$) rather than the mappings $\chi_{\alpha,\beta}$ and $\psi_{\alpha,\beta}$ (see the remark at the end of Section 2).

Let $\alpha > \beta$ in Y, and let $A = (a; i, \lambda) \in S_\alpha$ and $B = (b; j, \mu) \in S_\beta$. By (2.3),

$$AB = ((A\xi_{\alpha,\beta})\, p^{\beta}_{1A,j}b;\ Aj, \mu). \tag{5.2}$$

As a special case of this,

$$aB = ((a\xi_{\alpha,\beta})\, p^{\beta}_{1a,j}b;\ aj, \mu).$$

By Lemma 5.2, $AB = aB$. Hence $Aj = aj$ for all $j \in I_\beta$. Dually, we can show that $\mu A = \mu a$ for all $\mu \in \Lambda_\beta$. In particular, $1A = 1a$, and we then conclude from $AB = aB$ that $A\xi_{\alpha,\beta} = a\xi_{\alpha,\beta}$. Defining $\phi_{\alpha,\beta}: G_\alpha \to G_\beta$ to be the restriction of $\xi_{\alpha,\beta}$ to G_α, we have shown that

$$(a; i, \lambda)\xi_{\alpha,\beta} = a\phi_{\alpha,\beta} \qquad (\text{all } (a; i, \lambda) \in S_\alpha). \tag{5.3}$$

Equation (5.2) now reduces to the case of $\alpha > \beta$ of (5.1). The case $\alpha < \beta$ is dual, and the case $\alpha = \beta$ is just product in $\mathscr{M}(G_\alpha; I_\alpha, \Lambda_\alpha; P_\alpha)$. If we now apply (2.2) to the case $\alpha \neq \alpha\beta \neq \beta$, using (5.3), and note from Lemma 5.1 that $|I_{\alpha\beta}| = |\Lambda_{\alpha\beta}| = 1$, we obtain the fourth case in (5.1).

Let $\alpha > \beta$ in Y. As remarked after Theorem 2, S_α acts on I_β from the left and on Λ_β from the right. In particular, so does the subgroup G_α of S_α. By Lemma 5.1, e_α induces the identity transformation of I_β and Λ_β, and hence the action of G_α on I_β and Λ_β is by permutations.

All that remains is to show that (C1–C6) hold. From the third equation in (B3), replacing β by α and γ by β, we get (for $\alpha > \beta$, and $A, B \in S_\alpha$),

$$(AB)\,\xi_{\alpha,\beta} = (A\xi_{\alpha,\beta})\,p^{\beta}_{1A,B1}(B\xi_{\alpha,\beta}).$$

(C1) is immediate from this and the definition of $\phi_{\alpha,\beta}$. Similarly, (C2) is immediate from (B0).

Let $a = (p^{\alpha}_{\lambda,i})^{-1}$. Then $(a; i, \lambda)$ is an idempotent element of S_α, so, by Lemma 5.1, $(a; i, \lambda)(b; j, \mu) = (b; j, \mu)$ for all $(b; j, \mu) \in S_\beta$. Comparing this with the first case in (5.1), we find

$$(a\phi_{\alpha,\beta})\,p^{\beta}_{1a,j} = e_\beta, \qquad aj = j \qquad (\text{all } j \in I_\beta).$$

Hence the first half of (C4) holds, and the second half is dual thereto. Also, taking $j = 1$ and using the normalization of P_β, we obtain (C3).

Let $\alpha > \beta > \gamma$ in Y, and let $a \in G_\alpha$ and $i \in I_\gamma$. Let $b = a\phi_{\alpha,\beta}$. By (5.1),

$$\begin{aligned} ae_\beta(e_\gamma; i, 1_\gamma) &= (b; a1_\beta, 1_\beta)(e_\gamma; i, 1_\gamma) \\ &= ((b\phi_{\beta,\gamma})\,p^{\gamma}_{1b,i}; bi, 1_\gamma), \\ a(e_\gamma; i, 1_\gamma) &= (a\phi_{\alpha,\gamma}; ai, 1_\gamma). \end{aligned}$$

By Lemma 5.1, $e_\beta(e_\gamma; i, 1_\gamma) = (e_\gamma; i, 1_\gamma)$, so these are equal. Hence $ai = bi = (a\phi_{\alpha,\beta})i$, proving the first part of (C6), and the second is dual. Also, taking $i = 1_\gamma$, we have $p^{\gamma}_{1,b1} = e_\gamma$, and so $a\phi_{\alpha,\gamma} = b\phi_{\beta,\gamma} = a\phi_{\alpha,\beta}\phi_{\beta,\gamma}$, proving (C5).

Turning now to the direct half of Theorem 5, assume all the hypotheses and notation of the theorem. We state two consequences of these hypotheses as lemmas for later use.

LEMMA 5.3. *If $\alpha > \beta$ in Y, then*

$$(ap^{\alpha}_{\lambda,i}b)\,\phi_{\alpha,\beta} = (a\phi_{\alpha,\beta})\,p^{\beta}_{1a,b1}(b\phi_{\alpha,\beta}),$$

for all $a, b \in G_\alpha$; $i \in I_\alpha$, $\lambda \in \Lambda_\alpha$.

Proof. Writing c for $p^{\alpha}_{\gamma,i}$ and ϕ for $\phi_{\alpha,\beta}$, we have $c\phi = e_\beta$ by (C3), and $cj = j$, $\mu c = \mu$ for all $j \in I_\beta$, $\mu \in \Lambda_\beta$, by (C4). Hence, by (C1),

$$\begin{aligned}(acb)\phi &= (ac)\phi \cdot p^{\beta}_{1ac,b1}(b\phi)\\ &= (a\phi)\, p^{\beta}_{1a,c1}(c\phi)\, p^{\beta}_{1ac,b1}(b\phi)\\ &= (a\phi)\, p^{\beta}_{1a,b1}(b\phi)\end{aligned}$$

since $p^{\beta}_{1a,c1} = p^{\beta}_{1a,1} = e_\beta$ and $1ac = (1a)c = 1a$.

LEMMA 5.4. *If $\alpha > \beta$ in Y, then*

$$(ap^{\alpha}_{\lambda,i}b)j = abj \qquad \textit{and} \qquad \mu(ap^{\alpha}_{\lambda,i}b) = \mu ab$$

for all $a, b \in G_\alpha$; $i \in I_\alpha$, $j \in I_\beta$; $\lambda \in \Lambda_\alpha$, $\mu \in \Lambda_\beta$.

Proof. Immediate from (C4) and the hypothesis that G_α acts on I_β and Λ_β.

We proceed now to apply Theorem 2. We take condition (B1) as a definition of $\xi_{\alpha,\alpha}$, $\chi_{\alpha,\alpha}$, and $\psi_{\alpha,\alpha}$. For $\alpha > \beta$ and $A = (a; i, \lambda) \in S_\alpha$, we define $A\xi_{\alpha,\beta}$, $A\chi_{\alpha,\beta}$, and $A\psi_{\alpha,\beta}$ by

$$\begin{aligned}A\xi_{\alpha,\beta} &= a\phi_{\alpha,\beta},\\ (A\chi_{\alpha,\beta})j &= aj \qquad (\text{all } j \in I_\beta),\\ \mu(A\psi_{\alpha,\beta}) &= \mu a \qquad (\text{all } \mu \in \Lambda_\beta).\end{aligned}$$

(B0) is then immediate from (C2).

Let $A = (a; i, \lambda) \in S_\alpha$ and $B = (b; j, \mu) \in S_\beta$. If $\alpha > \beta$, then $(A\chi_{\alpha,\beta})(B\chi_{b,\beta}) = (A\chi_{\alpha,\beta})\langle j\rangle^* = \langle aj\rangle^*$, and $(A\psi_{\alpha,\beta})(B\psi_{\beta,\beta}) = (A\psi_{\alpha,\beta})\langle\mu\rangle = \langle\mu\rangle$. Hence (B2) holds with $k = aj$ and $\nu = \mu$. Moreover we see that (2.2) agrees with the first case in (5.1). The case $\alpha < \beta$ is dual, and the case $\alpha = \beta$ is trivial. (B2) also holds trivially for the case $\alpha \neq \alpha\beta \neq \beta$, since $|I_{\alpha\beta}| = |\Lambda_{\alpha\beta}| = 1$, and (2.2) agrees with the fourth case in (5.1) since $p^{\alpha\beta}_{1A,B1} = e_{\alpha\beta}$.

To show (B3), let $\alpha\beta > \gamma$, $A = (a; i, \lambda) \in S_\alpha$, $B = (b; j, \mu) \in S_\beta$. We must consider the cases $\alpha > \beta$, $\alpha < \beta$, $\alpha = \beta$, and $\alpha \neq \alpha\beta \neq \beta$. Let $k \in I_\gamma$, $\nu \in \Lambda_\gamma$. We are to show in each case that

$$\begin{aligned}(AB)k &= A(Bk) = a(bk),\\ \nu(AB) &= (\nu A)B = (\nu a)b,\\ (AB)\,\xi_{\alpha\beta,\gamma} &= (a\phi_{\alpha,\gamma})\, p^{\gamma}_{1a,b1}(b\phi_{\beta,\gamma}).\end{aligned}$$

We omit the second, which is dual to the first.

Case $\alpha > \beta$. $AB = ((a\phi_{\alpha,\beta})p^{\beta}_{1a,j}b;\, aj, \mu)$. We have

$$(AB)k = [(a\phi_{\alpha,\beta})\, p^{\beta}_{1a,j}b]k = [(a\phi_{\alpha,\beta})b]k$$

by Lemma 5.4. This is equal to $(a\phi_{\alpha,\beta})(bk)$, since G_β acts on I_γ, and this in turn to $a(bk)$ by (C6). Likewise,

$$(AB)\,\xi_{\beta,\gamma} = [(a\phi_{\alpha,\beta})\,p^{\beta}_{1a,j}b]\phi_{\beta,\gamma}$$
$$= [(a\phi_{\alpha,\beta})\,\phi_{\beta,\gamma}]\,p^{\gamma}_{1a,b1}(b\phi_{\beta,\gamma})$$

by Lemma 5.3 and the fact that $1(a\phi_{\alpha,\beta}) = 1a$ by (C6). The desired conclusion now follows from (C5).

Case $\alpha < \beta$. Dual to the preceding.

Case $\alpha = \beta$. $AB = (ap^{\alpha}_{\lambda,j}b;\, i, \mu)$. *We have*

$$(AB)k = (ap^{\alpha}_{\lambda,j}b)k = (ab)k = a(bk)$$

by Lemma 5.4. Likewise

$$(AB)\,\xi_{\alpha,\gamma} = (ap^{\alpha}_{\lambda,j}b)\,\phi_{\alpha,\gamma} = (a\phi_{\alpha,\gamma})\,p^{\gamma}_{1a,b1}(b\phi_{\alpha,\gamma})$$

by Lemma 5.3.

Case $\alpha \neq \alpha\beta \neq \beta$. $AB = (a\phi_{\alpha,\alpha\beta})(b\phi_{\beta,\alpha\beta})$. We have

$$(AB)k = (a\phi_{\alpha,\alpha\beta})[(b\phi_{\beta,\alpha\beta})k] = (a\phi_{\alpha,\alpha\beta})(bk) = a(bk)$$

by (C6). Likewise

$$(AB)\,\xi_{\alpha\beta,\gamma} = [(a\phi_{\alpha,\alpha\beta})(b\phi_{\beta,\alpha\beta})]\,\phi_{\alpha\beta,\gamma}$$
$$= [(a\phi_{\alpha,\alpha\beta})\,\phi_{\alpha\beta,\gamma}]\,p^{\gamma}_{1a,b1}[(b\phi_{\beta,\alpha\beta})\,\phi_{\alpha\beta,\gamma}]$$

by Lemma 5.3 and the facts that $1(a\phi_{\alpha,\alpha\beta}) = 1a$ and $(b\phi_{\beta,\alpha\beta})1 = b1$ by (C6). Using (C5), we reach the desired conclusion.

This concludes the proof of Theorem 5.

6. Special Cases

In this section we specialize Theorem 5 to orthogroups (Theorem 6.1) and bands of groups (Theorem 6.2).

A completely regular semigroup $S = \bigcup\{S_\alpha\colon \alpha \in Y\}$, where

$$S_\alpha = \mathscr{M}(G_\alpha; I_\alpha, \Lambda_\alpha; P_\alpha)$$

and each P_α is normalized, is orthodox if and only if $p^{\alpha}_{\lambda,i} = e_\alpha$ for all $\lambda \in \Lambda_\alpha$, $i \in I_\alpha$. If this is the case, each S_α is the direct product $G_\alpha \times (I_\alpha \times \Lambda_\alpha)$ of a group G_α and a rectangular band $I_\alpha \times \Lambda_\alpha$; such a semigroup is called a *rectangular group.*

The following theorem is immediate from Theorem 5. It was found by Fortunatov [3] and one of us [9] for the case Y totally ordered.

THEOREM 6.1. *Let $S = \bigcup\{S_\alpha : \alpha \in Y\}$ be a disjoint union of rectangular groups $S_\alpha = G_\alpha \times I_\alpha \times \Lambda_\alpha$, indexed by a semilattice Y. Assume that if $\alpha \neq \alpha\beta \neq \beta$ in Y, then $|I_{\alpha\beta}| = |\Lambda_{\alpha\beta}| = 1$; we then identify $S_{\alpha\beta}$ with $G_{\alpha\beta}$. For every pair $\alpha > \beta$ in Y, assume that G_α acts by permutations on I_β from the left and on Λ_β from the right, and that there exists a homomorphism $\phi_{\alpha,\beta}$: $G_\alpha \to G_\beta$ such that, for all $\alpha > \beta > \gamma$ in Y,*

(C5) $\phi_{\alpha,\beta}\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$,

(C6) $(a\phi_{\alpha,\beta})i = ai$ *and* $\lambda(a\phi_{\alpha,\beta}) = \lambda a$,

for all $a \in G_\alpha$, $i \in I_\gamma$, $\lambda \in \Lambda_\gamma$. Define product in S as follows. If $(a; i, \lambda) \in S_\alpha$ and $(b; j, \mu) \in S_\beta$, let

$$\begin{aligned}(a; i, \lambda)(b; j, \mu) &= ((a\phi_{\alpha,\beta})b;\, aj, \mu) && \text{if } \alpha > \beta,\\ &= (a(b\phi_{\beta,\alpha});\, i, \lambda b) && \text{if } \alpha < \beta,\\ &= (ab;\, i, \mu) && \text{if } \alpha = \beta,\\ &= (a\phi_{\alpha,\alpha\beta})(b\phi_{\beta,\alpha\beta}) && \text{if } \alpha \neq \alpha\beta \neq \beta. \end{aligned} \tag{6.1}$$

Then S becomes an orthogroup satisfying $\mathcal{D}$-covering. Conversely, every such orthogroup can be constructed in this way.

Turning now to the specialization of Theorem 5 to bands of groups, let S be a semigroup constructed as in Theorem 5. From (5.1) it is clear that S will be a band of groups if and only if, for every $\alpha > \beta$ in Y, G_α acts trivially on I_β and Λ_β (that is, $ai = i$ and $\lambda a = \lambda$ for all $a \in G_\alpha$, $i \in I_\beta$, $\lambda \in \Lambda_\beta$). If this is the case, $p^\beta_{1a,j} = p^\beta_{1,j} = e_\beta$ and $p^\alpha_{\lambda,b1} = p^\alpha_{\lambda,1} = e_\alpha$ in (5.1). Likewise, $p^\beta_{1a,b1} = p^\beta_{1,1} = e_\beta$ in (C1), so each $\phi_{\alpha,\beta}$ is a homomorphism. Also, (C2) becomes

$$(a\phi_{\alpha,\beta})\, p^\beta_{\lambda,i} = p^\beta_{\lambda,i}(a\phi_{\alpha,\beta}),$$

while (C4) and (C6) become superfluous. Thus the following theorem is immediate from Theorem 5.

THEOREM 6.2. *Let $S = \bigcup\{S_\alpha : \alpha \in Y\}$ be a disjoint union of Rees matrix semigroups $S_\alpha = \mathcal{M}(G_\alpha; I_\alpha, \Lambda_\alpha; P_\alpha)$, indexed by a semilattice Y, and with each P_α normalized. Assume that if $\alpha \neq \alpha\beta \neq \beta$ in Y, then $S_{\alpha\beta} = G_{\alpha\beta}$, and that, for every pair $\alpha > \beta$ in Y, there exists a homomorphism $\phi_{\alpha,\beta}$: $G_\alpha \to G_\beta$ such that the following conditions hold.*

(C2′) *For every pair $\alpha > \beta$ in Y, and for all $i \in I_\beta$, $\lambda \in \Lambda_\beta$, $p^\beta_{\lambda,i}$ belongs to the centralizer of $G_\alpha\phi_{\alpha,\beta}$ in G_β.*

(C3) *For every pair* $\alpha > \beta$ *in* Y, *and for all* $i \in I_\alpha$, $\lambda \in \Lambda_\alpha$, $p^\alpha_{\lambda,i}$ *belongs to the kernel of* $\phi_{\alpha,\beta}$.

(C5) *For every triple* $\alpha > \beta > \gamma$ *in* Y, $\phi_{\alpha,\beta}\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$.

Define product in S *as follows. If* $(a; i, \lambda) \in S_\alpha$ *and* $(b; j, \mu) \in S_\beta$, *let*

$$\begin{aligned}(a; i, \lambda)(b; j, \mu) &= ((a\phi_{\alpha,\beta})b; j, \mu) && \textit{if} \quad \alpha > \beta,\\ &= (a(b\phi_{\beta,\alpha}); i, \lambda) && \textit{if} \quad \alpha < \beta,\\ &= (ap^\alpha_{\lambda,j}b; i, \mu) && \textit{if} \quad \alpha = \beta,\\ &= (a\phi_{\alpha,\alpha\beta})(b\phi_{\beta,\alpha\beta}) && \textit{if} \quad \alpha \neq \alpha\beta \neq \beta.\end{aligned} \tag{6.2}$$

Then S *becomes a band of groups satisfying* $\mathscr{D}$*-covering. Conversely, every such band of groups can be constructed in this way.*

There is an interesting class of bands of groups satisfying $\mathscr{D}$-covering that arose in the work of Kacman [5].

Let Ω be a chain (=totally ordered set). Let $\{S_\alpha : \alpha \in \Omega\}$ be a set of mutually disjoint semigroups indexed by Ω. Define product in $S = \bigcup\{S_\alpha : \alpha \in \Omega\}$, extending the given product in each S_α, as follows. If $a \in S_\alpha$ and $b \in S_\beta$ with $\alpha \neq \beta$, then let

$$\begin{aligned}ab &= a \quad \text{if} \quad \alpha < \beta,\\ &= b \quad \text{if} \quad \alpha > \beta.\end{aligned}$$

Associativity is easily checked. We call S the *ordinal sum* of the chain Ω of semigroups S_α $(\alpha \in \Omega)$.

It is easily seen that a band is an ordinal sum of rectangular bands if and only if it satisfies $\mathscr{D}$-covering and its structure semilattice is a chain.

For any semigroup S, let $\Sigma'(S)$ denote the lattice of subsemigroups of S, empty set included. A surjective homomorphism of $\Sigma'(S)$ onto a lattice L is called special if $\varnothing$ [S] is the only element of $\Sigma'(S)$ mapped onto the least [greatest] element of L. Kacman [5] showed that there exists a special complete homomorphism of $\Sigma'(S)$ onto a relatively complemented lattice if and only if

(1) S is a band of groups,

(2) $S/\mathscr{H}$ is an ordinal sum of rectangular bands,

(3) S is idempotent generated, and

(4) S is periodic.

The following corollary of Theorem 6.2 gives the structure of the semigroups figuring in Kacman's theorem.

COROLLARY 6.3. *A semigroup satisfies Kacman's conditions* (1), (2), (3) *if and only if it is an ordinal sum of idempotent-generated, completely simple semigroups.*

Proof. The "if" part is evident. Conversely, let $S = \bigcup\{S_\alpha : \alpha \in Y\}$ be a band of groups satisfying (2) and (3). Then Y is a chain, and $S/\mathscr{H}$ satisfies $\mathscr{D}$-covering. The latter implies that S itself satisfies $\mathscr{D}$-covering. For suppose $e^2 = e \in S_\alpha$ and $f^2 = f \in S_\beta$, with $\alpha > \beta$. Let H_e denote the $\mathscr{H}$-class (maximal subgroup) of S containing e. Since $S/\mathscr{H}$ satisfies $\mathscr{D}$-covering, $H_eH_f \subseteq H_f$ and $H_fH_e \subseteq H_e$. This implies that $H_e \cup H_f$ is a subsemigroup of S which is a (two-element) semilattice of groups. But this implies $e > f$.

Hence we may assume that S has the structure described in Theorem 6.2. We recall that the idempotents of $S_\alpha = \mathscr{M}(G_\alpha; I_\alpha, \Lambda_\alpha; P_\alpha)$ are the elements of S_α of the form $((p^\alpha_{\lambda,i})^{-1}; i, \lambda)$. It follows that if G_α' is the subgroup of G_α generated by the set $\{p^\alpha_{\lambda,i} : i \in I_\alpha, \lambda \in \Lambda_\alpha\}$ of entries of P_α, then the subsemigroup S_α' of S_α generated by the idempotents of S_α is $\mathscr{M}(G_\alpha'; I_\alpha, \Lambda_\alpha; P_\alpha)$. By Kacman's condition (3), $S_\alpha' = S_\alpha$, and hence $G_\alpha' = G_\alpha$. From condition (C3) of Theorem 6.2, we conclude that $G_\alpha\phi_{\alpha,\beta} = \{e_\beta\}$, since G_α is generated by the entries $p^\alpha_{\lambda,i}$ of P_α. From this and (6.2) it follows that S is the ordinal sum of the S_α.

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