Mekler’s construction preserves CM-triviality

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Abstract

For every structure $M$ of finite signature Mekler (J. Symbolic Logic 46 (1981) 781) has constructed a group $G$ such that for every $\kappa$ the maximal number of $n$-types over an elementary equivalent model of cardinality $\kappa$ is the same for $M$ and $G$. These groups are nilpotent of class 2 and of exponent $p$, where $p$ is a fixed prime greater than 2. We consider stable structures $M$ only and show that $M$ is CM-trivial if and only if $G$ is CM-trivial. Furthermore, we obtain that the free group $F_2(p, \omega)$ in the variety of 2-nilpotent groups of exponent $p \geq 2$ with $\omega$ free generators has a CM-trivial $\omega$-stable theory. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let $S$ be a complete theory of finite signature. Let $\lambda_S(\kappa)$ be the maximal size $|S_1(M)|$ of the Stone space $S_1(M)$ for $M \models S$ with $\kappa = |M|$. For every such $S$ Mekler [11] constructed a complete group theory $T_S$ such that $S$ is interpretable in $T_S$ and $\lambda_S(\kappa) = \lambda_{T_S}(\kappa)$ for every cardinality $\kappa$. It follows that $S$ and $T_S$ have the same stability spectrum. Furthermore, it can be shown that $S$ is simple if and only if $T_S$ is simple.

The models of $T_S$ are nilpotent groups of class 2 and exponent $p$ ($p > 2$). Already Eršov [7] used similar ideas to Mekler’s to prove the undecidability of the variety of these groups. We call the models of such theories $T_S$ Mekler groups. If $S$ is superstable then $T_S$ is superstable. Under the assumption that $S$ is superstable it is shown in [3] that $S$ has NDOP if $T_S$ has NDOP. If Depth($S$) exists, then

$$\text{Depth}(S) \leq \text{Depth}(T_S) \leq \text{Depth}(S) + 1.$$
This is used to construct $\omega$-stable groups with NDOP and Depth $n$ for every $n < \omega$. In [4] it is used to construct superstable NDOP–NOTOP groups. Note that Mekler groups have infinite rank and are not $\aleph_0$-categorical. Since Mekler groups are not abelian by finite they are not one-based, as by Hrushorski and Pillay [10] one-based stable groups are abelian by finite. For stable theories one-basedness implies the non-interpretability of a field. For uncountably categorical theories Zil’ber [16] conjectured the converse. In [9] Hrushovski refuted this conjecture and introduced $CM$-triviality. We define this notion in the next section. It is weaker than one-basedness but it still implies the non-interpretability of a field. It is an open question whether Zil’ber’s conjecture is true if we replace one-basedness by $CM$-triviality. In [6] an $\omega$-stable, non-$CM$-trivial theory is constructed that does not allow the interpretation of a field. But this theory has infinite rank.

In this paper we show that $S$ is stable and $CM$-trivial if and only if the corresponding theory $T_S$ of Mekler-groups is stable and $CM$-trivial. Hence Mekler’s construction can be used to produce examples of stable $CM$-trivial groups. Pillay proved in [12] that $CM$-trivial groups of finite Morley rank are nilpotent by finite. Wagner [15] got similar results replacing the condition to be of finite Morley rank by stability plus several other conditions. Note that the new $\aleph_1$-categorical group constructed in [5] is $CM$-trivial.

Let $F_c(p, \omega)$ be the free group in the variety of nilpotent groups of class $c$ and of exponent $p$ (prime $> c$) with $\omega$ free generators. In [1, 2] it is shown that $Th(F_c(p, \omega))$ is $\omega$-stable and non-multidimensional with exactly $c$ dimensions. Our proof of the main result of this paper gives us, furthermore, that $Th(F_2(p, \omega))$ is $CM$-trivial. Hence it is not possible to interpret a field.

In Section 2 we define $CM$-triviality and consider interpretations without new information. In Section 3 we consider the bilinear maps $\mathcal{F}(G)$ for $G \models T_S$ that are given by the commutator. They live in $G^{eq}$. Often it is more convenient to work in $G^{eq}$. In Section 4 we follow Hodges [8] to describe Mekler’s construction. In Section 5 we prove that $S$ is simple if and only if $T_S$ is simple. This is common work with my student Alexander Pentzel. In Section 6 we introduce the notion of a good subspace of $V_G = G/Z(G)$ for $G \models T_S$. We show that every element $a$ of the commutator subgroup $G'$ is interalgebraic with a finite good subspace of $V_G$. In Section 7 we explain how each good subspace of $V_G$ corresponds to a good subspace that respects $V_G$ for $G \leq C \models T_S$. In Section 8 we formulate the main result and prove it for the tame case. The lemmas we show are important for the general case. In Section 9 we discuss the case of the mixed elements. In Section 10 the results of Sections 8 and 9 are used to prove the main result. In Section 11 $CM$-triviality of $F_2(p, \omega)$ is proved.

2. $CM$-triviality and interpretation

Let $T$ be any complete stable theory. We use $\mathbb{C}$ to denote the monster model. Often we work in $T^{eq}$ and $\mathbb{C}^{eq}$. In this paper the definable and the algebraic closure $dcl(A)$ and $acl(A)$, respectively, are considered in $\mathbb{C}^{eq}$. $Cb(tp(\vec{a}/A))$ is used to denote the
canonical base of the type $tp(\bar{a}/A)$. In fact, it is the canonical base of the strong type of $\bar{a}/A$, that means of $tp(\bar{a}/acl(A))$.

$CM$-triviality was introduced by Hrushovski [9]. It is a property of $T^{eq}$.

**Definition.** $T$ is **$CM$-trivial** if whenever $c \in C^{eq}$, $A \subseteq B$ are algebraically closed sets in $C^{eq}$ with $acl(cA) \cap B = A$, then $Cb(tp(c/A)) \subseteq acl(Cb(tp(c/B)))$.

Pillay [12] proved that it is sufficient to consider only models $M \preceq N$ of $T$ instead of $A \subseteq B$, and tuples $\bar{c}$ in $C$. As usual in $C^{eq}$ we often do not distinguish between models $M$ of $T$ and $M^{eq} = acl^{eq}(M)$. Note that it is also possible to assume that $M$ and $N$ are saturated.

Let $S$ and $T$ be theories, and $L_S$ and $L_T$ be the corresponding languages. At the moment we do not assume completeness as in the rest of the section. We use $G$ and $H$ to denote the models of $T$, and $M$ and $N$ to denote the models of $S$.

**Definition.** An interpretation $\Gamma$ of $S$ in $T$ is a uniform definition of an $S$-model $\Gamma(G)$ in every model $G^{eq}$ of $T^{eq}$ as a relativised reduct of a definitional expansion of a finite slice of $G^{eq}$.

As in Hodges book [8] an interpretation $\Gamma$ of $S$ in $T$ induces a functor $Func(\Gamma)$ from the category of models of $T$ and elementary embeddings to the category of models of $S$ and elementary embeddings.

We want to describe this notion in more detail. An unnested atomic formula of $L_S$ is a formula of the form $x_0 = x_1$, $x_0 = c$, $R(x_0, \ldots, x_{m-1})$ or $f(x_0, \ldots, x_{m-2}) = x_{m-1}$, where $c$ is a constant symbol of $L_S$, $R$ is a relation symbol of $L_S$, and $f$ is a function symbol of $L_S$. $\Gamma$ can be given by an $L_T$-formula $\gamma(\bar{x})$ and $L_T$-formulas $\gamma(\phi)(\bar{x}_0, \ldots, \bar{x}_{m-1})$ for every unnested atomic formula $\phi(x_0, \ldots, x_{m-1})$ of $L_S$, where all $\bar{x}, \bar{x}_i$ are of the same length $n$.

We write $\bar{x} \sim_\text{eq} \bar{y}$ instead of $\gamma(x = y)(\bar{x}, \bar{y})$. Since we want that these formulas describe an $L_S$-structure $\Gamma(G)$ in every model $G$ of $T$, the following elementary conditions must be satisfied by $\Gamma$:

(i) $\sim_\text{eq}$ is an equivalence relation.

(ii) $\forall \bar{x} \bar{y}(\bar{x} \sim_\text{eq} \bar{y} \land \gamma(\bar{x}) \rightarrow \gamma(\bar{y}))$.

(iii) $\forall \bar{x}_0, \ldots, \bar{x}_{m-1}, \bar{y}_0, \ldots, \bar{y}_{m-1}(\bigwedge_{i < m} \bar{x}_i \sim_\text{eq} \bar{y}_i \land \gamma(\phi)(\bar{x}_0, \ldots, \bar{x}_{m-1}) \rightarrow \gamma(\phi)(\bar{y}_0, \ldots, \bar{y}_{m-1}))$ for all unnested atomic formulas $\phi(x_0, \ldots, x_{m-1})$ of $L_S$. Let the domain of $\Gamma(G)$ be the set of all $\sim$-classes $a$ in $G^{eq}$, where $a = \bar{a}/\sim$ for some tuple $\bar{a}$ satisfying $\gamma(\bar{x})$. $\sim$ plays the role of equality. Then we can define the $L_S$-structure $\sim(\Gamma(G))$ on this set by

$$\Gamma(G) \models \phi(a_0, \ldots, a_{m-1}) \text{ iff } G \models \gamma(\phi)(\bar{a}_0, \ldots, \bar{a}_{m-1}) \quad (2.1)$$

for all $\bar{a}_i$ in $a_i$ and all unnested atomic formulas $\phi(x_0, \ldots, x_{m-1})$ of $L_S$. $\Gamma$ is an interpretation of $S$ in $T$ if $\Gamma(G)$ is a model of $S$. We can define $L_T$ formulas $\gamma(\phi)$ for all $L_S$ formulas $\phi$ in such a way that (2.1) remains true.

Let $\Gamma(x)$ be the $L_T^{eq}$-formula that describes the domain of $\Gamma(G)$. Using (2.1) we can define for every $\phi(x_0, \ldots, x_{m-1})$ of $L_S$ a $L_T^{eq}$-formula $\Gamma(\phi)(x_0, \ldots, x_{m-1})$ such that for
all $a_0, \ldots, a_{m-1}$ in $\Gamma(G)$

$$
\Gamma(G) \models \varphi(a_0, \ldots, a_{m-1}) \iff G^{\text{eq}} \models \Gamma(\varphi)(a_0, \ldots, a_{m-1}).
$$

(2.2)

Now we assume that $S$ and $T$ are complete.

**Definition.** $\Gamma$ is an interpretation without new information, if the following equivalent conditions are fulfilled:

(i) For every model $G$ of $T$ there is an elementary extension $H$ such that it is possible to extend every automorphism of $\Gamma(G)$ (with respect to $L_\text{S}$) to an automorphism of $H$ (with respect to $L_T$).

(ii) For every formula $\psi(x_0, \ldots, x_{m-1})$ of $L^{\text{eq}}$ with

$$
T^{\text{eq}} \models \forall x_0 \ldots x_{m-1} \left( \psi(x_0, \ldots, x_{m-1}) \rightarrow \bigwedge_{i< m} \Gamma(x_i) \right)
$$

there exists some formula $\varphi(x_0, \ldots, x_{m-1})$ of $L_\text{S}$ such that

$$
T^{\text{eq}} \models \forall x_0 \ldots x_{m-1} (\psi(x_0, \ldots, x_{m-1}) \leftrightarrow \Gamma(\varphi)(x_0, \ldots, x_{m-1})).
$$

Interpretations $\Gamma$ without new information are considered in [2, 3]. For the rest of this section we assume that $T$ is stable and therefore $S$ is stable. Furthermore, let $\Gamma$ be an interpretation without new information. If $T$ is stable, then for every model $M$ of $S$ there is some model $G$ of $T$ such that $M \equiv \Gamma(G)$. Using the Open Mapping Theorem for stable theories it is shown [3]:

**Lemma 2.1** ($T$ stable). Let $G$ be a model of $T$. Let $\psi(x_0, \ldots, x_{m-1}, \vec{a})$ be a formula of $(L_T)^{\text{eq}}$ with parameter $\vec{a}$ and

$$
G^{\text{eq}} \models \psi(x_0, \ldots, x_{m-1}, \vec{z}) \rightarrow \bigwedge_{i< m} \Gamma(x_i).
$$

Then there is some $L_\text{S}$ formula $\varphi(x_0, \ldots, x_{m-1}, \vec{b})$ with $\vec{b}$ in $\Gamma(G)$ such that

$$
G^{\text{eq}} \models \psi(x_0, \ldots, x_{m-1}, \vec{a}) \leftrightarrow \Gamma(\varphi)(x_0, \ldots, x_{m-1}, \vec{b}).
$$

Assume $\vec{c}, A \subseteq \Gamma(\mathbb{C})$. Then we define

$$
\text{tp}_T(\vec{c}/A) = \{ \Gamma(\varphi)(\vec{x}, \vec{a}) : \varphi(\vec{x}, \vec{y}) \in L_\text{S}, \ \vec{a} \subseteq A, \ \Gamma(\mathbb{C}) \models \varphi(\vec{c}, \vec{a}) \}.
$$

Here $\Gamma(\mathbb{C})$ is used to denote the $L_\text{S}$-structure defined in $\mathbb{C}^{\text{eq}}$. It is also used to denote the subset of $\mathbb{C}^{\text{eq}}$ defined by the formula $\Gamma(x)$. Clearly, $\text{tp}_T(\vec{c}/A)$ is a subset of $\text{tp}(\vec{c}/A)$. But the definition of an interpretation without new information gives us

**Lemma 2.2.** $\text{tp}_T(\vec{c}/A)$ implies $\text{tp}(\vec{c}/A)$.

We can consider $\text{tp}_T(\vec{c}/A)$ as a type for the theory $S$. In [3] it is shown
Lemma 2.3. (i) Assume \( \bar{c}, A \) and \( B \) are in \( \Gamma(\mathbb{C}) \subseteq \mathbb{C}^\mathbb{eq} \) and \( A \subseteq B \). In \( T^{\mathbb{eq}} \) we have
\[
\text{tp}(\bar{c}/B)\text{dnf}/A \text{ if and only if in } S \text{ } \text{tp}_T(\bar{c}/B)\text{dnf}/A.
\]
(ii) Assume \( \bar{c} \subseteq \Gamma(\mathbb{C}) \) and \( G \preceq H \preceq \mathbb{C} \). Then
\[
in T^{\mathbb{eq}} \text{ tp}(\bar{c}/H)\text{dnf}/G \text{ if and only if } in S \text{ } \text{tp}_T(\bar{c}/\Gamma(H))\text{dnf}/\Gamma(G).
\]
Again consider \( \bar{c} \) and \( A \) in \( \Gamma(\mathbb{C}) \subseteq \mathbb{C}^\mathbb{eq} \). According to this situation there is a subset \( \text{Cb}_T(\text{tp}_T(\bar{c}/A)) \) of \( \Gamma(\mathbb{C})^\mathbb{eq} \subseteq \mathbb{C}^\mathbb{eq} \) which is the canonical base of \( \text{tp}_T(\bar{c}/A) \) if we consider it as a type with respect to \( S \). For \( A \subseteq \Gamma(\mathbb{C}) \) the algebraic closure with respect to \( S \) is the algebraic closure in \( \Gamma(\mathbb{C})^\mathbb{eq} \) with respect to \( T \): \( \text{acl}_S(A) = \text{acl}_T(A) \cap \Gamma(\mathbb{C})^\mathbb{eq} \).

Lemma 2.4. Let \( S \) and \( T \) be stable and assume that \( \Gamma \) is an interpretation without new information of \( S \) in \( T \).
(i) For \( \bar{c}, A \subseteq \Gamma(\mathbb{C}) \subseteq \mathbb{C}^\mathbb{eq} \)
\[
\text{Cb}(\text{tp}(\bar{c}/A)) = \text{Cb}_T(\text{tp}_T(\bar{c}/A)).
\]
(ii) If \( T \) is CM-trivial, then \( S \) is CM-trivial.

Proof. (i) If we consider \( \text{tp}_T(\bar{c}/A) \) as a type with respect to \( S \), then \( \text{stp}_T(\bar{c}/A) \) is well defined over \( \text{acl}_S(A) \) and of course over \( \text{Cb}_T(\text{tp}_T(\bar{c}/A)) \). Now we consider the situation in \( T \). Since \( \Gamma \) is an interpretation without new information the defining scheme for \( \text{stp}_T(\bar{c}/A) \) induces a defining scheme for \( \text{stp}(\bar{c}/A) \). Hence \( \text{Cb}(\text{tp}(\bar{c}/A)) = \text{Cb}_T(\text{tp}_T(\bar{c}/A)) \).

(ii) Let \( M \preceq N \preceq \Gamma(\mathbb{C}) \) be models of \( S \) and \( \bar{c} \) be in \( \Gamma(\mathbb{C}) \) such that \( \text{acl}_S(\bar{c}M) \cap N = M \). We consider \( M \) and \( N \) as subsets of \( \mathbb{C}^\mathbb{eq} \) closed under \( \text{acl}_S \). Inside \( \Gamma(\mathbb{C}) \) \( \text{acl}_S \) and \( \text{acl}_T \) are the same. Note that \( \text{acl}_T(\bar{c}M) \cap \Gamma(\mathbb{C}) = \text{acl}_S(\bar{c}M) \). Hence,
\[
\text{acl}_S(\bar{c}M) \cap N = \text{acl}_T(\bar{c}M) \cap \Gamma(\mathbb{C}) \cap N = \text{acl}_T(\bar{c}M) \cap N.
\]
(2.3) To use the CM-triviality of \( T \) we need
\[
\text{acl}_T(\bar{c}M) \cap \text{acl}_T(N) = \text{acl}_T(M).
\]
(2.4) Note that there is some \( G \preceq \mathbb{C} \) such that \( \Gamma(G) = N \). To show this we use the fact from [3] that there is some \( T \)-model \( G' \) with \( \Gamma(G') \simeq N \). W.l.o.g. we can assume \( G' \preceq \mathbb{C} \). Since \( \Gamma \) is an interpretation without new information there is an \( L_T \) automorphism \( f \) of \( \mathbb{C} \) with \( f(\Gamma(G')) = N \). Then \( f(G') \) is the desired \( G \).

Let \( e \) be an element of \( \text{acl}_T(\bar{c}M) \cap \text{acl}_T(N) \). We choose a \( L_{T^{\mathbb{eq}}} \)-formula \( \theta(x, \bar{b}) \) with parameters \( \bar{b} \) from \( \Gamma(\mathbb{C}) \), such that
\[
\mathbb{C} \models \theta(e, \bar{b})
\]
and the number \( |\theta(\mathbb{C}, \bar{b})| \) is finite and minimal, where \( \theta(\mathbb{C}, \bar{b}) \) is used to denote the set of elements that satisfy \( \theta(x, \bar{b}) \). Since \( e \in G \) and \( G \preceq \mathbb{C} \) we can find \( \bar{b} \) in \( N \). Let \( \bar{b}_1 \) be any tuple in \( \Gamma(\mathbb{C}) \) such that \( \mathbb{C} \models \exists x(\theta(x, \bar{b}) \land \theta(x, \bar{b}_1)) \) and \( |\theta(\mathbb{C}, \bar{b}_1)| = |\theta(\mathbb{C}, \bar{b})| \).
By assumption \( \emptyset(C, \bar{b}_1) = \emptyset(C, \bar{b}) \). Let \( \emptyset(\bar{y}) \) be a formula that says that \( \bar{y} \) is in \( \Gamma(C) \), \( |\emptyset(C, \bar{y})| = |\emptyset(C, \bar{b})| \) and for every \( \bar{y}_1 \) with \( |\emptyset(C, \bar{y}_1)| = |\emptyset(C, \bar{b})| \) we have
\[
\emptyset(C, y_1) = \emptyset(C, y) \quad \text{or} \quad \emptyset(C, y_1) \cap \emptyset(C, y) = \emptyset.
\]
For \( \bar{b}_1 \) and \( \bar{b}_2 \) in \( \Gamma(C) \) with \( |\emptyset(\bar{b}_1) \wedge \emptyset(\bar{b}_2)| \) we define
\[
\bar{b}_1 \simeq \bar{b}_2 \quad \text{iff} \quad \exists x(\emptyset(x, \bar{b}_1) \wedge \emptyset(x, \bar{b}_2)).
\]
\( \simeq \) is an equivalence relation. Since \( \Gamma \) is an interpretation without new information there is a \( L_S \)-formula \( \chi(\bar{y}_1, \bar{y}_2) \) such that \( \Gamma(\chi)(\bar{y}_1, \bar{y}_2) \) defines \( \simeq \). Let \( \hat{\bar{b}} \) be the \( \simeq \)-class of \( \bar{b} \). \( \bar{b} \) is an element of \( N \). But it is also an element of \( \acl_T(e) \). By (2.3)
\[
\hat{\bar{b}} \in \acl_S(\bar{c}M) \cap N = M.
\]
W.l.o.g. we can choose \( \bar{b} \in M \). Hence \( e \in \acl_T(M) \), as desired in (2.4). The \( CM \)-triviality of \( T \) implies
\[
Cb(tp_T(\bar{c}/M)) \subseteq \acl_T(Cb(tp_T(\bar{c}/N))).
\]
By (i) we have
\[
Cb_T(tp_T(\bar{c}/M)) \subseteq \acl_T(Cb_T(tp_T(\bar{c}/N))).
\]
For \( S \) this means
\[
Cb(tp_S(\bar{c}/M)) \subseteq \acl_S(Cb(tp_S(\bar{c}/N))).
\]
The \( CM \)-triviality of \( S \) is proved. \( \Box \)

### 3. Alternating bilinear maps

Often it is helpful for our purposes to consider alternating bilinear maps besides nilpotent groups of class 2 and exponent \( p > 2 \). We follow similar ideas as in [5] but we do not assume that the centre of such a group is the commutator subgroup.

We fix a prime \( p \) greater than 2. Let \( G_{2,p} \) be the category of all nilpotent groups \( G \) of class 2 and exponent \( p > 2 \). We write \( G_{2,p}^P \) if we have an additional unary predicate \( P(x) \) such that \( P(G) \) is a subgroup between the commutator subgroup \( G' \) and the centre \( Z(G) \). The morphisms of \( G_{2,p}^P \) are the monomorphisms with respect to the signature \( "1, \cdot, P(x)" \). Later we fix a theory \( U \) in \( G_{2,p} \) and consider the subcategory of \( G_{2,p}^P \) of all substructures of models of \( U \), where \( P \) is considered as a predicate for the centre of the model in which the substructure is considered. So the typical situation is the following: \( U \) is complete, \( C \) is a monster model of \( U \), \( P(C) = Z(C) \) and we consider all small substructures of \( C \) with respect to the signature \( "1, \cdot, P(x)" \).

First some notation. We often write \( AB \) instead of \( A \cup B \). In groups \( G \) we use \( [a, b] = a^{-1}b^{-1}ab \). \( \langle X \rangle \) denotes the subgroup generated by \( X \subseteq G \). Then \( G' = \langle \{ [a, b] : a, \)
$b \in G_1$) is the commutator subgroup. $Z(G)$ denotes the centre of $G$. Since in this paper all considered groups $G$ are in $G_{2,p}$ we work with abelian subgroups (e.g. $Z(C)$) or abelian factor groups (e.g. $G/Z(G)$) as with vector spaces over the field $\mathbb{F}_p$ with $p$ elements. We speak about linear independence of elements of such abelian groups.

Let $\mathbb{B}_p$ be the category of all alternating bilinear maps $\langle V, W, \beta \rangle$ where $V$ and $W$ are vector spaces over the field $\mathbb{F}_p$ with $p$ elements and $\beta$ is an alternating bilinear map of $V \times V$ into $W$. Similarly, as in groups we call elements $\beta(x, y)$ in $W$ commutators. The morphisms of $\mathbb{B}_p$ are the embeddings of this class of structures. Hence, a morphism of $\langle V_1, W_1, \beta_1 \rangle$ into $\langle V_2, W_2, \beta_2 \rangle$ is a pair $(f, g)$ of vector space embeddings $f : V_1 \to V_2$ and $g : W_1 \to W_2$ such that the following diagram is commutative:

$$
\begin{array}{ccc}
V_1 \times V_1 & \xrightarrow{\beta_1} & W_1 \\
| & f \times f | & | \\
V_2 \times V_2 & \xrightarrow{\beta_2} & W_2
\end{array}
$$

Note that for every $\mathbb{F}_p$-vector space $V$ there is a free alternating bilinear map $\langle V, A^2V, \wedge \rangle$. It is called the exterior square of $V$ and it is defined by the following property:

If $\langle V, W, \beta \rangle$ is any alternating bilinear map over $V$, then there is a vector space homomorphism $f_{\beta}$ such that:

$$
\begin{array}{ccc}
V \times V & \xrightarrow{\wedge} & A^2V \\
| & \beta | & | \\
W & \xrightarrow{f_{\beta}} &
\end{array}
$$

In a canonical way we can define a functor $\mathcal{F}$ of $G_{2,p}^p$ into $\mathbb{B}_p$: For $G \in G_{2,p}^p$ let $\mathcal{F}(G)$ be $\langle G/P(G), P(G), [x, y] \rangle \in \mathbb{B}_p$. Here $G/P(G)$ and $P(G)$ are considered as vector spaces over $\mathbb{F}_p$. Note that $[x_0, y_0] = [x_1, y_1]$ if $x_0^{-1} \in P(G) \subseteq Z(G)$ and $y_0^{-1} \in P(G)$. For a $G_{2,p}^p$-embedding $f$ of $G$ in $H$ let $\mathcal{F}(f)$ be the pair $(\tilde{f}, f \downarrow P(G))$, where $\tilde{f}$ is the embedding of $G/P(G)$ into $H/P(H)$ induced by $f$ and $f \downarrow P(G)$ is the restriction of $f$ to $P(G)$.

Since $f$ is a monomorphism with respect to the signature “1, $\cdot, P(x)$”, we have

$$
\begin{array}{ccc}
G/P(G) \times G/P(G) & \xrightarrow{[\cdot]} & P(G) \\
| & f \times f | & | \\
H/P(H) \times H/P(H) & \xrightarrow{[\cdot]} & P(H)
\end{array}
$$

Note that $\mathcal{F}$ is a functor. Often we write $\mathcal{F}(G) = \langle V_G, W_G, \beta_G \rangle$.

**Lemma 3.1.** Let $G$ and $H$ be groups in $G_{2,p}^p$ and let $(g, h)$ be an embedding of $\mathcal{F}(G)$ into $\mathcal{F}(H)$. Let $\{c_x : x < v\}$ be a subset of $G$ such that $\{c_x/P(G) : x < v\}$ is
Proof. To apply Lemma 3.1 we choose $a_\alpha : \alpha < \nu$ to be a subset of $H$ with $g(c_\alpha/P(G)) = a_\alpha/P(H)$. Then $f(\prod_x c_x^2 \cdot d) = \prod_x a_\alpha^2 h(d)$ where $d \in P(G)$ and $r_\alpha = 0$ for all but finitely many $\alpha$, defines an $G^2_{2,p}$-embedding of $G$ into $H$ with $F(f) = (g,h)$.

Every element of $G$ can be written as $\prod_x c_x^2 \cdot d$ where $d \in P(G)$ and $r_\alpha = 0$ for all but finitely many $\alpha$. We use the uniqueness of such representations $\prod_x c_x^2 \cdot d$ and $\prod_x a_\alpha^2 h(d)$ of elements of $G$ and $H$, respectively. Since $h$ is an embedding of $P(G)$ into $P(H)$ it shows that $f$ is a well-defined injection.

To produce the standard representation of a product on both sides the “same” commutators are used:

$$c_\alpha c_\beta = c_\beta c_\alpha [c_\gamma, c_\zeta]$$

and

$$a_\alpha a_\beta = a_\beta a_\alpha [a_\gamma, a_\zeta] = a_\beta a_\gamma [g(c_\gamma), g(c_\zeta)] = a_\beta a_\gamma h([c_\gamma, c_\zeta]).$$

This proves that $f$ is a homomorphism. □

Corollary 3.2. Assume $G_0 \subseteq G$ and $H_0 \subseteq H$ are in $G^p_{2,p}$, and there are a $G^p_{2,p}$-isomorphism $f_0$ of $G_0$ onto $H_0$ and a $B_p$-isomorphism $(g,h)$ of $F(G)$ onto $F(H)$ that extends $F(f_0)$. Then there is a $G^p_{2,p}$-isomorphism $f$ of $G$ onto $H$ that extends $f_0$ and fulfills $F(f) = (g,h)$.

Proof. To apply Lemma 3.1 choose $\{c_\alpha : \alpha < \nu\}$ in $G$ such that $\{c_\alpha/P(G) : \alpha < \mu\}$ is a basis for $V_{G_0}$ for some $\mu \leq \nu$. Furthermore choose $a_\alpha = f_0(c_\alpha)$ for $\alpha < \mu$. Then Lemma 3.1 gives the desired isomorphism. □

We can use a similar proof to obtain the following consequence of Lemma 3.1 that will be used in the proof of the main theorem of the paper.

Corollary 3.3. Let $G$ be a substructure of $\mathbb{C}$ in $G^p_{2,p}$, where $P(\mathbb{C}) = Z(\mathbb{C})$. Let $A$ be a subset of $\mathbb{C}$ linearly independent modulo $(Z(\mathbb{C}) \cup G)$. Let $f_0$ be an automorphism of $G$ and let $(g,h)$ be an automorphism of $F(\mathbb{C})$ that extends $F(f_0)$ with $g(a/Z(\mathbb{C})) = a/Z(\mathbb{C})$ for $a \in A$. Then there is an automorphism $f$ of $\mathbb{C}$ that extends $f_0$ with $f(a) = a$ for $a \in A$ and $F(f) = (g,h)$.

Proof. To apply Lemma 3.1 we choose $\{c_\alpha : \alpha < \nu\}$ such that $\{c_\alpha/Z(\mathbb{C}) : \alpha < \nu\}$ is a basis of $V_{G_0}$, $\{c_\alpha/Z(\mathbb{C}) : \alpha < \mu\}$ is a basis of $V_G$, and $A \subseteq \{c_\alpha : \mu \leq \alpha < \nu\}$. Then we choose $a_\alpha = f_0(c_\alpha)$ for $\alpha < \mu$ and $a_\alpha = c_\alpha$ for $c_\alpha \in A$. Lemma 3.1 provides the desired result. □

Lemma 3.4. Let $G$ be in $G^p_{2,p}$. Assume that there is a $B_p$-embedding $(g,h)$ of $F(G)$ into an alternating bilinear map $(V,W,\beta)$. Then there are a group $H$ with a predicate $P(H)$ in $G^p_{2,p}$, a $B_p$-isomorphism $i = (i_0,i_1) : F(H) \cong (V,W,\beta)$, and a $G^p_{2,p}$-embedding $f$ of $G$ into $H$ such that $i \circ F(f) = (g,h)$. 

If \((g,h)\) is surjective, then \(f\) is surjective, and \(H\) is uniquely determined up to isomorphisms respecting the embedding of \(G\).

**Proof.** First we define \(H\). There are ordinals \(v \leq \mu\) and a basis \(\{a_x : x < \mu\}\) of \(V\) such that \(\{a_x : x < v\}\) is a basis of \(g(G/P(G))\). The elements of \(H\) are pairs \((\sum z r_z a_z, b)\) where \(b \in W\) and \(r_z = 0\) for all but finitely many \(z\). The group multiplication is defined by

\[
\left(\sum _z r_z a_z, b_1\right) \left(\sum _z s_z a_z, b_2\right) = \left(\sum _z (r_z + s_z) a_z, b_1 + b_2 + \sum _{\gamma < z} r_z s_{\gamma} \beta(a_{\gamma}, a_{\gamma})\right).
\]

Furthermore, we define \(P(H) = \{(0,b) : b \in W\}\). It is easy to show that \(H\) is a group in \(G_{2,p}\) and that \(H' \subseteq P(H) \subseteq Z(H)\). It follows that \((\sum z r_z a_z, b) = \prod z (a_z,0)^{\gamma}(0,b)\) and the image of \(\{(a_z,0) : x < \mu\}\) is a basis of \(H/P(H)\). We have in \(H\)

\[
\left[\left(\sum _z r_z a_z, b_1\right), \left(\sum _z s_z a_z, b_2\right)\right] = \left(0, \sum _{\gamma < z} (r_z s_{\gamma} - s_z r_{\gamma}) \beta(a_{\gamma}, a_{\gamma})\right).
\]

Hence \(i = (i_0, i_1)\) with

\[
i_0 \left(\left(\sum _z r_z a_z, b\right)/P(H)\right) = \sum z r_z a_z
\]

and

\[
i_1((0,b)) = b
\]

is the desired isomorphism of \(\mathcal{F}(H)\) onto \(\langle V, W, \beta \rangle\).

To define \(f\) we choose \(\{c_x : x < v\}\) in \(G\) with \(g(c_x/P(G)) = a_x\). Then \(\{c_x/P(G) : x < v\}\) is a basis for \(G/P(G)\). Hence, every element of \(G\) has a unique representation of the form \(\prod z c_z^x d\) where \(d \in P(G)\). Then we define

\[
f \left(\prod z c_z^x d\right) = \left(\sum z r_z a_z, h(d)\right).
\]

By the uniqueness of the representation of the elements of both sides and since \(h\) is a vector space embedding, \(f\) is a well-defined injection into \(H\). Because of the definition
of the multiplication in $H$ by (3.1) it is an embedding. $i \circ \mathcal{F}(f) = (g, h)$ follows from the definitions. They also imply the surjectivity of $f$, if $(g, h)$ is surjective.

Assume that $H^*$, $f^*$, and $i^*$ also satisfy the conditions for $H$, $f$ and $i$. Then $(i^*)^{-1}i$ defines an $\mathbb{B}_p$-isomorphism of $\mathcal{F}(H)$ onto $\mathcal{F}(H^*)$. By Lemma 3.1 we can lift $(i^*)^{-1}i$ to an $\mathbb{G}_{2,p}$-isomorphism $k$ of $H$ onto $H^*$. We can choose $k$ in such a way that $kf = f^*$:

$$
\begin{array}{ccc}
H & \xrightarrow{k} & H^* \\
\downarrow{f} & & \downarrow{f^*} \\
G & & \\
\end{array}
$$

This implies the uniqueness of $H$ over $G$. □

Lemma 3.4 shows that for every bilinear map $M$ in $\mathbb{B}_p$ there is a group $G$ in $\mathbb{G}_{2,p}$ with $\mathcal{F}(G) = M$. Furthermore, it follows that all possible $G$ are isomorphic.

**Corollary 3.5.** The functor $\mathcal{F}$ gives a bijection between the isomorphism types of structures in $\mathbb{G}_{2,p}$ and $\mathbb{B}_p$.

The functor $\mathcal{F}$ provides an interpretation of $\mathbb{B}_p$ in $\mathbb{G}_{2,p}$. If $G \equiv H$ are in $\mathbb{G}_{2,p}$, then $\mathcal{F}(G) \equiv \mathcal{F}(H)$.

Hence, we can define for every complete theory $T$ of structures in $\mathbb{G}_{2,p}$ a complete theory $\mathcal{F}(T)$ of the bilinear maps that lives in the models of $T$. Again Lemma 3.1 implies:

**Corollary 3.6.** The interpretations of complete theories given by $\mathcal{F}$ are interpretations without new information.

Especially, we can apply this to $\mathbb{G}_{2,p}$ if we define $P(G) = Z(G)$. At the end of the section we show that there are stronger model-theoretic connections between $\mathbb{G}_{2,p}$ and $\mathbb{B}_p$.

**Corollary 3.7.** Let $G$ and $H$ be structures in $\mathbb{G}_{2,p}$. Let $A \subseteq G$ and $f_0(A) = B \subseteq H$ be substructures, where $f_0$ is an isomorphism. Then $\text{tp}_G(A) = \text{tp}_H(B)$ if and only if $\text{tp}_{\mathcal{F}(G)}(\mathcal{F}(A)) = \text{tp}_{\mathcal{F}(H)}(\mathcal{F}(B))$.

**Proof.** We add constant symbols $\{c_a : a \in A\}$ to the language of $\mathbb{G}_{2,p}$ and constant symbols $\{c_e : e \in \mathcal{F}(A)\}$ to the language of $\mathbb{B}_p$.

Let $\kappa$ be a cardinal such that for a theory of size $\aleph_0 + |A|$ there exists a special model. Since $\mathcal{F}$ is an interpretation $\text{tp}_G(A) = \text{tp}_H(B)$ implies $\text{tp}_{\mathcal{F}(G)}(\mathcal{F}(A)) = \text{tp}_{\mathcal{F}(H)}(\mathcal{F}(B))$.

Let $\langle G^*, A \rangle$ be an elementary extension of $\langle G, A \rangle$ that is a special model of cardinality $\kappa$. Let $\langle H^*, B \rangle$ be an elementary extension of $\langle H, B \rangle$ that is a special model of
cardinality \( \kappa \). Then \( \langle \mathcal{F}(G^*), \mathcal{F}(A) \rangle \) is an elementary extension of \( \langle \mathcal{F}(G), \mathcal{F}(A) \rangle \) and it is a special model of cardinality \( \kappa \). Analogously, \( \langle \mathcal{F}(H^*), \mathcal{F}(B) \rangle \) is an elementary extension of \( \langle \mathcal{F}(H), \mathcal{F}(B) \rangle \) and it is a special model of cardinality \( \kappa \). They are all models of the same theory, since \( \text{tp}_{\mathcal{F}(G)}(\mathcal{F}(A)) = \text{tp}_{\mathcal{F}(H)}(\mathcal{F}(B)) \). Hence, the special models \( \langle \mathcal{F}(G^*), \mathcal{F}(A) \rangle \) and \( \langle \mathcal{F}(H^*), \mathcal{F}(B) \rangle \) are isomorphic and the isomorphism extends \( \mathcal{F}(f_0) \). We can lift the automorphism to a group automorphism \( f \) of \( G^* \) and \( H^* \) that extends \( f_0 \) by Corollary 3.2. Hence \( \text{tp}_G(A) = \text{tp}_H(B) \), as desired. \( \square \)

**Corollary 3.8.** For \( G \) and \( H \) in \( \mathbb{G}_{2,p} \), we have

\[ G \equiv H \text{ if and only if } \mathcal{F}(G) \equiv \mathcal{F}(H). \]

If we consider \( \mathbb{G}_{2,p} \) and interpret \( P(x) \) as \( Z(x) \), then \( \mathcal{F}(G) \) for \( G \in \mathbb{G}_{2,p} \) is well defined. If \( T \) is a complete theory of groups in \( \mathbb{G}_{2,p} \), then let \( \mathcal{F}(T) = \text{Th}(\mathcal{F}(G) : G \in \text{Mod}(T)) \). It is clear that stability or simplicity of \( T \) implies stability or simplicity of \( \mathcal{F}(T) \), respectively. We prove the converse.

**Corollary 3.9.** Let \( T \) be a complete theory of groups in \( \mathbb{G}_{2,p} \). Then \( T \) is \( \lambda \)-stable if and only if \( \mathcal{F}(T) \) is \( \lambda \)-stable.

**Proof.** To show the non-trivial direction assume that there are \( \lambda^+ \) types over a model \( G \) of \( T \) with \( |G| = \lambda \). W.l.o.g. we can assume that these types have the form \( \text{tp}(a/G) \) with \( a \in \mathbb{C} \), where \( \mathbb{C} \) is a large elementary extension of \( G \). If we have \( \lambda^+ \)-types \( \text{tp}(a/G) \) with \( a \in \langle G \cup Z(\mathbb{C}) \rangle \), then their “images” \( \text{tp}(aZ(\mathbb{C})/\mathcal{F}(G)) \) are pairwise different by Corollary 3.7. To prove this we assume that \( \text{tp}(a, G) \) and \( \text{tp}(c, G) \) are two different types of this kind. If \( \text{tp}(aZ(\mathbb{C})/\mathcal{F}(G)) \) and \( \text{tp}(cZ(\mathbb{C})/\mathcal{F}(G)) \) are equal, then we consider \( \langle G \cup a \rangle \) and \( \langle G \cup c \rangle \). By Lemma 3.1 we can extend the identity on \( G \) to an isomorphism \( f_0 \) of these subgroups with \( f_0(a) = c \). Now, we can apply Corollary 3.7 and obtain a contradiction. Otherwise we have \( \lambda^+ \) types \( \text{tp}(a/G) \) with \( a \in \langle G \cup Z(\mathbb{C}) \rangle \). Then we have \( \lambda^+ \) types \( \text{tp}(a/G) \) with \( a \in Z(\mathbb{C}) \). Similarly as above by Corollary 3.7 their “images” are different. \( \square \)

Note \( \lambda \) is a strong limit cardinal, if \( \lambda \) is a limit cardinal and \( 2^\kappa < \lambda \) for all \( \kappa < \lambda \). For strong limit cardinals \( \lambda \) we have \( \lambda^{\text{cf}(\lambda)} = 2^\lambda \) and \( \lambda^{<< \text{cf}(\lambda)} = \lambda \). For every regular \( \kappa \) there are arbitrary large strong limit cardinals \( \lambda \) with \( \text{cf}(\lambda) = \kappa \). The following characterization of simplicity is essentially due to Shelah [14]:

**Theorem 3.10.** Let \( \kappa \) be \( |T|^+ \) and \( \lambda > 2^\kappa \) be a strong limit cardinal with \( \text{cf}(\lambda) = \kappa \). \( T \) is not simple if and only if there are \( 2^\lambda \) pairwise contradictory 1-types of power \( \kappa \) over a set \( A \) of cardinality \( \lambda \).

**Corollary 3.11.** Let \( T \) be a complete theory of infinite groups in \( \mathbb{G}_{2,p} \). Then \( T \) is simple if and only if \( \mathcal{F}(T) \) is simple.
Proof. The tree property for $F(T)$ implies the tree property for $T$. Hence simplicity of $T$ implies simplicity of $F(T)$. Now assume that $T$ is not simple. We have the situation described in Theorem 3.10. W.l.o.g. we assume that $A$ is a model and the types $p_i$ are over elementary submodels $A_i$ of $A$ ($i < 2^\lambda, |A_i| = \kappa$). Then we can assume that either all $p_i$ are realized outside of $\langle A \cup \mathbb{C} \rangle$ or all $p_i$ are realized inside. In the second case we can assume that there is some $a \in A$ such that the realizations have the form $ab_i$ where $b_i \in \mathbb{C}$, since $\text{cf}(2^\lambda) > \lambda$. Hence w.l.o.g. $p_i = \text{tp}(b_i/A_i)$ where either $b_i \notin \langle A \cup \mathbb{C} \rangle$ for all $i$, or $b_i \in \mathbb{C}$ for all $i$. In both cases Corollary 3.7 implies that the $F(T)$-images of the types are also contradictory. Hence Theorem 3.10 implies that $F(T)$ is not simple. □

4. Mekler’s construction

Let $p$ be a prime greater than 2. Let $S^*$ be any theory of finite similarity type. Mekler [11] has given a uniform construction of groups $G(M)$ for every model $M$ of $S^*$, a theory $T^*$ of all groups $G(M)$ for $M$ in $S^*$, and an interpretation $I$ of $S^*$ in $T^*$ such that

(i) $T^*$ is a theory of nilpotent groups of class 2 and of exponent $p$.
(ii) If $G \models T^*$, then there is an $M \models S^*$ such that $G(M) \equiv G$.
(iii) For $S^*$-models $M$ and $N$ we have $M \equiv N$ if and only if $G(M) \equiv G(N)$.
(iv) $I(G(M)) \equiv M$.
(v) $\text{Th}(M)$ is $\lambda$-stable iff $\text{Th}(G(M))$ is $\lambda$-stable.

The aim of this paper is to prove that

(vi) $\text{Th}(M)$ is stable and $CM$-trivial if and only if $\text{Th}(G(M))$ is stable and $CM$-trivial.

Before we show:

(vii) $\text{Th}(M)$ is simple if and only if $\text{Th}(G(M))$ is simple.

Mekler has done his construction for the theory $S^*$ of nice graphs. This restriction is possible since this theory is universal for biinterpretation (in the notation of Hodges [8]).

In this section we want to describe Mekler’s construction. We follow the detailed presentation in [8]. The notation is slightly changed. We use $T^*$ instead of $T_{\text{ng}}$. For Theorem 4.1 until Corollary 4.7 you find detailed proofs in [8]. Furthermore, there is an explicit axiomatization of $T^*$.

A nice graph is a structure with only one binary symmetric and irreflexive relation $R(x, y)$ such that

(a) $\exists x_0 x_1 (x_0 \neq x_1)$.
(b) $\forall x_0 x_1 \exists y (x_0 \neq x_1 \rightarrow y \neq x_0 \land y \neq x_1 \land R(x_0, y) \land \neg R(x_1, y))$.
(c) $\forall x_0 x_1 x_2 (\bigwedge_{1 \leq j < 3} x_i \neq x_j \rightarrow \neg (\bigwedge_{1 \leq j < 3} R(x_i, x_j)))$.
(d) $\forall x_0 x_1 x_2 x_3 (\bigwedge_{1 \leq j < 4} x_i \neq x_j \rightarrow \neg (\bigwedge_{1 \leq j < 3} R(x_i, x_{i+1}) \land R(x_3, x_0)))$.

Conditions (c) and (d) say that there are no triangles and squares. Let $S^*$ be the theory of nice graphs. We use $M, N, \ldots$ to denote graphs and often models of $S^*$ and
Now let \( M \) be any graph. Then let \( F(M) \) be the free nilpotent group of class 2 and exponent \( p \) that is freely generated by the domain of \( M \). Let \( Z(M) \) be the centre of \( F(M) \). We can consider \( Z(M) \) and \( F(M)/Z(M) \) as vector spaces over the field \( \mathbb{F}_p \) with \( p \) elements. \( F(M)/Z(M) \) has a basis \( \{a/Z(M) : a \in M\} \). We assume that \( M \) is ordered by a relation \( < \) that is not in the language. Then \( \{[a,b] : a,b \in M, a < b \} \) is a basis of \( Z(M) \), where \([a,b] = a^{-1} b^{-1} aba \). Let \( H(M) \) be the subgroup \( \langle \{[a,b] : a,b \in M, M \models R(a,b) \} \rangle \). Then Mekler defined \( G(M) = F(M)/H(M) \).

If the meaning is clear, then we write \( Z \) instead of \( Z(G(M)) \) or \( Z(G) \). First, we study the elements of \( G(M) \). Note that \( \{[a,b] : a,b \in M, a < b \text{ and } M \models \neg R(a,b) \} \) is a basis of the centre of \( G(M) \). Furthermore \( (G(M))' = Z(G(M)) \). We define, for any elements \( g,h \) in any group \( G \)
\[
g \sim h, \quad \text{if } C(g) = C(h),
g \approx h, \quad \text{if there is some } r \ (0 < r < p) \text{ such that } g = h^r \cdot c \text{ where } c \in Z.
g \equiv_Z h, \quad \text{if } g/Z = h/Z.
\]
These are three \( \emptyset \)-definable equivalence relations and we have \( g \equiv_Z h \Rightarrow g \approx h \Rightarrow g \sim h \).
\( g/Z \) is the \( \equiv_Z \)-class of \( g \), \( g^\approx \) is the \( \approx \)-class of \( g \), and \( g^\sim \) is the \( \sim \)-class of \( g \). Each \( g^\sim \) is a union of \( \approx \)-classes. Hence we can define:

**Definition.** Let \( g \) be an element of any nilpotent group \( G \) of class 2 and exponent \( p \). Assume \( g \notin Z \).

- \( g \) is of type 0, if \( q \) is the number of different \( \approx \)-classes in \( g^\sim \).
- \( g \) is isolated if \( G \models [g,h] = 1 \) implies \( h \approx g \) or \( h \in Z \).
- \( g \) is of type \( q' \), if \( g \) is of type \( q \) and isolated.
- \( g \) is of type \( q^v \) if \( g \) is of type \( q \) and not isolated.

If \( q \) is finite, then all notions above are first order definable. Now we consider nice graphs \( M \). In the notation of Hodges [8], Mekler [11] proved:

**Theorem 4.1.** Let \( M \) be a nice graph.

(i) Every non-central element in \( G(M) \) is of type 1, \( p - 1 \), or \( p \).

(ii) An element of \( G(M) \) is \( \approx \)-equivalent to a (unique) vertex of \( M \) if and only if it is of type \( 1' \).

(iii) For every element \( g \) in \( G(M) \) of type \( p \), there is an element \( b \) of type \( 1' \) such that \( G(M) \models [g,b] = 1 \), and \( b^\approx \) is uniquely determined.

There is a formula \( \gamma(x) \) that says “\( x \) is of type \( 1' \)”. Hence we can recover \( M \) from \( G(M) \). There is an injective map of the elements of \( M \) onto the \( \approx \)-classes of the non-central elements of \( G(M) \) of type \( 1' \). We speak about vertexes \( g^\approx \) and \( h^\approx \) in \( G(M) \) (or better \( G(M)^{eq} \)). If \( X \) is a set of such elements of type \( 1' \) that are in different
≈-classes, then \( X \) is linearly independent modulo \( Z \). Two such vertexes \( g^≈ \) and \( h^≈ \) are joined in \( M \) if and only if \( g \) and \( h \) commute. Hence

**Corollary 4.2.** There is an interpretation \( \Gamma \) of the theory \( S^* \) of nice graphs in the theory of nilpotent groups of class 2 and exponent \( p \) such that \( \Gamma(G(M)) \cong M \).

Let \( T^* \) be \( \text{Th}(\{G(M) : M \models S^*\}) \). Hodges [8] has given 10 axiom schemes that axiomatize \( T^* \). Below we give some of the properties of the models of \( T^* \).

Let \( G \) be a model of \( T^* \).

- Every non-central element in \( G \) is of type \( 1, p - 1, \) or \( p \) (Theorem 4.1(i)).
- \( \Gamma(G) \) is a nice graph.
- Elements of type \( p \) or \( p - 1 \) in \( G \) are not isolated. Hence we distinguish only types \( \lambda', \lambda, p \) and \( p - 1 \).
- Elements of type \( p - 1 \) are the product of two \( \sim \)-inequivalent elements of type \( \lambda' \).
- If \( g \) is an element of type \( p \), then by Theorem 4.1 there is some \( b \) of type \( \lambda' \) such that \( g \) and \( b \) commute.

In the last paragraph \( b^\sim = b^≈ \) is uniquely determined this properties. We call the element \( b \) a handle for \( g \). We also speak about the handle \( b^\sim = b^≈ \) of \( g \).

Note that in this case \( C(g) = \{g^a b^b c : c \in Z, 0 \leq \alpha, \beta < p\} \). Then \( b^≈ \in \text{dcl}(g) \) and \( b/Z \in \text{acl}(g) \).

**Definition.** Let \( M \) be a nice graph. A graph \( M^+ \supseteq M \) is a cover of \( M \) if for every vertex \( g \) in \( M^+ \setminus M \) either there is a vertex \( b \) in \( M \) such that \( b \) is the unique vertex in \( M^+ \) which is joined to \( g \) and \( b \) is joined to infinitely many vertexes in \( M \), or \( g \) is not joined to any other vertex.

A cover \( M^+ \) of \( M \) is a \( \lambda \)-cover if for every vertex \( b \in M \) the number of vertexes \( g \) in \( M^+ \setminus M \) joined to \( b \) is \( \lambda \), if \( b \) is joined to infinitely many vertexes in \( M \), and zero otherwise, and the number of new vertexes in \( M^+ \setminus M \), which are not joined to any other vertex, is \( \lambda \) if \( M \) is infinite and 0 otherwise.

Note that a cover of a nice graph is not nice in general. Now we introduce the notion of a transversal following Hodges [8].

**Definition.** Let \( G \) be a model of \( T^* \).

- An \( \lambda' \)-transversal \( X_{\lambda'} \) of \( G \) is a set consisting of one representative of each \( \sim \)-class of elements of type \( \lambda' \).
- An element of \( G \) is proper if it is not a product of elements of type \( \lambda' \).
- A \( p \)-transversal \( X_p \) of \( G \) is a set of representatives of \( \sim \)-classes of proper elements of type \( p \), which is maximal with the property that if \( Y \) is a finite subset of \( X_p \) and all elements of \( Y \) have the same handle, then \( Y \) is independent modulo the subgroup generated by the elements of type \( \lambda' \) and \( Z(G) \).
- An \( \lambda' \)-transversal is a set of representatives of \( \sim \)-classes of proper elements of type \( \lambda' \), which is maximal independent modulo the subgroup generated by the elements of type \( \lambda' \) and \( p \) and \( Z(G) \).
A subset $X$ of $G$ is a transversal if $X = X_1 \cup X_p \cup X_V$ where $X_1$ is an $1^v$-transversal, $X_p$ is a $p$-transversal, and $X_V$ is an $1^v$-transversal.

$X_{p,e}$ is used to denote the subset of the elements with handle $e$ of $X_p$.

$\gamma(x)$ was the formula that says that $x$ of type $1^v$. Hence $\langle \gamma(G) \rangle$ is the subgroup generated by the elements of type $1^v$. Then $\langle \gamma(G) \rangle \supseteq Z(G)$. If $X_1$ is an $1^v$-transversal, then $\langle \gamma(G) \rangle = Z(G) = \bigoplus_{a \in X_V} \langle a \rangle Z(G) = Z(G)$.

**Lemma 4.3.** Assume $G \models T^*$ and $X = X_1 \cup X_p \cup X_V$ is a transversal of $G$. We can consider $X$ as a graph $M^+$ with $R(a,b)$ if and only if $G \models [a,b] = 1$. Then $M^+$ is a cover of $M^+ \upharpoonright X_V$ and $M^+ \upharpoonright X_V$ can be identified with $\Gamma(G)$.

Note that transversals exist in models of $T^*$. As shown by Mekler transversals can be used to describe the structure of models of $T^*$:

**Theorem 4.4.** Let $G$ be a model of $T^*$. Let $X$ be a transversal of $G$. Let us consider $X$ as a cover $M^+$ of $\Gamma(G)$ as in Lemma 4.3. Then there is an elementary abelian $p$-group $H$ such that $G = \langle X \rangle \oplus H$ and $\langle X \rangle \cong G(M^+)$. 

**Corollary 4.5.** (i) If in Theorem 4.4 $G$ is an infinite special model of $T^*$ (e.g. $G$ is saturated), then $M^+$ is a $|G|$-cover and $|H| = |G|$.

(ii) If $G_1$ and $G_2$ are special models of the same infinite cardinality, then every isomorphism from $\Gamma(G_1)$ to $\Gamma(G_2)$ lifts to an isomorphism from $G_1$ onto $G_2$.

**Corollary 4.6.** If $M$ and $N$ are elementarily equivalent nice graphs, then $G(M) \equiv G(N)$.

Let $G \models T^*$ and $X$ be a transversal of $G$. Assume $G = \langle X \rangle \oplus H$ as in Theorem 4.4. Let $< \subseteq$ be a well-ordering of $X$. We may assume $X_F < X_P < X_V$. Let $X = \{x_\alpha : \alpha < \kappa\}$ be an enumeration of $X$ with respect to $<$. If $g$ is an element of $G$, then we call

$$g = \prod_{x < \kappa} x^x \prod_{x < \beta < \kappa} (x, x_\beta)^{s_{x\beta}} \cdot h,$$

where $h \in H$ and $0 \leq r_x$, $s_{x\beta} < p$, a representation of $g$ with respect to $X$, $<$, and $H$.

**Corollary 4.7.** In the situation described above every element $g \in G$ has a unique representation with respect to $X$, $<$, and $H$.

For the rest of the paper we consider completions $S$ of $S^*$ with an infinite model. By Corollary 4.6 the theory $T_S = \text{Th}(\{G(M) : M \models S\})$ is complete. Using Theorem 4.4 Mekler showed:
Corollary 4.8. For all infinite cardinals \( \lambda \) we have that \( S \) is \( \lambda \)-stable if and only if \( T_S \) is \( \lambda \)-stable.

If necessary we consider transversals \( X \) in models \( G \) of \( T_S \) as covers over its \( 1' \)-part \( X_1' \). Similarly as above we have:

Lemma 4.9. Let \( G \models T_S \).
(i) Two transversals of \( G \) are isomorphic as graphs. The isomorphism respects the \( 1' \), \( p' \), and \( 1' \)-parts.
(ii) If there is a graph isomorphism of a transversal \( X \) of \( G \models T_S \) onto a transversal \( Y \) of \( G \), then it lifts to an automorphism of \( G \).

Corollary 4.10. \( \Gamma \) is an interpretation without new information.

Corollary 4.11. If \( T_S \) is stable and \( CM \)-trivial, then \( S \) is stable and \( CM \)-trivial.

Proof. Use Lemmas 2.4 and 4.10.

To prove the converse of Corollary 4.11 we need some further results about Mekler-groups.

Definition. A subset \( Y \) of a transversal \( X \) is called a part of \( X \) if for every element \( g \in X_p \cap Y \) there is a handle of \( g \) in \( Y \).

Note that a part of a transversal can be defined similarly as a transversal:

Let \( G \) be a model of \( T^* \). \( Y \) is a part of a transversal if \( Y = Y_1' \cup Y_p \cup Y_1 \) where \( Y_1 \) is a subset of elements of type \( 1' \) where two different elements are in different \( \sim \)-classes, \( Y_p \) is a set of elements of type \( p \) that is linearly independent modulo the subgroup generated by all elements of type \( 1' \) and \( Z(G) \), and for every \( y \in Y_p \) there is a handle in \( Y_1' \), and \( Y_1 \) is a set of elements of type \( 1' \) that is linearly independent modulo the subgroup generated by all elements of type \( 1' \) and \( p \) and \( Z(G) \).

Lemma 4.12. In a model \( G \) of \( T_S \) let \( h \) be a bijection between two parts of transversals \( Y \) and \( h(Y) \) such that \( h \) respects the \( 1' \), \( p' \), and \( 1' \)-parts, the handles, and \( tp_\Gamma(\bar{Y}_1') = tp_\Gamma(h(\bar{Y}_1')) \). Then \( tp(Y) = tp(h(Y)) \).

Proof. Let \( G^* \) be an elementary extension of \( G \) that is a special model of \( Th(G) \). Also in \( G^* \) the sets \( Y \) and \( h(Y) \) are parts of a transversal and \( tp_\Gamma(\bar{Y}_1') = tp_\Gamma(h(\bar{Y}_1')) \).

Let \( X_1' \supseteq Y_1' \) be an \( 1' \)-transversal of \( G^* \). Since \( tp_\Gamma(\bar{Y}_1') = tp_\Gamma(h(\bar{Y}_1')) \) we can extend \( h \) to a graph-automorphism \( g \) of \( \Gamma(G^*) \). As in Corollary 4.5(ii) we can lift \( g \) to a group-automorphism \( f \) of \( G^* \). Using Theorem 4.4 it is easy to do this in such a way that \( f \) extends \( h \).

The next lemma is shown in [3].
Lemma 4.13. Let \( G \subseteq H \) be models of \( T_S \). Then \( G \preceq H \) if and only if \( \Gamma(G) \preceq \Gamma(H) \) and every transversal of \( G \) can be extended to a transversal of \( H \).

For the rest of the paper we use \( C \) to denote a monster model of \( T_S \).

Definition. (i) Assume \( G \preceq C \models T_S \). A transversal \( X \) of \( C \) respects \( G \) if \( X = X^0 \cup X^1 \), where \( X^0 = X \cap G \) and \( X^0 \) is a transversal of \( G \).

(ii) \( X \) is a part of a transversal of \( C \) that respects \( G \), if \( X \) can be extended to a transversal of \( C \) that respects \( G \).

Lemma 4.14. Let \( T_S \) be stable. Assume \( G \preceq C \models T_S \) and \( Y \) is a finite part of a transversal of \( C \) that respects \( G \). Then

\[
\text{Cb}(\text{tp}(Y/G)) = (Y \cap G) \cup \text{Cb}_f(\text{tp}_f(Y^\infty_1/G)/\Gamma(G)).
\]

Proof. W.l.o.g. we can assume that \( G \) is saturated. To show the non-trivial direction consider some automorphism \( f \) of \( C \) that fixes \( G \) setwise and \( (Y \cap G) \cup \text{Cb}_f(\text{tp}_f(Y^\infty_1/G)/\Gamma(G)) \) pointwise. Then \( \text{tp}_f(Y^\infty_1/G) = \text{tp}_f(f(Y^\infty_1)/\Gamma(G)) \). Let \( X \) be a part of a transversal of \( C \) such that \( Y \subseteq X \), \( X \cap G \) is a transversal of \( G \), and \( X = (X \cap G) \cup Y \). Let \( h \) be a map of \( X \) into \( C \) such that \( h \upharpoonright (X \cap G) = \text{id} \) and \( h(a) = f(a) \) for \( a \in Y \). By Lemma 4.12 \( \text{tp}(X) = \text{tp}(h(X)) \). Hence \( \text{tp}(Y/G) = \text{tp}(h(Y)/G) = \text{tp}(f(Y)/G) \), as desired.

Since we consider only nice graphs we have the following fact:

Lemma 4.15. If \( G \preceq C \models T_S \), then the following is impossible: \( a, b, \) and \( c \) are elements of type \( 1^v \) that are pairwise not \( \approx \)-equivalent, \( a, b \in G, \ c \notin G, \) and \( [a, c] = [b, c] = 0 \).

Proof. Assume we have the above situation. Since \( G \preceq C \) there is some \( c' \) in \( G \) such that \( c' \) is not \( \approx \)-equivalent to \( a \) and \( b \) and \( [c', a] = 0, [c', b] = 0 \). We have a square in \( \Gamma(C) \), which is forbidden.

Hence up to \( \approx \)-equivalence there is at most one \( a \in G \) with \( \beta(c, a) = 0 \) for \( c \) of type \( 1^v \) not in \( G \). We call \( a^\approx \) the root of \( c \) in \( G \).

Corollary 4.16. If \( G \preceq C \models T_S \) and \( c \) is an element of type \( 1^v \) not in \( G \), then \( c \) has at most one root in \( G \).

5. Simplicity

As before \( S \) is a complete theory of nice graphs, and \( T_S \) is the corresponding Mekler-theory. Of course, simplicity of \( T_S \) implies simplicity of \( S \). We show the converse. This is joint work with my student Alexander Pentzel.
Theorem 5.1. $S$ is simple if and only if $T_S$ is simple.

Proof. To show the non-trivial direction we assume that $T_S$ is not simple. We use Theorem 3.10. Let $\lambda$ and $\kappa$ be cardinals chosen as in Theorem 3.10. Let $I(\Bbb C)$ be a monster-model of $S$ living in $\Bbb C \models T_S$. We have to find a subset $A \subseteq I(\Bbb C)$ of cardinality $\lambda$ and $2^\lambda$ pairwise contradictory $1$-types of power $\kappa$ over $A$. By assumption we have this situation in $\Bbb C \models T_S$. There are a model $G \subseteq \Bbb C$ with $|G| = \lambda$ and $2^\lambda$ pairwise contradictory types $p_i \in S_1(\langle D^i \rangle)$ with $|D^i| \leq \kappa$, $D^i \subseteq G$, and $i < 2^\lambda$. Let $X$ be a transversal of $G$. By Lemma 4.13 we find a transversal of the form $XY$ of $\Bbb C$. By Theorem 4.4 there are elementary abelian p-groups $\langle I \rangle \subseteq Z(G)$ and $\langle J \rangle \subseteq Z(\Bbb C)$ such that $G = \langle X \rangle \oplus \langle I \rangle$ and $\Bbb C = \langle XY \rangle \oplus (\langle I \rangle \oplus \langle J \rangle)$, where $I$ and $J$ are bases of the abelian p-groups. W.l.o.g. we can assume that $D^i \subseteq X \cup I$ and $\langle D^i \rangle \leq G$ for all $i < 2^\lambda$. Then we have that the $D^i$ are closed under handles: If $d$ is an element of type $p$ in $D^i$, then the handle of $d$ is in $D^i$. Now we consider realizations of the $p_i$. We can assume that

$$p_i = \text{tp}(t_i(\langle X^i Y^i I^i J^i \rangle)/\langle D^i \rangle),$$

where $X^i \subseteq X$, $Y^i \subseteq Y$, $I^i \subseteq I$, and $J^i \subseteq J$ are finite, $X^i Y^i$ is closed under handles, and the $t_i$ are terms. W.l.o.g. $X^i I^i \subseteq D^i$.

Let $Y^i_i$ be the $i$-part of $Y^i$, $Y^i_{p,e}$ be the $p$-part of $Y^i$ with handle $e \in X^i Y^i_i$, and $Y^i_i$ be the $i$-part of $Y^i$.

Since $\text{cf}(2^\lambda) > \lambda$ we can assume w.l.o.g. that there are a term $t$ and finite subsets $X^* \subseteq X$ and $I^* \subseteq I$ such that $t_i = t$, $X_i = X^*$, and $I^i = I^*$ for all $i < 2^\lambda$. By the same argument we can, furthermore, assume that all $J^i$ have the same length and there are bijections $h_{ij}$ of $Y^i$ onto $Y^j$ such that

$$h_{ij}(Y^i_{p,e}) = Y^j_{p,e}, \quad h_{ij}(Y^i_{p,h,e}) = Y^j_{p,h,e}, \quad \text{and} \quad h_{ij}(Y^i) = Y^j.$$

Finally, we can w.l.o.g. assume that there is some $J^* \subseteq J$ with $J^i = J^*$ for all $i < 2^\lambda$.

Now we show:

Claim. If $\text{tp}_r(Y^i_{p,e}/\langle D^i \rangle) \cup \text{tp}_r(Y^j_{p,e}/\langle D^j \rangle)$ is consistent, then $p_i \cup p_j$ is consistent.

The claim implies that the $S$-types $q_i = \text{tp}_r(Y^i_{p,e}/\langle D^i \rangle)$ are pairwise contradictory. Hence, we have the desired $2^\lambda$ pairwise contradictory types $q_i$ over $\Gamma(G)$ where $|\Gamma(G)| = \lambda$ and $|q_i| = \kappa$.

Proof of the Claim. Since $\Gamma(\Bbb C)$ is a monster model of the theory $S$ there is a set $E_{p}$ of elements of type $1^*$ in $\Bbb C$ such that $E_{p}$ is a common realization of $\text{tp}_r(Y^i_{p,e}/\langle D^i \rangle)$ and $\text{tp}_r(Y^j_{p,e}/\langle D^j \rangle)$.

Choosing the right elements in the $\approx$-classes we can assume that $E_{p} \subseteq X_{p} Y_{p}$. We have $E_{p} \cap D_{p} = E_{p} \cap D_{p} = \emptyset$. Now we choose $E_{p} \subseteq Y_{p}$, $E_{p} \subseteq Y_{p}$, and $E = E_{p} E_{p} E_{p}$ such that there is a bijection $h'$ of $Y^i D^i$ onto $ED^i$ that is the identity on $D^i$, $h'$ respects the
1*- p-, and 1*-parts and the handles, and
\[ \text{tp}(Y^{*\ell}_i D^{*\ell}_i) = \text{tp}(h^{*\ell}(Y^{*\ell}_i D^{*\ell}_i)) = \text{tp}(E^{*\ell}_i D^{*\ell}_i). \]
Then we have also such a bijection \( h^i \) between \( Y^i D^i \) and \( ED^i \). By Lemma 4.12 we have
\[ \text{tp}(E^{*\ell}_i) = \text{tp}(Y^{*\ell}_i D^{*\ell}_i), \quad \text{and} \quad \text{tp}(ED^i) = \text{tp}(Y^i D^i) \]
and therefore
\[ \text{tp}(E^{*\ell}_i J^*) = \text{tp}(Y^{*\ell}_i D^{*\ell}_i J^*), \quad \text{and} \quad \text{tp}(E^i D^i J^*) = \text{tp}(Y^i D^i J^*). \]
Hence \( E^{*\ell}_i \) is a common realization of \( \text{tp}(Y^{*\ell}_i J^*) \) and \( \text{tp}(Y^i J^*/D^i) \). But then \( p_i = \text{tp}(t(X^i Y^i D^i J^*))/(D^i) \) and \( p_j = \text{tp}(t(X^i Y^i D^i J^*))/(D^i) \) have the common realization \( t(X^i E^i J^*) \). Note that by construction \( X^i I^* \subseteq D^i \) for all \( \ell \).

6. Lifting elements of the commutator subgroup

Let \( S \) be a complete theory of nice graphs and let \( T_S \) be the corresponding theory of Mekler-groups \( G \). Often it is convenient to work in \( \mathcal{F}(G) = \langle V_G, W_G, \beta_G \rangle \) (see Section 3). We assume that \( \mathcal{F}(G) \) lives in \( G^{eq} \). If we work in \( \mathcal{F}(G) \) then we use additive notation for \( V_G = G/Z(G) \) and \( W_G = Z(G) \). If \( G = \mathbb{C} \) is the monster model of \( T_S \), then we leave out the index \( G = \mathbb{C} \).

**Definition.** A subset \( X \) of \( V_G \) is (a part of) a transversal if a set of representatives in \( G \) is (a part of) a transversal.

Note that this is equivalent to say that all sets of representatives in \( G \) are (parts of) transversals.

In \( V_G \) we also use other notions for \( G \) like \( \sim \)-equivalence, \( \approx \)-equivalence, type 1*- .

**Definition.** If \( a, b, c \in V \) are of type 1*, \( a \not\approx b, c \not\approx a, c \not\approx b, \beta(a, c) = 0, \) and \( \beta(b, c) = 0, \) then \( c \approx \) is called the connection between \( a \) and \( b \).

If such a connection \( c \approx \) of \( a \) and \( b \) exists, then it is uniquely determined by the niceness of \( \Gamma(\mathbb{C}) \).

**Definition.** A subspace \( U \subseteq V \) is called good, if it has a basis \( X \) such that \( X \) is a part of a transversal and for the handles in \( X \) all connections are represented in \( X \).

Let \( V_{1^*} \) be the subspace of \( V \) generated by all elements of type 1*. Then \( V_{1^*} = \bigoplus_{a \in X_{1^*}} \langle a \rangle \) for every 1*-transversal \( X_{1^*} \) of \( V \).

Let \( V_{1^*, p} \) be the subspace of \( V \) generated by all elements of type 1* or \( p \).
Lemma 6.1. Let $v$ be an element of $V_{v,p}$. Then there is a finite part $X \cup Y \cup \{u_e: e \in Y\}$ of a transversal such that

(i) $X \cup Y$ is a part of an $V$-transversal and $X \cap Y = \emptyset$.
(ii) $\{u_e: e \in Y\}$ is a part of a $p$-transversal and $e$ is the handle of $u_e$.
(iii) If $d \in X$ and $e \in Y$, then $\beta(d,e) \neq 0$.
(iv) $v = \sum_{d \in X} d + \sum_{e \in Y} u_e$.

Furthermore, let $C(Y)$ be a set of representatives of the connections between the elements of $Y$. Then $U(v) = \langle X \cup Y \cup C(Y) \cup \{u_e: e \in Y\} \rangle$ is definable over $v$ and it is the smallest good subspace that contains $v$.

Proof. Let $Q$ be any transversal of $V$. Then $v$ is in

$$V_{v,p} = \bigoplus_{a \in Q_v} \langle a \rangle \oplus \bigoplus_{e \in Q_v} \langle Q_{p,e} \rangle.$$ 

If we replace some elements of $Q_v$ by $\approx$-equivalent elements in a suitable manner, then there are finite subsets $X^*$ and $Y$ of $Q_v$ such that

$$v = \sum_{a \in X^*} a + \sum_{e \in Y} w_e,$$

where $w_e \neq 0$, and $w_e \in \langle Q_{p,e} \rangle$. Let $X$ be $\{a \in X^* : \beta(a,e) \neq 0 \text{ for all } e \in Y\}$. Let $f$ be an automorphism of $\langle V,W,\beta \rangle$ that fixes $v$. Then $f$ permutes $Q_v^\approx$ and

$$f(Q_{p,e}) \subseteq \bigoplus_{\substack{a \in Q_v \\beta(a,f(e))=0}} \langle a \rangle \oplus \langle Q_{p,f(e)} \rangle$$

is a $p$-transversal for the handle $f(e)$. The inclusion follows since $\beta(b,f(e)) = 0$ for every $b \in f(Q_{p,e})$ and the right side is the subspace of all $d$ with $\beta(d,f(e)) = 0$.

Since $v = f(v)$ has a unique representation as a linear combination over $Q$, we get $f(X^\approx) = X^\approx$ and $f(Y^\approx) = Y^\approx$ and $X$ and $Y$ are contained in every good subspace of $V$ that contains $v$. It follows $f(\langle C(Y) \rangle) = \langle C(Y) \rangle$ and this subspace is also contained in every good subspace that contains $v$. Now we add every summand $a$ for $a \in X^* \setminus X$ to $w_a$ if $a \in \langle Y \rangle$ or to some $w_e$ with $\beta(e,a) = 0$ otherwise. The result we call $u_e$. W.l.o.g. we assume that $u_e = w_e$ and $X = X^*$.

Now we have

$$v = \sum_{d \in X} d + \sum_{e \in Y} u_e,$$

where $X \cap Y = \emptyset$, $X \cup Y \cup \{u_e: e \in Y\}$ is part of a transversal $Q$, and $u_e \in Q_{p,e}$. Then $v = f(v)$ implies as above

$$\sum_{d \in X} d + \sum_{e \in Y} u_e = \sum_{d \in X} f(d) + \sum_{e \in Y} f(u_e).$$
We have the stronger conclusion that \( f \) permutes \( X \) and the set \( \{ u_e : e \in Y \} \) modulo \( \langle C(Y) \rangle \). Hence

\[
v \in \bigoplus_{a \in X \cup Y \cup CY} \langle a \rangle \oplus \bigoplus_{e \in Y} \langle u_e \rangle = U(v)
\]

and for automorphisms \( f \) with \( f(v) = v \) we have \( f(U(v)) = U(v) \), as desired. \( \square \)

**Corollary 6.2.** In Lemma 6.1 representation (iv) \( \forall v \in V \), \( v = \sum_{d \in X} d + \sum_{e \in Y} u_e \) is unique modulo \( \langle C(Y) \rangle \) by properties (i) – (iii).

**Corollary 6.3.** For every \( v \in V \) there exists a smallest good subspace \( U(v) \) that contains \( v \). It is finite and definable over \( v \).

**Proof.** If \( v \in U_{1r,p} \), then the result is proved in Lemma 6.1. If \( v \not\in V_{1r,p} \), then \( v \) is an element of type \( 1' \). Therefore \( U(v) = \langle v \rangle \). \( \square \)

**Corollary 6.4.** If \( V^0 \) is a subspace of \( V \), then there is a smallest good subspace \( U(V^0) \) that contains \( V^0 \). If \( V^0 \) is finite, then \( U(V^0) \) is finite and definable over \( V^0 \).

**Proof.** By Lemma 6.1 for every \( v \in V^0 \cap V_{1r,p} \) there are \( X(v) \cup Y(v) \cup CY(v) = X^*_p(v) \) and \( \{ u(v)_e : e \in Y(v) \} = X^*_p(v) \) such that \( v \in \langle X^*_p(v) \rangle \) and \( \langle X^*_p(v) \rangle \) is finite, good, definable over \( v \), and contained in each good subspace that contains \( v \). There is a part \( X^1 \) of a \( 1' \)-transversal such that \( \langle X^1 \rangle = \langle \bigcup_{v \in V^0 \cap V_{1r,p}} X^*_p(v) \rangle \). Let \( X^1 \subseteq V^0 \) be a basis of \( V^0 \) modulo \( V_{1r,p} \). For every \( e \in X^1 \) we choose a maximal subset \( X_{p,e} \) of \( \{ u(v)_e : v \in V^0 \cap V_{1r,p} \} \) linearly independent modulo \( V_{1r} \). Then it is possible to represent every \( u(v)_e \) as a linear combination \( u^*(v)_e \) over \( X_{p,e} \) modulo \( V_{1r} \). We obtain \( X^1 \), if we add all necessary elements of \( V_{1r} \) to \( X^1 \) to represent \( u(v)_e - u^*(v)_e \) for all \( v \in V^0 \cap V_{1r,p} \).

Now \( X^1 \cup \bigcup_{e \in X^1} X_{p,e} \cup X_{1r} = X \) is a finite part of a transversal, \( V^0 \subseteq \langle X \rangle \), and \( X \) is contained in every good subspace \( U \) with \( V^0 \subseteq U \). \( \square \)

If we analyse the proof of Corollary 6.4, then we obtain:

**Lemma 6.5.** For a subspace \( V^0 \subseteq V \)

\[
U(V^0) = \left( \bigcup_{v \in V^0} U(v) \cup R(V^0) \right),
\]

where \( R(V^0) \) is a set of non-\( \simeq \)-equivalent elements of type \( 1' \) that occur in the representation of elements of \( \langle \bigcup_{e \in V^0} U(v) \rangle \cap V_{1r} \).

Now, we easily show:

**Lemma 6.6.** The intersection of two good subspaces is good.
Proof. Let \( U_0 \) and \( U_1 \) be two good subspaces of \( V \). Then \( U_0 \cap U_1 \) contains all \( U(v) \) for \( v \in U_0 \cap U_1 \) and the set \( R(U_0 \cap U_1) \) described in Lemma 6.5. Hence \( U(U_0 \cap U_1) = U_0 \cap U_1 \). □

Let \( X \) be a transversal for \( V \). Assume that \( X \) is ordered by an additional relation \(<\). By Corollary 4.7

\[
\{ \beta(a,b) : a, b \in X, \ a < b, \ \beta(a,b) \neq 0 \}
\]

is a basis for the vector space \( \langle \beta(V,V) \rangle \). That means that elements of \( \beta(V,V) \) have a unique representation as a linear combination over this basis. For \( a \in \langle \beta(V,V) \rangle \) (or \( a \in G' \)) define \( U(a) \) as the intersection of all good subspaces \( V_0 \) of \( V \) with \( a \in \langle \beta(V_0, V_0) \rangle \). Since by Corollary 6.4 every finite subspace is contained in a finite good subspace, \( U(a) \) is finite; it is good by Lemma 6.6. We want to show that \( a \in \beta(U(a), U(a)) \) and that \( U(a) \subseteq \text{acl}(a) \). For the last assertion it is sufficient to show that \( U(a) \) is definable over \( a \).

As for groups we use the following notation:

Definition. Let \( U \) be a good subspace of \( V = V_C \) and let \( X \) be a part of a transversal. \( X \) respects \( U \) if \( X \) can be extended to a transversal \( Y \) of \( V_C \) such that \( Y \cap U \) is a transversal for \( U \). A good subspace respects \( U \) if it is generated by a part of a transversal that respects \( U \).

Let \( X \) be a part of a transversal and let \( U \) be a good subspace of \( V \). Then \( X_U \) represents all elements of type 1 up to \( \approx \)-equivalence and it is linearly independent. If \( u \in U \) is a linear combination \( u = \sum r_i a_i \) with \( a_i \in X_U \), then \( a_i \in U \) since \( U \) is good (Lemma 6.1). Hence we have \( X_U = X_U^U \cup X_U^+ \) where \( X_U^U \subseteq U \) and \( X_U^+ \) is linearly independent over \( U \).

Note that \( X \) respects \( U \) if and only if

\[
X_{p,e} = X_{p,e}^U \cup X_{p,e}^+ \quad \text{for every } e \in X_U^U,
\]

and

\[
X_U = X_U^U \cup X_U^+,
\]

where \( X_{p,e}^U \subseteq U \), \( X_U^U \subseteq U \), \( X_{p,e}^+ \) is linearly independent modulo \( \langle V_U \cup U \rangle \)

and

\[
X_U^+ \text{ is linearly independent modulo } \langle V_U, p \cup U \rangle.
\]

Lemma 6.7. If \( U_1 \subseteq U_2 \subseteq V \) are good subspaces, then every part of a transversal \( X \) that generates \( U_1 \) can be extended to a part of a transversal \( XY \) that generates \( U_2 \).
Then there is a linear combination $y'$ over $X_{p,e} \cup Y_{p,e}$ such that

$$y = y' \mod V_{1,} \cap U_{2}.$$  

Hence $y \in \langle X_{1}, Y_{1}, X_{p,e}, Y_{p,e} \rangle$. We have shown that

$$X_{1} \cup Y_{1} \cup \bigcup_{e \in X_{1} \cap Y_{1}} (X_{p,e} \cup Y_{p,e})$$

is a part of a transversal that generates $U_{2} \cap V_{1,}$. Then we find a maximal $Y_{1} \subseteq X_{1}^{0}$ that is linearly independent modulo $\langle V_{1,} \cup U_{1} \rangle$. Similarly as above $X_{1}^{0} \subseteq \langle (U_{2} \cap V_{1,}) \cup X_{1} \cup Y_{1} \rangle$.  

**Lemma 6.8.** If $U_{0}$ and $U_{1}$ are good subspaces of $V$ with $a \in \beta(U_{0}, U_{0})$ and $a \in \beta(U_{1}, U_{1})$, then $a \in \beta(U_{0} \cap U_{1}, U_{0} \cap U_{1})$.

**Proof.** $U_{0} \cap U_{1}$ is good by Lemma 6.6. Let $X$ be a generating part of a transversal for $U_{0} \cap U_{1}$. By Lemma 6.7 there are $Y_{0}$ and $Y_{1}$ such that $XY_{0}$ and $XY_{1}$ are parts of transversals that generate $U_{0}$ and $U_{1}$, respectively. W.l.o.g. we can assume, that there is no part $XY^{*}$ of a transversal in $U_{1}$ such that $|Y_{1}^{*}| < |Y_{1}|$ and $a \in \beta(\langle XY^{*} \rangle, \langle XY^{*} \rangle)$. First we show $Y_{1}^{1} \subseteq U_{0}$ modulo $V_{1,}$. Otherwise we can choose $Y_{1}^{1}$ in such a way that $Y_{1}^{1} = Y_{1}^{2} \cup Y_{1}^{3}$ where $Y_{1}^{2}$ is linearly independent modulo $\langle U_{0} \cap V_{1,} \rangle$ and $Y_{1}^{3} \subseteq \langle U_{0} \cap V_{1,} \rangle$. Let $\tilde{X}$ be a transversal that contains $XY_{0}Y_{1}^{2}$. By assumption (minimality of $|Y_{1}^{1}|$) the representation of $a$ over $XY_{1}$ contains each $y \in Y_{1}^{2}$ in some commutator. But then we have the same for the unique representation of $a$ over $\tilde{X}$. This contradicts $a \in \beta(U_{0}, U_{0})$. Hence $Y_{1}^{2} = \emptyset$ and $Y_{1}^{1} \subseteq U_{0}$ modulo $V_{1,}$. For each $y \in Y_{1}^{1}$ there is some $y^{0} \in U_{0}$ such that $y^{1} - y^{0} \in V_{1,}$ and hence

$$y^{1} - y^{0} \in V_{1,} \cap \langle U_{0} \cup U_{1} \rangle.$$  

If we add suitable $u_{0} \in V_{1,} \cap U_{0}$ and $u_{1} \in V_{1,} \cap U_{1}$ to $y^{0}$ and $y^{1}$, respectively, then we have w.l.o.g. $Y_{1}^{1} \subseteq U_{0}$ and therefore $Y_{1}^{1} \subseteq U_{0} \cap U_{1}$. Hence $Y_{1}^{1} = \emptyset$. Similarly, we can assume w.l.o.g. that $Y_{0}^{1} = \emptyset$.

For all transversals $\tilde{X} \supseteq X$ the set $I^{\equiv}$ of handles $e^{\equiv}$ such that $\tilde{X}_{p,e}$ is used in representation of $a$ over $\tilde{X}$ as a linear combination of commutators is the same. Hence, we can assume that $I \subseteq X$ and $C(I) \subseteq X$. Now $Y^{1} = \bigcup_{e \in I} Y_{p,e} \cup Y_{1}^{1}$. We fix some $e \in I$ and assume that we have chosen a part $XY^{*}$ of a transversal in $U_{1}$ such that $|Y_{p,e}^{*}|$ is minimal and $a \in \beta(\langle XY^{*} \rangle, \langle XY^{*} \rangle)$. W.l.o.g. $Y^{1} = Y^{*}$. As above for $Y_{1}^{1}$ and $Y_{0}^{1}$ we have $Y_{p,e}^{1} = Y_{p,e}^{0} = \emptyset$. Then $Y^{0} = Y_{1}^{0}$, $Y^{1} = Y_{1}^{1}$, and hence $XY_{0}Y^{1}$ is a part of a transversal.
Then all elements of any transversal $\bar{X} \supseteq \chi^{Y_0}Y^1$ that are used in commutators of the representation of $a$ over $\bar{X}$ are in $X$. Since $I$ and $CI$ are also in $X$ we get $Y_0 = Y^1 = \emptyset$, as desired. \textcircled{QED}

We have shown that $a \in \beta(U(a), U(a))$. Finally we prove:

**Theorem 6.9.** For every $a \in \beta(V, V')$ ($a \in \mathbb{C}'$ resp.) there is a smallest good subspace $U(a) \subseteq V$ such that $a \in \beta(U(a), U(a))$. $U(a)$ is finite and definable over $a$.

**Proof.** It is clear that $a \in \beta(V^0, V'^0)$ for some finite good subspace. By Lemma 6.6 we can define $U(a)$ as the intersection of all good subspaces $U$ that contain $a$ in $\beta(U, U')$. By Lemma 6.8 $a \in \beta(U(a), U(a))$. $U(a)$ is finite, hence definable, and $\{a\}$-automorphism invariant, hence definable over $a$. \textcircled{QED}

Let $\text{Aut}_{\mathbb{C}}(\mathbb{G})(\mathbb{C})$ be the subgroup of all automorphisms of $\mathbb{C}$ that fix $d/V\mathbb{G}$ and fix $G$ setwise.

**Lemma 6.10.** Let $G \subseteq \mathbb{C} = T_S$. Let $d = \sum_{1 \leq i \leq n} c_i + \sum_{1 \leq i \leq m} d_i + \sum_{1 \leq i \leq \ell} w_i + x$ be an element of $\langle V\mathbb{G} \cup V\mathbb{I} \cup P \rangle$ where $c_1, \ldots, c_n, e_1, \ldots, e_m$ are elements of type $V'$ not in $V\mathbb{G}$ in different $\approx$-classes, each $d_i$ is an proper element of type $p$ with handle $e_i$, for every $i$ either $\beta(c_i, e_j) \neq 0$ for all $j$ or $c_i$ is a connection of two $e_j$, any $w_i \notin \langle V\mathbb{G} \cup V\mathbb{I} \rangle$ and it is a proper element of type $p$ with handle $f_i \in V\mathbb{G}$, different $f_i$ are in different $\approx$-classes, $\beta(c_i, f_j) \neq 0$ for all $i, j$, and $x \in V\mathbb{G}$. Then there is a subgroup $\mathbb{K}$ of $\text{Aut}_{\mathbb{C}}(\mathbb{G})(\mathbb{C})$ of finite index such that every $f \in \mathbb{K}$ fixes $\{c_1, \ldots, c_n, e_1, \ldots, e_m, d_1, \ldots, d_m, f_1, \ldots, f_\ell\}$ pointwise and $f(w_i) = w_i$ modulo $V\mathbb{G}$.

**Proof.** Let $C$ be the set of connections of the handles $e_i$ and $f_j$. By Lemma 6.1

$$\{c_1, \ldots, c_n, e_1, \ldots, e_m, d_1, \ldots, d_m, f_1, \ldots, f_\ell, w_1, \ldots, w_\ell\} \cup C$$

generates $U(d - x)$ the smallest good subspace that contains $d - x$.

Hence $\langle V\mathbb{G} \cup U(d - x) \rangle$ is the smallest good subspace that contains $V\mathbb{G}$ and $d$. It is fixed setwise by every $f \in \text{Aut}_{\mathbb{C}}(\mathbb{G})(\mathbb{C})$. Such an automorphism $f$ determines an automorphism of the finite space $\langle V\mathbb{G} \cup U(d - x) \rangle / V\mathbb{G}$. Hence there is a subgroup $\mathbb{K}$ of $\text{Aut}_{\mathbb{C}}(\mathbb{G})(\mathbb{C})$ of finite index such that $f \in \mathbb{K}$ fixes $\{c_1, \ldots, c_n, e_1, \ldots, e_m, d_1, \ldots, d_m, w_1, \ldots, w_\ell\}$ pointwise modulo $V\mathbb{G}$ and therefore $f_1, \ldots, f_\ell$ pointwise. By Lemma 4.15 $|\{u \in V\mathbb{G}; \beta(u, e_i) = 0\}| \leq p$. Hence we can decrease $\mathbb{K}$ in such a way that it is still of finite index but every $f \in \mathbb{K}$ fixes $\{c_1, \ldots, c_n, e_1, \ldots, e_m, d_1, \ldots, d_m\}$ pointwise. To show this first assume $c = c_i$ or $c = e_j$. Then $f(c) = c + v$ where $v \in V\mathbb{G}$ and $f \in \text{Aut}_{\mathbb{C}}(\mathbb{G})(\mathbb{C})$ implies $f(c) = c$ since $c + v$ with $v \neq 0$ is not of type $V'$. Now consider $f(d_i) = d_i + v$ where $v \in V\mathbb{G}$ and $f \in \text{Aut}_{\mathbb{C}}(\mathbb{G})(\mathbb{C})$. Then $\beta(v, e_i) = 0$. Since $e_i \notin V\mathbb{G}$ there is no proper element of type $p$ with handle $e_i$ in $V\mathbb{G}$. By Lemma 4.15 $\beta(v, e_i) = 0$ has at most one nontrivial solution up to $\approx$-equivalence. \textcircled{QED}
7. Good subspaces that respect \( V_G \)

Let \( \mathbb{C} \) be a monster model of \( T_S \) and \( G \preceq \mathbb{C} \). Let \( D \) be a finite good subspace of \( V = V_\mathbb{C} \). Our aim is to have a good subspace \( D^* \) that contains \( D \), is definable over \( D \), and respects \( V_G \). This is impossible in general. We will develop a substitute. First we give some notations and definitions.

Let \( M \) be a model of any elementary theory \( T \) and \( D \) be a subset of \( M \). Then
\[
\text{Aut}(D)(M) = \{ f \in \text{Aut}(M) : f(d) = d \text{ for } d \in D \}
\]
is the pointwise stabilizer of \( D \) and
\[
\text{Aut}\{D\}(M) = \{ f \in \text{Aut}(M) : f(D) = D \}
\]
is the setwise stabilizer of \( D \).

**Definition.** Two subgroups of some group are commensurable, if their intersection has finite index in both of them.

**Fact 7.1.** Let \( \mathbb{C} \) be a monster model of \( T; M \preceq \mathbb{C} \) saturated; \( A \subseteq \mathbb{C} \); and \( E \subseteq M_{eq} \). If \( \text{Aut}(E)(M) \) and \( J = \{ f \in \text{Aut}(M) : \text{tp}(A=M) = f(\text{tp}(A=M)) \} \) are commensurable, then
\[
\text{acl}(E) = \text{acl}(\text{Cb}(\text{tp}(A=M))).
\]

Note that \( J = \{ f \upharpoonright M : f \in \text{Aut}_{\{M\}(A)}(\mathbb{C}) \} \).

Furthermore:

**Fact 7.2.** If \( A \subseteq B \subseteq \text{acl}(A) \), then \( \text{Cb}(\text{tp}(A/M)) \subseteq \text{Cb}(\text{tp}(B/M)) \subseteq \text{acl}(\text{Cb}(\text{tp}(A/M))) \). If \( A \subseteq B \subseteq \text{dcl}(A) \), then \( \text{Cb}(\text{tp}(A/M)) = \text{Cb}(\text{tp}(B/M)) \).

**Fact 7.3.** If in the situation of Fact 7.1 \( A = (A \cap M) \cup B \), then \( \text{Cb}(\text{tp}(A/M)) = \text{dcl}((A \cap M) \cup \text{Cb}(\text{tp}(B/M))) \).

Now we come back to \( T_S \) and \( G \preceq \mathbb{C} \models T_S \). As before we often use \( \mathcal{F}(G) = \langle V_G, W_G, \beta_G \rangle \preceq \mathcal{F}(\mathbb{C}) = \langle V, W, \beta \rangle \) in \( \mathbb{C}^{eq} \).

**Definition.** \( \square = (X, Y, D) \) is a special triple with respect to \( V_G \) of subsets of \( V \), if

(i) \( D \) is linearly independent modulo \( V_G \).

(ii) \( X \subseteq Y \subseteq V \), \( Y \) is linearly independent modulo \( V_G \).

\[
X = X_1^\square \cup X_2^\square, \quad Y = Y_p^\square = \bigcup_{e \in X_1^\square} Y_{p,e}^\square
\]

and all \( Y_{p,e} \) for \( e \in X_1^\square \) are non-empty.

(iii) There are a bijection \( d \to x_d \) of \( D \) onto \( X_1^\square \) and an injection \( d \to y_d \) of \( D \) into \( \langle Y \rangle \) such that
\[
d = x_d + y_d \quad \text{for every } d \in D.
\]

(iv) \( \langle V_G, Y \rangle \) is the smallest good subspace that contains \( V_G \) and \( D \).
If $X$ is a subset of $V$, then we use $\langle X \rangle$ to denote the subspace of $V$ that is generated by $X$ as above. $\langle X \rangle_\varnothing$ denotes the restriction of the considered alternating bilinear map to the subspace $\langle X \rangle$. Given a special triple $\square = (X^0, Y^0, D^0)$ as above we introduce an equivalence relation $\theta_\square(X^0, X^1)$:

**Definition.** Let $\bar{x}^0$ and $\bar{x}^1$ be subsequences of $V$ of the length of $D^0$, namely $\bar{x}^i = (x^i_d : d \in D^0)$. Define $\theta_\square(\bar{x}^0, \bar{x}^1)$ to hold if there is a vector space homomorphism $f$ of $\langle Y^0 \rangle$ into $V$ that can be extended to an $\mathbb{B}_p$-homomorphism $\tilde{f}$ of $\langle X^1_0 Y^0 \rangle_\varnothing$ into $\mathcal{F}(\mathbb{C})$ with $\tilde{f}(x) = x$ for $x \in X^1_0$ and

$$x^0_d = x^1_d + f(y_d).$$

Note:

- In the definition above we have $\beta(e, f(y)) = 0$ for $e \in X^1_0$ and $y \in Y^0_{p,e}$.
- We obtain $\langle X^1_0 Y^0 \rangle_\varnothing$ from the free alternating bilinear map over the vector space $\langle X^1_0 Y^0 \rangle$, if we factorize it by all $\beta(a_0, a_1) = 0$ for $a_0, a_1 \in X^1_0$ that come from $\langle X^1_0 \rangle_\varnothing$ and by $\beta(y, e) = 0$ for $y \in Y^0_{p,e}$. Hence a vector space homomorphism $f$ of $\langle Y^0 \rangle$ in $V$ can be extended to an $\mathbb{B}_p$-homomorphism $\tilde{f}$ of $\langle X^1_0 Y^0 \rangle_\varnothing$ into $\mathcal{F}(\mathbb{C})$ with $\tilde{f}(x) = x$ for $x \in X^1_0$ if and only if $\beta(e, f(y)) = 0$ for all $y \in Y^0_{p,e}$.
- $\theta_\square(\bar{x}^0, \bar{x}^1)$ defines an equivalence relation.
- $\theta_\square$ is definable over $X^1_0$.
- In a situation where $X^1_0$ is fixed we can consider the classes of $\theta_\square$ as elements of $\mathbb{C}^\varnothing$.
- We use $\square(X^1_0)$ to denote the $\theta_\square$-class of $X^1_0$. Often we only write $\square$.

**Theorem 7.4.** Let $D$ be a finite good subspace of $V$. Then there are a finite good subspace $\langle XX^0 Y^0 \rangle \supseteq D$, where $XX^0 Y^0$ is a part of a transversal that respects $G$ with $\langle XX^0 Y^0 \rangle \cap V_G = \langle XX^0 \rangle$, and a subset $D^0$ of $\langle X^0 Y^0 \rangle$ such that

(i) $\langle V_G Y^0 \rangle$ is the smallest good subspace that contains $V_G$ and $D$. $D \subseteq \langle X^0 Y^0 \rangle$.

(ii) $\square = (X^0, Y^0, D^0)$ is a special triple and

$$X^0 \cap X = X^1_0, \quad Y^0 \cap Y = \emptyset.$$ 

(iii) $\text{Aut}_{\{G\}_{(D)}}(\mathbb{C})$ and $\text{Aut}_{\{G\}_{(XYD^0)}}(\mathbb{C})$ are commensurable.

(iv) If $H$ is a saturated model of $T_8$ with $G \preceq H \preceq \mathbb{C}$ and $\text{acl}(D \cup G) \cap H = G$, then $\text{Aut}_{\{H\}_{(D)}}(\mathbb{C})$ and $\text{Aut}_{\{H\}_{(XYD^0)}}(\mathbb{C})$ are commensurable.

We call $(X, X^0, Y, Y^0, D^0)$ with the properties in Theorem 7.4 a platform for $D$.

**Proof.** By Lemma 6.6 $D \cap V_G$ is good. By Lemma 6.7 there is a part of a transversal $I$ for $D$ such that $I \cap V_G$ is a part of a transversal for $D \cap V_G$. It is possible that $I$ does not respect $V_G$. Starting from $I$ we construct $X$, $Y$, $X^0$, $Y^0$, and $D^0$. In each step we say which constructed elements belong to which set.
Assume \( I = I^0 \cup I^1 \) where \( I^0 = I \cap V_G \). Then \( I^0 \) will be part of \( X \) and \( I^1 \) will be part of \( Y \). Note that \( e \in I^1 \) implies that \( I^1_{p,e} \subseteq I^1 \). Then \( I^1_{p,e} \) is linearly independent modulo \( \langle V_G \cup V_T \rangle \). \( I^1_{p,e} \) will be part of \( Y \). Now we consider \( I^1_{p,e} \) for \( e \in I^0 \). W.l.o.g. \( I^1_{p,e} = I^2_{p,e} \cup I^3_{p,e} \) where \( I^2_{p,e} \subseteq \langle V_G \cup V_T \rangle \) and \( I^3_{p,e} \) is linearly independent over \( \langle V_G \cup V_T \rangle \). \( I^3_{p,e} \) will be \( Y_{p,e} \). Up to now all constructed elements of \( X \) and \( Y \) are elements of \( D \). Therefore they are fixed by all automorphisms that fix \( D \) pointwise. If \( w \in I^2_{p,e} \), then

\[
 w = \sum_{1 \leq i \leq n} c_i + x, \quad (7.1)
\]

where \( c_1, \ldots, c_n \) are elements of type 1' not in \( V_G \) in different \( \approx \)-classes with \( \beta(e, c_i) = 0 \) and \( x \in V_G \) is a proper element of type \( p \) with handle \( e \) in \( V_G \). The \( \approx \)-classes of the \( c_i \) and \( e \) are uniquely determined by \( w \) (Lemma 6.10).

Let \( X^2_{p,e} \subseteq V_G \) be the set of the \( x \)'s that correspond to the \( w \)'s in \( I^2_{p,e} \) in (7.1). Then \( X^2_{p,e} = I^2_{p,e} \cup X^2_{p,e} \) is linearly independent modulo \( V_{1'} \). It will be the \( p \)-transversal for the handle \( e \) in \( X \subseteq V_G \). Furthermore all \( c_i \) will be elements of \( \langle Y_T \rangle \). In situation (7.1) we have by Lemma 6.10

\[
 \text{Aut}_{(G(w)(C))} \text{ and } \text{Aut}_{(G((x_{1}, \ldots, x_n)))(C)} \text{ are commensurable.} \quad (7.2)
\]

By Lemma 6.1 \( \{ c_1, \ldots, c_n \} \subseteq \text{acl}(D \cup V_G) \). If \( H \) is as in (iv), then \( \text{acl}(D \cup V_G) \cap H \subseteq G \) implies that \( \{ c_1, \ldots, c_n \} \) is linearly independent modulo \( V_H \). We obtain by Lemma 6.10

\[
 \text{Aut}_{(H(w)(C))} \text{ and } \text{Aut}_{(H((x_{1}, \ldots, x_n)))(C)} \text{ are commensurable.} \quad (7.2^H)
\]

Finally we consider \( I^1 \). Again we split w.l.o.g. \( I^1 = I^2 \cup I^3 \cup I^4 \) where \( I^2 \) is linearly independent modulo \( \langle V_G \cup V_H \cup \bigcup_{e \in V_G} Y^{0}_{p,e} \rangle \), \( I^3 \subseteq \langle V_H \cup V_{1'} \rangle \), and \( I^4 \) is linearly independent modulo \( \langle V_H \cup V_{1'} \rangle \). Again w.l.o.g. we can assume that \( I^4 \subseteq Y_{1'} \). Similarly as in (7.1) we have for \( w \in I^2_{p,e} \)

\[
 w = \sum_{1 \leq i \leq n} c_i + \sum_{1 \leq i \leq m} d_i + x, \quad (7.3)
\]

where the \( c_i \)'s are elements of type 1' not in \( V_G \) in different \( \approx \)-classes, the \( d_i \)'s are proper elements of type \( p \) not in \( V_G \) with different handles \( e_i \) not in \( V_G \), \( \beta(c_i, e_i) \neq 0 \) for all \( i \) and \( j \), and \( x \in V_G \) is an element of type 1'. By Lemma 6.10 we have for (7.3):

\[
 \text{Aut}_{(G(w)(C))} \text{ and } \text{Aut}_{(G((x_{1}, \ldots, x_n, e_{1}, \ldots, e_{m}, d_{1}, \ldots, d_{m}))(C))} \text{ are commensurable.} \quad (7.4)
\]

Let \( X^2_1 \) be the set of the \( x \)'s in (7.3) for the \( w \)'s in \( I^2_1 \). Then \( X_1 = I^0 \cup X^2_1 \) is linearly independent modulo \( V_{1'} \). \( X_1 \) is the \( 1' \)-part of \( X \).

We will put \( c_1, \ldots, c_n, e_1, \ldots, e_{m}, d_1, \ldots, d_{m} \) into \( \langle Y \rangle \). Note (7.4) implies \( \{ c_1, \ldots, c_n, e_1, \ldots, e_{m}, d_1, \ldots, d_{m} \} \subseteq \text{acl}(V_G \cup D) \). By \( \text{acl}(V_G \cup D) \cap H \subseteq G \) we have that this set is linearly independent modulo \( V_H \). Since the handle \( e_i \) of \( d_i \) is not in \( V_H \) we have \( d_i \notin \langle V_H \cup V_{1'} \rangle \). Otherwise we would get a proper element of type \( p \) in \( V_H \) with
handle $e_i \notin V_H$. Again we can apply Lemma 6.10:

$$\text{Aut}_{(H)(w)}(C) \quad \text{and} \quad \text{Aut}_{(H)(c_1,...,c_n,e_1,...,e_m,d_1,...,d_m,\{v\})}(C)$$

are commensurable. (7.4)

The set of elements we have collected for $X$ up to now we call $X^0$. We will have $X^0_1 = X_p$ and $X^0_0 = X_0$. Using all elements we have chosen for $(Y)$ we form $Y^0$ such that $X^0_0 Y^0$ is a part of a transversal that respects $V_G$ with $V_G \cap (X^0 Y^0) = (X^0)$. The elements of $X^0 Y^0$ are either elements of $D$ or they occur in (7.1) or (7.3). By (7.2) and (7.4) we get therefore a subgroup $\mathbb{K}^0$ of $\text{Aut}_{(H)(D)}(C)$ of finite index such that each $f \in \mathbb{K}^0$ fixes $X^0_0 Y^0$ pointwise. Analogously by (7.2) and (7.4) we have $\mathbb{K}^0_H$ in $\text{Aut}_{(H)(D)}(C)$ of finite index such that $f \in \mathbb{K}^0_H$ fixes $X^0_0 Y^0$ pointwise.

Finally we consider $w \in D_3$. We have

$$w = \sum_{1 \leq i \leq n} c_i + \sum_{1 \leq i \leq m} d_i + \sum_{1 \leq i \leq \ell} w_i + x,$$

where $c_1, ..., c_n, e_1, ..., e_m$ are elements of type $1'$ not in $V_G$ in different $\approx$-classes, each $d_i$ is a proper element of type $p$ with handle $e_i$, $\beta(c_i, e_j) \neq 0$ for all $i, j$, the element $w_i \notin (G \cup V_T)$ is proper of type $p$ with handle $f_i \in V_G$, different $f_i$ are in different $\approx$-classes, $\beta(c_i, f_j) \neq 0$ for $i, j$, and $x \in V_G$.

Then $x$ is a proper element of type $1'$. By Lemma 6.10 there is a subgroup $\mathbb{K}^w$ of $\text{Aut}_{(G)(w)}(C)$ of finite index such that every $f \in \mathbb{K}^w$ fixes $\{c_1, ..., c_n, e_1, ..., e_m, d_1, ..., d_m, f_1, ..., f_\ell\}$ pointwise and $f(w_i) = w_i$ modulo $V_G$.

Let $d$ be $x + \sum_{1 \leq i \leq \ell} w_i$. We have $f(d) = d$ for $f \in \mathbb{K}^w$. So

$$\text{Aut}_{(G)(w)}(C)$$

and

$$\text{Aut}_{(G)(d,e_1,...,e_n,d_1,...,d_m,\{v\})}(C),$$

are commensurable.

(7.6)

By Lemma 6.1 $\{c_1, ..., c_n, e_1, ..., e_m, d_1, ..., d_m, w_1, ..., w_\ell\}$ are in $\text{acl}(V_G \cup D)$. As $\text{acl}(DG) \cap H = G$ we Know that these elements are linearly independent modulo $V_H$. We have almost situation (7.5) for $w$ with respect to $V_H$. The only difference is that it is possible that some $w_i$ are in $\langle V_H V_T \rangle$. In this case $w_i = w_i^* + x_i^*$ where $x_i^* \in V_H \cap V_{p,f_i}$ and $w_i^*$ is a sum of elements $v$ of type $1'$ that are not in $V_H$ and satisfy $\beta(v, f_i) = 0$.

By Lemma 4.15 we have $\beta(v, f_j) \neq 0$ for $i \neq j$. If $\beta(v, f_j) = 0$, then $v$ is a connection between $f_i$ and $e_j$. Let $w_i^* = w_i$ if $w_i \notin \langle V_H \cup V_T \rangle$. Otherwise we work with $w_i^*$ and add $x_i^*$ to $x$. Let $x^*$ be sum of $x$ and these $x_i^*$. Then we have

$$w = \sum_{1 \leq i \leq n} c_i + \sum_{1 \leq i \leq m} d_i + \sum_{1 \leq i \leq \ell} w_i^* + x^*, \quad (7.5)$$

where $c_1, ..., c_n, e_1, ..., e_m, d_1, ..., d_m, f_1, ..., f_\ell$ behave with respect to $V_H$ as in (7.5) with respect to $V_G$. Furthermore, $x^* \in V_H$; if $w = w_i^*$ then $w_i^*$ is a proper element of type $p$ with handle $f_i$ and $w_i^* \notin V_H$, and if $w_i \neq w_i^*$ then $w_i^*$ is a sum of non-\approx-equivalent elements $v$ of type $1'$ not in $V_H$ with $\beta(v, f_i) = 0$, $\beta(v, f_j) \neq 0$ for $i \neq j$, and $\beta(v, e_j) \neq 0$.
for all \(j\) or \(v\) is a connection of two handles \(e_j, f_i\). In the last case \(f_i\) is uniquely determined by \(v\) (Lemma 4.15). In this case we define \(w_i^{**} = w_i^* - v\) and \(d_i^{**} = d_i + v\). It can happen that we add several \(v\) to \(d_i\). We call the final result again \(d_i^{**}\). We can write (7.5) as

\[
w = \sum_{1 \leq i \leq n} c_i + \sum_{1 \leq i \leq m} d_i^{**} + \sum_{1 \leq i \leq \ell} w_i^{**} + x^*.
\]

Now we apply Lemma 6.10 and obtain \(\mathbb{K}^{wH} \subseteq \text{Aut}(\langle H \rangle(w))\) of finite index such that every \(f \in \mathbb{K}^{wH}\) fixes all \(c_i\), all \(e_i\), all \(d_i^{**}\), all \(v\) in \(w_i^{**}\) if \(w_i^* \neq w_i\), and for \(w_i = w_i^*\) the only \(\approx\)-class in \(V_H\) with \(\beta(v, f_i) = 0\). Hence w.l.o.g. we can assume that \(f(f_i) = f_i\) also in this case. Finally w.l.o.g. \(f(v) = v\) for a connection \(v\) of \(f_i\) and some \(e_j\). Hence \(f(d_i) = d_i\) and \(f(d) = d\) for \(d = x + \sum_{1 \leq i \leq \ell} w_i = x^* + \sum_{1 \leq i \leq \ell} w_i^*\). We have shown:

\[
\text{Aut}(\langle H \rangle(w)) \text{ and } \text{Aut}(\langle H \rangle(d,e_1,\ldots,e_n,d_1,\ldots,d_m,f_1,\ldots,f_p),\mathbb{C}) \text{ are commensurable. (7.6)}
\]

Let \(\mathbb{K}\) be the intersection of \(\mathbb{K}^0\) and all \(\mathbb{K}^w\) for \(w \in I^3_h\) and \(\mathbb{K}^H\) the intersection of \(\mathbb{K}^{0H}\) and all \(\mathbb{K}^{wH}\).

Let \(X_{V'}^D \subseteq V_G\) be a part of a \(1^v\)-transversal that consists of representatives of all \(\approx\)-classes of the \(f_i\) in (7.5) for \(w \in I^3_h\).

Now we define \(X\) as \(X^0 \cup X_{V'}^D\), where we assume that \(X\) and \(X^0\) contain the same element \(a\) of type \(1^v\), if they both represent the \(\approx\)-class of \(a\). We enlarge \(Y^0\) to \(Y^1\) in such a way that all \(c_1,\ldots,c_n,e_1,\ldots,e_m\), and \(d_1,\ldots,d_m\) from (7.5) for \(w \in I^3_h\) generate \(Y^1\) over \(Y^0\) and \(XY^1\) is a part of a transversal with \(\langle XY^1 \rangle \cap V_G = \langle X \rangle\). \(Y^1\) is fixed pointwise by \(\mathbb{K}\) and \(\mathbb{K}^H\).

Let \(f_i\) be an element of \(X_{V'}^D\). For each \(w\) in \(I^3_h\) either there is no element of type \(p\) with handle \(f_i\) in (7.5) or there is some \(w_i\) that we call \(w_i(w)\). Let \(X_{V'}^D\) be a subset of \(\{w_i(w): w \in I^3_h\}\) such that \(X_{V'}^D, Y_{V'}^D = \{X_{V'}^D\}, Y_{V'}^D\) modulo \(\langle V_G \cup V_{I^3_h} \rangle\). Define \(Y^D = \bigcup_{c \in X_{V'}^D} Y_{V',c}\). The elements of \(Y^D\) are fixed modulo \(V_G\) by the automorphisms of \(\mathbb{K}\) and fixed modulo \(V_{I^3_h}\) by the automorphisms of \(\mathbb{K}^H\), since this was true for the \(w_i\).

We consider again (7.5). By definition \(d = x + \sum_{1 \leq i \leq \ell} w_i\). Hence \(d = x + v + y + y_d\) where \(y_\delta \in \langle Y^D\rangle\), \(y \in \langle Y^1\rangle\), and \(v \in V_{I^3_h}\) modulo \(V_G\). We have \(y = \sum_{1 \leq i \leq \ell} w_i - y_d\). Then \(y\) is fixed by every \(f \in \mathbb{K}\) and \(g \in \mathbb{K}^H\). We have \(f(v) = v\) modulo \(V_{I^3_h}\) for \(f \in \mathbb{K}\) and \(g(v) = v\) modulo \(V_H\) since this was true for the \(w_i\) and \(Y^1\). Then the elements of type \(1^v\) in \(v\), which are not in \(V_G\), and therefore not in \(V_H\), are w.l.o.g. fixed by the automorphisms of \(\mathbb{K}\) and \(\mathbb{K}^H\) (Lemma 6.10). We enlarge \(Y^1\) to \(Y\) such that the \(\approx\)-classes of these elements are represented in \(Y_{I^3_h}\). Now we substract all summands of \(d\) that are from \(Y\) and call the result again \(d\). We have \(d = x_d + y_d\) with \(x_d \in V_G\) is of type \(1^v\) and \(y_d \in \langle Y^D\rangle\). \(y_d = 0\) is impossible since this would imply \(w \in I^3_h\).
Let $D^\square$ be the set of these elements $d$ we obtain from the $w$'s in $I_T^*$ this way via (7.5). We define $X^d_T = \{x_d : d \in D^\square\}$ and $Y^d = X^d_T \cup Y^d_T$. Assume w.l.o.g. that $X_T$ and $Y_T$ contain all connections of the handles. By construction it is easily seen that $(X, Y^d, Y, Y^d, D^\square)$ fulfills the conditions of Theorem 7.4:

$$D \subseteq \langle XX^dYY^d \rangle, \quad D \subseteq \langle XYD^\square \rangle,$$

$XX^dYY^d$ is a part of a transversal that respects $V_G$, $\langle XX^dYY^d \rangle \cap V_G = \langle XX^d \rangle$. Every element $y$ in $YY^d$ is either an element of $D$ or it was the result of an application of Lemma 6.10 in (7.1), (7.3) and (7.5). But if we consider $w - x$ in (7.1), (7.3), and (7.5) then we can apply Lemma 6.1 and obtain the element $y$ as an element of $U(w - x)$. Hence $\langle V_G YY^d \rangle$ is the smallest good subspace that contains $V_G$ and $D$. This proves (i).

Condition (ii) is clear by construction. We obtain that $\langle V_G YY^d \rangle$ is the smallest good subspace that contains $V_G$ and $D^\square$ by a similar argument as above for (i).

Finally, we show (iii) and (iv). Since $D \subseteq \langle XYD^\square \rangle$ we have that $\text{Aut}_{\langle G \rangle}(XYD^\square)(C)$ is a subgroup of $\text{Aut}_{\langle G \rangle}(D^\square)(C)$ and $\text{Aut}_{\langle H \rangle}(XYD^\square)(C)$ is a subgroup of $\text{Aut}_{\langle H \rangle}(D^\square)(C)$. Conversely, $\mathcal{K}$ is a subgroup of finite index of $\text{Aut}_{\langle G \rangle}(D^\square)(C)$. Analogously, $\mathcal{K}^H$ is a subgroup of finite index in $\text{Aut}_{\langle H \rangle}(D^\square)(C)$. In fact, (7.2), (7.4), (7.6) and (7.2)$^H$, (7.4)$^H$, (7.6)$^H$ imply $\mathcal{K} \subseteq \text{Aut}_{\langle G \rangle}(XYD^\square)(C)$ and $\mathcal{K}^H \subseteq \text{Aut}_{\langle H \rangle}(XYD^\square)(C)$. □

**Corollary 7.5.** Let $D$ be a finite good subspace of $V$. Let $(X, X^d, Y, Y^d, D^\square)$ be a platform for $D$. Then the following are true:

(i) For every $f \in \text{Aut}_{\langle G \rangle}(XYD^\square)(C)$ there is some $g \in \text{Aut}_{\langle G \rangle}(D^\square)(C)$ such that $f \upharpoonright V_G = g \upharpoonright V_G$.

Proof. (v) Since $D \subseteq \langle XX^dYY^d \rangle$ we have $d = x(d) + y(d)$ where $x(d) \in \langle XX^d \rangle$ and $y(d) \in \langle YY^d \rangle$. By Corollary 6.4 $U(\{y(d) : d \in D\}) \subseteq \text{acl}(XX^d)$. If $C$ is the set of handles and connections of $YY^d$ in $XX^d$, then $\langle CYD^\square \rangle$ is a good subspace that respects $V_G$ and contains $U(\{y(d) : d \in D\})$. Since $\langle CYD^\square \rangle \cap V_G = \langle C \rangle$ and $U(\{y(d) : d \in D\}) \subseteq \langle CYD^\square \rangle$ we have $U(\{y(d) : d \in D\}) \subseteq V_G \subseteq \langle C \rangle$. If some $y \in YY^d$ is not in $U(\{y(d) : d \in D\})$, then $\langle V_G \cup U(\{y(d) : d \in D\}) \rangle$ is a proper good subspace of $\langle V_G YY^d \rangle$ that contains $D$. This contradicts Theorem 7.4(i).

(vi) By Theorem 7.4(i) we have for every $f \in \text{Aut}_{\langle G \rangle}(D^\square)(C)$ that $f(\langle V_G YY^d \rangle) = \langle V_G YY^d \rangle$. Hence there is some subgroup $\mathcal{K}$ of $\text{Aut}_{\langle G \rangle}(D^\square)(C)$ of finite index fixing $YY^d$ modulo $V_G$ pointwise.

(vii) Theorem 7.4(iii) implies this since $\text{Aut}_{\langle G \rangle}(XYD^\square)(C) \subseteq \text{Aut}_{\langle G \rangle}(XYD^\square)(C)$. To show this let $h$ be an element of $\text{Aut}_{\langle G \rangle}(XYD^\square)(C)$. For $d \in D^\square$ we have $d = x_d + y_d$ with $x_d \in X^d_T$ and $y_d \in \langle Y^d \rangle$, as $(Y^d, Y^d, D^\square)$ is special. So $d = h(d)$ implies $h(y_d) =$
$y_d = x_d - h(x_d) \in V_G$. Furthermore, we have $h(x) = x$ for $x \in X_D^D \subseteq X$. Hence for $y \in Y^D$ the elements $y$ and $h(y)$ are elements of type $p$ with the same handle. The map $j(x) = x$ for $x \in X_D^D$ and $j(y) = y - h(y)$ for $y \in Y^D$ induces an $\mathbb{B}_p$-homomorphism $\tilde{j}$ of $(X_D^D)^G$ into $\mathcal{F}(G)$. Hence $h(x_d) = x_d + \tilde{j}(y_d)$ and therefore $(x_d : d \in D^D)$ and $(h(x_d) : d \in D^D)$ are in the same $\theta$-class: $h(\Box) = \Box$.

To show (viii) let $f$ be an automorphism in $\text{Aut}_{(G)}(X_D^D)(C)$. We have to find an extension $g$ of $f \upharpoonright G$ with $f(d) = d$ for $d \in D^D$. Then the assertion follows from $D \subseteq \langle XYD_D \rangle$. Again we use $d = x_d + y_d$ with $x_d \in X_D^D$ and $y_d \in \langle Y_D \rangle$. Since $(x_d : d \in D^D)$ and $(f(x_d) : d \in D^D)$ are in the same $\theta$-class $x_d - f(x_d) \in V_G$ is an homomorphic image $\tilde{j}(y_d)$, where $j$ is an $\mathbb{B}_p$-homomorphism of $(X_D^D)^{\Box}$ into $\mathcal{F}(G)$ with $\tilde{j}(x) = x$ for $x \in X_D^D$. We define the desired $g$ for a transversal $X^0Y^0Y^*$. Then $X^0Y^0X$ be any transversal for $V_G$ and $X^0Y^0Y^*$ be a part of a transversal of the smallest good subspace that contains $V_G \cup D$. On $X^0Y^* \ g$ is defined like $f$ and for $y \in Y^D$ we define $g(y) = y + \tilde{j}(y)$. By Lemma 4.9(ii) we define an automorphism of $C$ in this way. For this it is important that $y$ and $g(y)$ are proper elements of type $p$ with the same handle. Then $g(d) = g(x_d) + g(y_d) = f(x_d) + y_d + \tilde{j}(y_d) = f(x_d) + y_d + x_d - f(x_d) = x_d + y_d = d$ for $d \in D^D$ as desired. □

**Corollary 7.6.** Let $G$ be a saturated model of $T_5$, $G \preceq C$, and $D$ be a good subspace of $V$. Then

$$
\text{acl}(\text{Cb}(\text{tp}(D/G))) = \text{acl}(\text{Cb}(\text{tp}(XYD_D/G))) = \text{acl}(\text{Cb}(\text{tp}(XY^0\Box/G))) = \text{acl}(\text{Cb}(\text{tp}(Y/G)) \cup X \cup \Box) = \text{acl}(\text{Cb}(\text{tp}(Y^0\Box/G)) \cup X \cup \Box).
$$

**Proof.** By Facts 7.1 and 7.2 and Theorem 7.4(iii) we have the first equality. The second equality follows from Corollary 7.5(vii) and (viii). The next equality follows from Fact 7.3 and the last from Lemma 4.14. □

Finally, we show the main assertion for good subspaces of $V$.

**Corollary 7.7.** Assume $S$ is stable and $CM$-trivial. Let $G \preceq H \preceq C \models T_5$ be saturated models of $T_5$ and $D$ be a finite good subspace of $V$ such that $\text{acl}(D \cup G) \cap H = G$. Then $\text{Cb}(\text{tp}(D/G)) \subseteq \text{acl}(\text{Cb}(\text{tp}(D/H)))$.

**Proof.** Let $(X, X^D, Y, Y^D, D^D, \Box)$ be the platform for $D$ with respect to $G$ given by Theorem 7.4. By Corollary 6.6 we have to show $\text{Cb}(\text{tp}(Y^\Box/G)) \cup X \cup \Box \subseteq \text{acl}(\text{Cb}(\text{tp}(D/H)))$.

By the last assertion in Theorem 7.4 there is a subgroup $\mathbb{K}$ of finite index in $\text{Aut}_{(H)}(C)$ such that all $f \in \mathbb{K}$ fix $X$, $Y$, and $D^D$ pointwise. As in the proof of Corollary 7.5(vii) it follows $f(\Box) = \Box$. Since $S$ is $CM$-trivial $\text{Cb}(\text{tp}(Y^\Box/G)) \subseteq \text{acl}(\text{Cb}(\text{tp}(Y^\Box/H))) \subseteq \text{acl}(\text{Cb}(\text{tp}(Y^\Box/H)))$. □
8. The tame case

The aim of this paper is to show:

**Theorem 8.1.** Let $S$ be stable and CM-trivial. Assume that $G \trianglelefteq H \trianglelefteq C$ are saturated models of $T_S$ and $A$ is a finite subgroup of the monster model $C$ such that

$$\text{acl}(A \cup G) \cap H = G.$$ 

Then $\text{Cb}(\text{tp}(A/G)) \subseteq \text{acl}(\text{Cb}(\text{tp}(A/H))).$

**Definition.** In the situation of Theorem 8.1 we say that $A$ is tame if

$$A \cap (G \cdot C') = (A \cap G)(A \cap C').$$

We show Theorem 8.1 for the tame case. The lemmas we prove are also useful for the general case. By Theorem 6.9 we get

**Lemma 8.2.** There is a finite good subspace $D_0 \subseteq V$ such that

$$A \cap C' \subseteq \text{dcl}(D_0) \quad \text{and} \quad D_0 \subseteq \text{acl}(A \cap C').$$

Now we work in the general situation that $G \trianglelefteq C$ are any nilpotent groups of class 2 and exponent $p.$

**Lemma 8.3.** Assume $G \trianglelefteq C$ are in $\mathbb{G}_{2,p},$ $G$ is saturated, and $\text{Th}(C)$ is stable. Furthermore, assume $A \cap (G \cdot Z(C)) = A \cap (G \cdot C'),$ $c_1, \ldots, c_n \in G$ ($c_1 = 1$ is possible), and $d_1, \ldots, d_n \in Z(C)$ linearly independent modulo $G \cdot C'.$ Then $\text{Cb}(\text{tp}(A \cup \{c_1d_1, \ldots, c_nd_n\})/G) = \text{dcl}(\text{Cb}(\text{tp}(A/G)) \cup \{c_1/Z(C), \ldots, c_n/Z(C)\}).$

**Proof.** Every $f \in \text{Aut}_{\{G\cup\{c_1d_1, \ldots, c_nd_n\}\}}(C)$ fixes $\text{Cb}(\text{tp}(A/G))$ and $\{c_1/Z(C), \ldots, c_n/Z(C)\}$ pointwise. Hence

$$\text{Cb}(\text{tp}(A/G)) \cup \{c_1/Z(C), \ldots, c_n/Z(C)\} \subseteq \text{Cb}(\text{tp}(A \cup \{c_1d_1, \ldots, c_nd_n\}/G)).$$

To prove the converse let $g$ be an automorphism of $G$ that fixes each $c_i/Z(G)$ and $\text{Cb}(\text{tp}(A/G))$. It is sufficient to find an extension $f^* \in \text{Aut}_{\{G\}}(C)$ of $g$ that fixes $A \cup \{c_1d_1, \ldots, c_nd_n\}$ pointwise. Assume $X$ is a basis for $G$ modulo $Z(G).$ There is a basis $XY$ for $C$ modulo $Z(C).$ Then $G = \langle X \rangle \oplus K^G$ and $C = \langle XY \rangle \oplus K^G \oplus K$ where $K^G$ and $K$ are elementary abelian $p$-groups. Since $A \cap (G \cdot Z(C)) = A \cap (G \cdot C')$ we have $A \subseteq \langle XY \rangle \oplus K^G.$ Since $g$ fixes $\text{Cb}(\text{tp}(A/G))$ there is an $f \in \text{Aut}_{\{G\\}}(C)$ that extends $g$ such that $f(a) = a$ for $a \in A.$ We have

$$f(C' \oplus K^G) = C' \oplus K^G.$$ 

Now $d_i = a_ib_i$ where $a_i \in C' \oplus K^G$ and $b_i \in K.$ By assumption $b_1, \ldots, b_n \in K$ are linearly independent. $c_ia_i$ and $f(c_ia_i)$ are in $\langle XY \rangle \oplus K^G.$ Now we define $f^* \in \text{Aut}_{\{G\}}(C)$ in
such a way that \( f^*(a) = f(a) \) for \( a \in \langle XY \rangle \oplus K^G \) and
\[
f^*(b_i) = f(c_i a_i)^{-1} c_i a_i b_i = f(c_i)^{-1} c_i f(a_i)^{-1} a_i b_i.
\]
Note \( f(c_i)^{-1} c_i \in G \cap Z(\mathbb{C}) = Z(G) \) since \( g \) fixes \( c_i / Z(\mathbb{C}) \) and \( f(a_i)^{-1} a_i \in \mathbb{C}' \cdot Z(G) \) by definition. \( f^* \) extends \( g \) and fixes \( A \) pointwise since \( A \subseteq \langle XY \rangle \oplus K^G \). Finally, \( f^*(c_i d_i) = f^*(c_i a_i b_i) = f^*(c_i a_i) f^*(b_i) = f(c_i a_i) f(c_i a_i)^{-1} (c_i a_i) b_i = c_i a_i b_i = c_i d_i \), as desired. \( \square \)

**Lemma 8.4.** Let \( G \leq \mathbb{C} \) be in \( \mathbb{G}_{2,p} \). For every finite subgroup \( A \) of \( \mathbb{C} \) there is a subgroup \( B \subseteq A \) such that \( B \cap G = B \cap G' \), \( B \cap (G \cdot Z(\mathbb{C})) = B \cap (G \cdot \mathbb{C}') \), and
\[
A = \langle B \cup (A \cap G) \cup \{c_1 d_1, \ldots, c_n d_n\} \rangle,
\]
where \( c_1, \ldots, c_n \in G \) and \( d_1, \ldots, d_n \in Z(\mathbb{C}) \) are linearly independent modulo \( G \cdot \mathbb{C}' \). Moreover, \( A \) is tame if and only if \( B \) is tame.

**Proof.** We consider the subgroups \( \mathbb{C}' \subseteq G \cdot \mathbb{C}' \subseteq G \cdot Z(\mathbb{C}) \) of \( \mathbb{C} \). Since \( G \leq \mathbb{C} \) we have \( \mathbb{C}' \cap G = G' \). Let \( X_1 \) be a basis of \( A \) modulo \( G \cdot Z(\mathbb{C}) \). Let \( X_2 \) be a basis of \( A \cap G \cdot Z(\mathbb{C}) \) modulo \( G \cdot \mathbb{C}' \). Let \( X_3 \) be a basis of \( A \cap (G \cdot \mathbb{C}') \) modulo \( (A \cap G)(A \cap \mathbb{C}') \). Let \( B \) be \( \langle X_1 \cup X_2 \cup (A \cap \mathbb{C}') \rangle \). Let \( u_0, \ldots, u_{r-1} \) be the elements of \( X_1 \), \( v_0, \ldots, v_{s-1} \) be the elements of \( X_3 \), and \( w_0, \ldots, w_{t-1} \) be a basis of \( A \cap \mathbb{C}' \). Every element of \( B \) can be written as \( \prod_{i<k} u_i^{a_i} \prod_{j<i} w_i^{b_i} \prod_{k<i} v_i^{c_i} \), since every product of elements of \( X_1 \cup X_3 \cup \{w_0, \ldots, w_{t-1}\} \) can be brought into this form using some additional elements of \( (A \cap \mathbb{C}') \). It follows \( B \cap G = A \cap \mathbb{C}' \cap G = A \cap G' \). Hence \( B \cap G = B \cap G' \). Furthermore by construction \( \langle B \cup (A \cap G) \cup X_2 \rangle = A \) and \( B \cap (G \cdot Z(\mathbb{C})) = B \cap (G \cdot \mathbb{C}') \). \( A \) is tame if \( X_3 = \emptyset \) iff \( B \) is tame. \( \square \)

**Corollary 8.5.** Let \( G \leq \mathbb{C} \) be groups in \( \mathbb{G}_{2,p} \), \( G \) is saturated, and \( Th(\mathbb{C}) \) is stable. In the situation of Lemma 8.4 we have
\[
\text{Cb}(tp(A/G)) = \text{dcl}[(A \cap G) \cup \text{Cb}(tp(B/G)) \cup \{c_1/Z(\mathbb{C}), \ldots, c_n/Z(\mathbb{C})\}].
\]

**Proof.** This follows from Fact 7.3 and Lemma 8.3. \( \square \)

**Lemma 8.6.** Assume \( G \leq H \leq \mathbb{C} \) are in \( \mathbb{G}_{2,p} \) and \( B \) is a subgroup of \( \mathbb{C} \) with \( B \cap G = B \cap G' \), \( B \cap (G \cdot Z(\mathbb{C})) = B \cap (G \cdot \mathbb{C}') \), and \( \text{acl}(B \cup G) \cap H = G \). Then \( B \cap H = B \cap G' \) and \( B \cap (H \cdot Z(\mathbb{C})) = B \cap (G \cdot \mathbb{C}') \).

**Proof.** By assumption we have \( B \cap H = B \cap G = B \cap G' \). If \( c \cdot d \in B \cap (H \cdot Z(\mathbb{C})) \) with \( c \in H \) and \( d \in Z(\mathbb{C}) \), then \( c/Z(\mathbb{C}) \in H \cap \text{acl}(B) \subseteq G \). Hence \( c = c_0 d_0 \) with \( c_0 \in G \) and \( d_0 \in Z(\mathbb{C}) \). We have \( cd = c_0 d_0 d \in (G \cdot Z(\mathbb{C})) \cap B \) and therefore \( cd \in (G \cdot \mathbb{C}') \cap B \). \( \square \)

**Corollary 8.7.** Let \( G \leq H \leq \mathbb{C} \) be in \( \mathbb{G}_{2,p} \), \( G \) and \( H \) are saturated, and \( Th(\mathbb{C}) \) is stable. If we want to show Theorem 8.1 for finite (tame) subgroups, then it is sufficient to
show it for finite (tame) subgroups $A$ with $A \cap G = A \cap G'$ and $A \cap (G \cdot Z(C)) = A \cap (G \cdot C')$.

**Proof.** Assume an arbitrary (tame) $A$ is given. We choose $B$ according to Lemma 8.4. We have $acl(B \cup G) \cap H \subseteq acl(A \cup G) \cap H = G$ since $B \subseteq A$. (If $A$ is tame, then $B$ is tame.) Since Theorem 8.1 is true for $B$ by assumption we obtain

$$Cb(tp(B/G) \subseteq acl(Cb(tp(B/H))).$$

By Corollary 8.5

$$Cb(tp(A/G)) = dcl[(A \cap G) \cup Cb(tp(B/G)) \cup \{c_1/Z(C), \ldots, c_n/Z(C)\}].$$

Since $(A \cap G) \cup \{c_1/Z(C), \ldots, c_n/Z(C)\} \subseteq Cb(tp(A/H))$, the assertion follows. 

Now we return to Mekler groups.

**Theorem 8.8.** Let $S$ be stable and $CM$-trivial. Let $G \leq H \leq C = T_S$ be saturated. Assume $A$ is finite subgroup of $C$, $A$ is tame with respect to $G$, $A \cap G = A \cap G'$, $A \cap (G \cdot Z(C)) = A \cap (G \cdot C')$, and $acl(A \cup G) \cap H \subseteq G$. Then there is some finite good subspace $D$ of $V$ such that $D \subseteq acl(A)$, $acl(Cb(tp(A/G))) = acl(Cb(tp(D/G)))$, and $acl(Cb(tp(A/H))) \subseteq acl(Cb(tp(D/H)))$. We have

$$Cb(tp(A/G)) \subseteq acl(Cb(tp(A/H))).$$

**Proof.** By assumption $A \cap (G \cdot Z(C)) = A \cap (G \cdot C') = (A \cap G)(A \cap C') = (A \cap G')(A \cap C') = A \cap C'$. Hence $A = \langle (A \cap C') \cup E \rangle$ where $E$ is a basis of $A$ modulo $A \cap G \cdot Z(C)$. Let $\bar{E}$ be the image of $E$ in $V = C/Z(C)$. By Theorem 6.9 for every $a \in (A \cap C')$ there is a good subspace $U(a) \subseteq V$ such that $a \in \beta(U(a),U(a))$ and $U(a) \subseteq acl(a)$. By Corollary 6.4 there is a smallest good subspace $D \subseteq V$ such that $\bar{E} \cup \bigcup \{U(a) : a \in A \cap C'\} \subseteq D$ and $D \subseteq acl(\bar{E} \cup \bigcup \{U(a) : a \in A \cap C'\})$. $D$ is finite. Hence $D \subseteq acl(A)$ and therefore

$$Cb(tp(D/G)) \subseteq acl(Cb(tp(A/G))).$$

and

$$Cb(tp(D/H)) \subseteq acl(Cb(tp(A/H))).$$

To show

$$acl(Cb(tp(D/G))) = acl(Cb(tp(A/G))),$$

let $g$ be any automorphism of $C$ that fixes $D$ pointwise and $G$ setwise. Let $\mathcal{F}(g)$ be the $H_p$-automorphism of $(V,W,\beta)$ induced by $g$. Then $\mathcal{F}(g)$ fixes $D$ and therefore
A ∩ C′ ⊆ β(D, D) and E ⊆ D pointwise. By Corollary 3.3 there is an automorphism f ∈ Aut\(_{(G)}\)(C) such that g|G = f|G, \(\mathcal{F}(f) = \mathcal{F}(g)\) and f(e) = e for e ∈ E. \(\mathcal{F}(f) = \mathcal{F}(g)\) implies f(a) = a for a ∈ A ∩ C'. Hence f ∈ Aut\(_{(G)}\)(C) with f|G = g|G, as desired.

We have acl(D ∪ G) ∩ H = G. Hence by Corollary 7.7 Cb(tp(D = G)) ⊆ acl(Cb(tp(D = H))). This implies the last assertion of Theorem 8.8. □

**Corollary 8.9.** Theorem 8.1 is true for finite tame subgroups A.

**Proof.** Apply Corollary 8.7 and Theorem 8.8. □

### 9. Mixed elements

In Theorem 8.8 we have excluded elements a in A with a ∉ G, a ∉ C' but a ∈ G · C'.

We call these elements mixed elements with respect to G.

As before we consider saturated T\(_3\)-models G ≪ H ≪ C, where C is a monster model. Let A be a finite subgroup of C such that acl(A ∪ G) ∩ H = G.

We only work with mixed elements under the assumption that

\[
A ∩ (G · Z(C)) = A ∩ (G · C').
\]

(9.1)

If a ∈ A is a mixed element with respect to H, then a/Z(C) ∈ H ∩ acl(A). Hence a/Z(C) ∈ G, a ∈ G · Z(C) and therefore by (9.1) a ∈ G · C'. We have shown that a is a mixed element with respect to G. If a is a mixed element with respect to G, then a is a mixed element with respect to H. From now we will speak about mixed elements. Our aim is to show the following result.

**Theorem 9.1.** Let a be a mixed element of A. Then there are elements a\(_{G}\) ∈ G, a′ ∈ C' and a part of a transversal \(XU\) that respects G with \(⟨XU⟩ ∩ V_G = ⟨X⟩\) such that

\[
a = a_{G} · \prod_{x ∈ X} {x, u}^{x_{u}} a'.
\]

and the following is true:

There are a subgroup \(K^G\) of Aut\(_{(G)}\)(C) of finite index and a subgroup \(K^H\) of Aut\(_{(H)}\)(C) of finite index such that for every \(f ∈ K^G\) and every \(g ∈ K^H\) we have \(f(a') = a', g(a') = a', f(x) = x, g(x) = x\) for \(x ∈ X\), and \(f(u) = u \mod V_G\) and \(g(u) = u \mod V_H\) for \(u ∈ U\).

The elements of U are not of type Y. If they are of type p, then their handle is in X. U is linearly independent modulo V_H.

By Theorem 9.1 we can assume w.l.o.g. that a′ ∈ A for a mixed element a ∈ A. That means we can restrict us to mixed elements of the form \(a^{G} \prod_{x ∈ X} [x, u]^{x_{u}}\).
We will study mixed elements to prove Theorem 9.1. Assume \( a = cd = c_1d_1 \) where \( c, c_1 \in G \) and \( d, d_1 \in C' \). Then \( dd_1^{-1} = c^{-1}c_1 \in G \cap C' = G' \). Here we use \( G \subseteq C \). Hence \( d \) is uniquely determined modulo \( G' \). By Theorem 6.9 there is a finite good subspace \( D \subseteq V \) with \( d \in \beta(D,D) \) and \( D \subseteq \text{acl}(d) \). \( D \) is the smallest good subspace such that \( d \in \beta(D,D) \). It follows that \( \text{Aut}_{\{d\}}(\mathbb{C}) \) and \( \text{Aut}_{\{D\}}(\mathbb{C}) \) are commensurable. By Theorem 7.4 there is a platform \( (X,Y,D,Y,D,D) \) for \( D \) with respect to \( G \). Then \( \langle V_G YD, Y \rangle \) is the smallest good subspace that contains \( V_G \) and \( D \). If \( f \in \text{Aut}_{\{G\}}(\mathbb{C}) \), then \( a = cd = f(c) \cdot f(d) \). Then \( f(D) \) is the smallest good subspace with \( f(d) \in \beta(f(D), f(D)) \) and \( f(D) \subseteq \text{acl}(f(d)) \). Since \( d = f(d) \) modulo \( G' \) it follows \( f(d) \in \beta((V_G D) \cap \langle V_G D \rangle) \). Hence we have \( f(D) \subseteq \langle V_G YD, Y \rangle \). Since \( f((\langle V_G YD, Y \rangle) \) is the smallest good subspace that contains \( V_G \) and \( f(D) \) we have \( f((\langle V_G YD, Y \rangle) \subseteq (\langle V_G YD, Y \rangle) \). By similar arguments \( D \subseteq f((\langle V_G YD, Y \rangle) \) and \( f((\langle V_G YD, Y \rangle) = (\langle V_G YD, Y \rangle) \). It follows the existence of a subgroup \( \mathbb{K} \) of \( \text{Aut}_{\{G\}}(\mathbb{C}) \) of finite index such that

\[(a) \quad f(y) = y \text{ modulo } V_G \text{ for all } y \in Y \] and all \( f \in \mathbb{K} \).

Let \( (X^H, X^H Y^H, Y^H, Y^H, D^H, D^H) \) be a platform for \( \langle XYD \rangle \) with respect to \( H \). The smallest good subspace that contains \( V_G \) and \( D \) is \( \langle V_G YD \rangle \) and it contains \( \langle XYD \rangle \). Hence the smallest good subspace that contains \( V_H \) and \( D \) contains \( \langle XYD \rangle \). Since \( D \subseteq \langle XYD \rangle \) the subspace \( \langle V_H Y^H Y^H \rangle \) is the smallest good subspace that contains \( V_H \) and \( D \). By Theorem 7.4 for every saturated \( H \) with \( H \subseteq H \) and \( \text{acl}(D \cup H) \cap H = H \) \( \text{Aut}_{\{H\}}(\mathbb{C}) \) and \( \text{Aut}_{\{H\}}(\mathbb{C}) \) are commensurable. Note \( \text{acl}(D \cup H) \cap H = H \) implies \( \text{acl}(D \cup G) \cap H \subseteq H \) and therefore \( \text{acl}(D \cup G) \cap H = H \). Since \( (X^H, X^H Y^H, Y^H, Y^H, D^H, D^H) \) is a platform for \( \langle XYD \rangle \) with respect to \( H \) we get that \( \text{Aut}_{\{H\}}(\mathbb{C}) \) and \( \text{Aut}_{\{H\}}(\mathbb{C}) \) are commensurable. Hence \( (X^H, X^H Y^H, Y^H, Y^H, D^H, D^H) \) is a platform for \( D \) with respect to \( H \).

Let \( f \in \text{Aut}_{\{H\}}(\mathbb{C}) \). Then \( f(d) = d \text{ modulo } H \). Hence

\[\langle V_H \cup D \rangle = \langle V_H \cup f(D) \rangle = f(\langle V_H \cup D \rangle) \text{ as above.} \]

As shown above \( \langle V_H \cup Y^H \cup Y^H \rangle \) is the smallest good subspace that contains \( V_H \) and \( D \). Similarly, \( f((\langle V_H \cup Y^H \cup Y^H \rangle) \) is the smallest good subspace that contains \( V_H \) and \( f(D) \). Since \( \langle V_H \cup D \rangle = \langle V_H \cup f(D) \rangle \)

\[f((\langle V_H \cup Y^H \cup Y^H \rangle) = (\langle V_H \cup Y^H \cup Y^H \rangle). \]

Note that

\[YY \subseteq (\langle V_G YD \rangle \subseteq (\langle V_H YD \rangle \subseteq (\langle V_H Y^H Y^H \rangle). \]

Hence there exists a subgroup \( \mathbb{K}^H \) of \( \text{Aut}_{\{H\}}(\mathbb{C}) \) of finite index such that

\[(a)^H f(y) = y \text{ modulo } V_H \text{ for every } y \in Y \] and all \( f \in \mathbb{K}^H \).

Using (a), Lemma 6.10, and Corollary 4.16 we obtain a subgroup \( \mathbb{K} \) of \( \text{Aut}_{\{G\}}(\mathbb{C}) \) of finite index such that (a) is true and for every \( f \in \mathbb{K} \).
(b) $f(y) = y$ for every $y \in Y_f$ and every $y \in Y_{p,e}$ with $e \in Y_f$,
and $f(x) = x$ for every handle $x \in X$ of some $y \in Y_{p,x} \cup Y_{p,x}^\square$.
By Corollary 7.5
$$YY^\square \subseteq \text{acl}(XX^\square D) \subseteq \text{acl}(GD).$$
Hence $YY^\square$ is linearly independent modulo $V_H$. Then we can show:

(b) $H$

There is a subgroup $K_H$ of $\text{Aut} \{H\} \subseteq (\mathbb{C})$ of finite index such that $(a)^H$ is true and for every $g \in K_H$ we have
$$g(y) = y \text{ for } y \in Y_f \cup \bigcup_{e \in Y_f} Y_{p,e}$$
and
$$g(x) = x \text{ for every handle } x \text{ of some } y \in Y_{p,x} \cup Y_{p,x}^\square$$
and every root $x$ of some $y \in Y_f$.

Condition (b) $H$ follows from (a) $H$ by Lemma 6.10 and Corollary 4.16 in most cases. The critical case concerns the handle $x$ of some $y \in Y_{p;x} \cup Y_{p;x}^\square$ with $y \in \langle V_H \rangle$. Then $y = \sum_{1 \leq i \leq r} v_i$ modulo $V_H$ where the $v_i$ are $\approx$-inequivalent elements of type $1^r$ not in $V_H$. The handle $e$ of $y$ is an element of $V_H$. We have $\beta(e, v_i) = 0$. By Lemma 6.10 we can choose $K_H$ such that $g(v_i) = v_i$, whence $g(e) = e$ for $g \in K_H$ by Corollary 4.16.

We call $(X, Y, Y^\square, D^\square)$ a support for $d$ with respect to $G$ if

(i) $XX^\square YY^\square$ is a part of a transversal that respects $V_G$,
(ii) $\langle XX^\square YY^\square \rangle \cap V_G = \langle XX^\square \rangle$, $X \cap Y = X^\square \cap Y = \emptyset$,
(iii) $D^\square$ is linearly independent modulo $(V_G Y, X^\square = X^\square_1 \cup X^\square_2, Y^\square = Y^\square = \bigcup_{e \in X^\square_1} Y_{p,e}$, for $e \in X^\square_1$ the set $Y_{p,e}^\square \neq \emptyset$, and every $d \in D^\square$ has the form $d = x_d + y_d$,

where $x_d \in X^\square_1$ and $y_d \in \langle Y^\square \rangle$.

(iv) $d$ is a linear combination of commutators $\beta(u, v)$ with $u, v \in XYD^\square$.

(v) $YY^\square \subseteq \text{acl}(aG) = \text{acl}(dG)$.

Note that a support is not necessarily a platform.

We are interested in all situations $a = c \cdot d$, $(X, X^\square, Y, Y^\square, D^\square), K, K^H$ where $(X, X^\square, Y, Y^\square, D^\square)$ is a support of $d$ and (a), (a) $H$, (b), (b) $H$ are true. We consider $XYD^\square$ as an ordered sequence, where $<$ denotes this ordering. After moving some summands $s \beta(x_1, x_2)$ with $x_1, x_2 \in X$ to $c$, we can replace (iv) by the following:

$$d = \sum_{x \in X} s_{x,y} \beta(x, y) + \sum_{y_1 < y_2} s_{y_1,y_2} \beta(y_1, y_2) \quad (9.2)$$

Our aim is to show that we can choose $d$, the support $(X, X^\square, Y, Y^\square, D^\square), K, K^H$ in such way that (a), (a) $H$, (b), (b) $H$, (9.2) are true and furthermore

(c) $f(x) = x$ for all $x \in X$ and $f \in K$.
and
\[(c)^H \quad g(x) = x \text{ for all } x \in X \text{ and } g \in K^H.\]

We need further refinements to get (c) and \((c)^H\). After deleting some elements of \(X\) we can assume:

For every \(x \in X\) we have \(s_{xy} \beta(x, y) \neq 0\) for some \(y \in Y\) in (9.2) or \(x\) is a handle of some element of type \(p\) in \(XXY\) or \(x\) is a connection between two handles.

If \(\beta(x, y) = 0\) for \(x\), then \(s_{xy} = 0\).

Finally, we consider only situations such that

\[X^c = \{x \in X : s_{xy} \beta(x, y) \neq 0 \text{ for some } y \in Y \text{ and } \beta(x, y) \neq 0 \text{ for all } y \in Y\}\]

is of minimal size.

Let us consider again

\[d = \sum_{x \in X} s_{xy} \beta(x, y) + \sum_{y_1 < y_2} s_{y_1 y_2} \beta(y_1, y_2). \quad (9.2)\]

If we use \(D \subseteq \langle X Y \rangle\), then we obtain

\[d = \sum_{x \in X} r_{xy} \beta(x, y) + \sum_{y_1 < y_2} r_{y_1 y_2} \beta(y_1, y_2) \mod (\beta(V_G, V_G)). \quad (9.2)^*\]

Since \(XXY\) is a part of a transversal that respects \(V_G\) we have that \(\{\beta(v_1, v_2) : \beta(v_1, v_2) \neq 0, v_1, v_2 \in X \\mod Y\} \subseteq \langle Y \rangle\). Hence

\[\sum_{y \in Y} r_{xy} \beta(x, y) = \sum_{y \in Y} s_{xy} \beta(x, y) \neq 0 \mod (\beta(V_G, V_G))\]

for \(x \in X^c\). We get that

\[X^c = \left\{x \in X : \sum_{y \in Y} r_{xy} \beta(x, y) \neq 0 \mod (\beta(V_G, V_G)) \text{ and } \beta(x, y) \neq 0 \text{ for all } y \in Y\right\}.\]
By property (v) of a support we have $YY \subseteq acl(aG)$ and therefore

$$\{\beta(x, y) : y \in YY, x \in X\} \subseteq acl(aG).$$

Since $acl(aG) \cap H = G$ we have that $\{\beta(x, y) : y \in YY\}$ is linearly independent modulo $\beta(V_H, V_H)$ for $x \in X^e$. Hence,

$$X^e = \left\{ x \in X : \sum_{y \in YY} r_{xy} \beta(x, y) \neq 0 \text{ modulo } \langle \beta(V_H, V_H) \rangle \text{ and } \beta(x, y) \neq 0 \text{ for all } y \in YY \right\}.$$ 

We can reformulate (9.4) in the following way: $(X, X^\square, Y, Y^\square, D^\square)$ is chosen in such a way that with respect to (9.2)$^\square$ we have that

$$X^e = \left\{ x \in X : \sum_{y \in YY} r_{xy} \beta(x, y) \neq 0 \text{ modulo } \langle \beta(V_H, V_H) \rangle \text{ and } \beta(x, y) \neq 0 \text{ for all } y \in YY \right\}$$

is of minimal size. (9.4)$^+$

Under all the assumptions above we give

**Proof of (c).** Let $Q$ be a part of a $1^\prime$-transversal that is up to $\approx$-equivalence the set of roots of $Y_T$. We assume that elements of $Q$ that represent $\approx$-classes in $XX^\square$ are elements of $XX^\square$. See Corollary 4.16. W.l.o.g. we can assume

$$(c)^{\text{weak}}$$

Let $x$ be an element of $XQ$. If $f(x) \in \langle XX^\square Q \rangle$ for all $f \in K$, then

$$f(x) = x \text{ for all } f \in K.$$ 

By (b) and the definition of $Q$ for all $f \in K$ we have $f(x) = x$, if $x \in Q$, or $x$ is a handle of an element in $Y_p$ or $Y^\square_p$. Note that $\beta(x, y) = 0$ for some $y \in YY^\square$ implies that $x$ is a root or $x$ is a handle. Hence it is sufficient to show that $f(x) = x$ for $f \in K$ and $x \in X^e$. Then we can choose $K$ such that $K$ fixes also handles of elements of $X_p$ and connections. By (9.3) it follows (c). Our main argument will be that $f(d) = d$ modulo $\beta(V_G, V_G)$ for $f \in K$. By $(c)^{\text{weak}}$ it is sufficient to show that $f(x) \in \langle XX^\square Q \rangle$. 
Hence we assume that:

(*) There are some \( f \in K \) and \( x \in X^c \) such that

\[
f(x) \notin \langle XX^c Q \rangle.
\]

Under our assumptions we show that:

There are a transversal \( \tilde{X}Y \) of \( V, x, u \) in \( \tilde{X} \), and \( f \in \mathbb{K} \) such that \( \tilde{X} \) is a transversal of \( V_G, QXX^c \subseteq \tilde{X}, YY^c \subseteq \tilde{Y}, x \in X^c, u \in \tilde{X} \setminus XX^c Q, \)

\[
f(x) = u + v \text{ with } v \in \langle \tilde{X} \setminus \{u\} \rangle.
\]

and for all \( x' \in XX^c \setminus \{x\} \) we have

\[
f(x') \in \langle \tilde{X} \setminus \{u\} \rangle.
\]

Furthermore \( \beta(u, y) \neq 0 \) for all \( y \in YY^c \).

The last assertion follows since all roots and handles of \( YY^c \) are in \( \langle XX^c Q \rangle \) and \( u \) is outside. To prove that we can assume (9.5) we distinguish several cases. \( f \) and \( x \) are given by (*). We choose a transversal \( \tilde{X} \tilde{Y} \) for \( V \) where \( \tilde{X} \) is a transversal of \( V_G, XX^c Q \subseteq \tilde{X}, \) and \( YY^c \subseteq \tilde{Y} \). We will only vary \( X \) and \( \tilde{X} \) to get (9.5).

**Case 1:** \( x \in X^c \). We choose \( \tilde{X} \) in such a way that \( f(x) \in \tilde{X} \setminus \langle XX^c Q \rangle \). Let \( u \) be \( f(x) \).

If \( tu \) with \( t \neq 0 \) occurs in the representation of \( f(x') \) for \( x' \neq x \) in \( XX^c \), then we replace \( x' \) by \( x' - tx \). We call the result of this process \( \tilde{X}^*X^c \). We have to show that \( X^*X^c \) is again a part of a transversal. Note that \( X^*_U = X^* \) since \( f \) induces a permutation of the \( \approx \)-classes of the elements of \( X^*_U \).

It remains to consider \( x' \in X^c_p,e \) with \( tu \) in the representation of \( f(x') \) over \( \tilde{X} \). Note that there is no \( X^c_p,e \). We have to show \( \beta(x, e) = 0 \). Since \( tu \) occurs in \( f(x') \) we have \( \beta(u, f(e)) = 0 \). Hence \( \beta(x, e) = 0 \), as desired.

Hence \( X^*X^c \) is a part of a transversal and \( X^*_p,e = X^*_U \). If we replace \( X \) by \( X^* \), \( X^c \)

by \( X^*X^c \), and \( \tilde{X} \) by \( (\tilde{X} \setminus XX^c Q) \cup X^*X^c \), then \( (X^*,X^cX^c, Y, Y^c, D) \) is a support for \( d \),

(a), \( (a)^H \), (b), \( (b)^H \), and (9.2) remain true. By (9.4) for \( X \) it follows (9.3) and (9.4) for \( X^* \).

**Case 2:** \( x \in X^c_p,e \) and (c) is true for all \( x' \in X^c_p,e \). Remember that for handles \( v \in X^*_U \)

with \( Y_{p,e, v} \neq 0 \) or \( Y_{p,e, v} \neq 0 \) we know \( f(v) = v \) by (b).

**Case 2.1:** \( f(x) \notin \langle XX^c Q \rangle \). This case includes the case \( f(e) \neq e \), since \( f(e) \neq e \) and (c)\(^{\text{weak}} \) imply \( f(e) \notin \langle XX^c Q \rangle \). Then \( f(x) \) is a proper element of type \( p \) with a handle

\( f(e) \notin \langle XX^c Q \rangle \). The proper elements of type \( p \) in \( \langle XX^c Q \rangle \) have a handle in \( X \) since they are of the form \( w + v \) with \( w \in X_p \) and \( v \in V \). We define \( u = f(x) \) and choose \( \tilde{X} \) such that \( u \in \tilde{X} \). The element \( u \) is proper of type \( p \). It cannot occur in the representation of \( f(x') \) for \( x' \in X^*_U \) because these elements are in \( \tilde{X}^*_U \). Let \( x' \in XX^c \) be such that \( u \) occurs in the representation of \( f(x') \). Assume \( x' \in X^c_p,e \). Then \( \beta(u, f(e')) = 0 \). We have \( \beta(u, f(e)) = 0 \). Hence \( f(e) \approx f(e') \) since \( u \) was a proper element of type \( p \). We get \( e \approx e' \) and \( x' \in X^c_p,e \). A suitable substitution \( x' - tx \) is possible. If \( x' \in X^*_U \), then we have no difficulties with such a substitution.

Now we can assume that \( f(x) = x \) for all handles \( x \) and therefore for all connections in \( X \) and \( f \in \mathbb{K} \). Using (9.3) and (b) as above we have \( f(x) = x \) for all \( x \in X^*_U \) and all \( f \in \mathbb{K} \).
Case 2.2: Condition (c) is true for all \( x' \in X_p \), \( f(x') \in \langle XX \rangle V_p \) for \( x' \in X_p \), and there is some \( x \in X_{p,e} \) with \( f(x) \in \langle XX \rangle Q \) but \( f(x) \in \langle XX \rangle V_p \). By assumption we have \( f(e) = e \). We choose \( u \) in the representation of \( f(x) \) such that \( u \in \tilde{X}_p \setminus XXQ \). By assumption \( u \notin f(X_p) \). If \( u \) occurs in the representation of \( f(x') \) for some \( x' \in X_{p,e} \) with \( e' \neq e \), then \( \beta(u,e) = 0 = \beta(u,e') \) since \( f(e) = e \) and \( f(e') = e' \) by assumption. Hence \( u \) is a connection of \( e \) and \( e' \) and \( u \in X \), a contradiction. Therefore, the occurrence of \( u \) in the representation of \( f(x') \) implies \( x' \in X_{p,e} \cup X_X \cup X_p^{\Box} \). A suitable substitution \( x' \mapsto x' - tu \) gives the desired conclusion (9.5).

Case 3: \( x \in X_p \) and \( f(x') = x' \) for all \( x' \in X_{p,p}X_p^{\Box} \).

Case 3.1: \( f(x) \notin \langle XX \rangle V_{p,p} \). Again we set \( f(x) = u \in \tilde{X} \) w.l.o.g. If \( u \) occurs in the representation of \( f(x') \), then \( x' \in X_X \cup X_p^{\Box} \). Hence suitable substitutions \( x' \mapsto x' - tu \) give us (9.5).

Case 3.2: \( f(x) \in \langle XX \rangle V_{p,p} \). Then we choose

\[
 f(x) = u + v \quad \text{where} \quad u \in \tilde{X}_{p,p} \setminus XXQ \quad \text{and} \quad v \in \langle \tilde{X}\setminus \{u\} \rangle.
\]

If \( u \) occurs in the representation of \( f(x') \), then the assumption in Case 3 implies that \( x' \in X_p^{\Box} \cup X_p^{\Box} \). The substitutions as given above ensure (9.5).

We continue the proof of (c). We choose \( f, x, u \) according to (9.5). There is some \( y \in XD \) with \( s_{xy} \beta(x,y) \neq 0 \) in (9.2) and \( \beta(x,y) \neq 0 \) for all \( y \in YY \) since \( x \in X^e \). By (9.4) we have \( \sum y \in YY \beta_{xy}(x,y) \neq 0 \) modulo \( \langle \beta(V_G, V_G) \rangle \). We will modify the \( y \)'s in \( YY \) that do not occur in (b) inside their cosets modulo \( V_G \). This way we will get a contradiction to (9.4)'

By (9.2) we obtain

\[
 f(d) = \sum_{x_1 \in XX} \sum_{y_1 \in YY} r_{x_1,y_1} \beta(f(x_1), f(y_1))
 + \sum_{y_1 \in YY} r_{y_1,y_2} \beta(f(y_1), f(y_2)) \mod \langle \beta(V_G, V_G) \rangle.
\]

Using the linear representations of \( f(x_1) \) and \( f(y_1) \) for all \( x_1 \in XX \) and \( y_1 \in YY \) over \( \tilde{X} \tilde{Y} \) we compute the unique linear representation of \( f(d) \) over the set of all basic commutators \( \beta(a,b) \) with \( a, b \in \tilde{X} \tilde{Y} \). For \( y \in YY \) let us summarize all summands of the form \( r \beta(u,y) \). From \( \sum_{x_1 \in XX} \sum_{y_1 \in YY} r_{x_1,y_1} \beta(f(x_1), f(y_1)) \) we get only \( r_{y_1} \beta(u,y) \) since by (b) \( f(y_1) \equiv y_1 \mod V_G \) and by (9.5) \( f(x_1) \) for \( x_1 \neq x \) does not contain \( u \) in its linear representation over \( \tilde{X} \tilde{Y} \). To find the summands \( r \beta(u,y) \) in \( \sum_{y_1 \in YY} r_{y_1,y_2} \beta(f(y_1), f(y_2)) \) we write \( f(y_1) = y_1 + t_{y_1}u + w_{y_1} \) for \( y_1 \in YY \) where \( w_{y_1} \in \langle \tilde{X}\setminus \{u\} \rangle \).
Then we obtain
\[ \sum_{y_1 < y} r_{y_1} t_{y_1} \beta(u, y) - \sum_{y < y_1} r_{y} t_{y_1} \beta(u, y) \]
as the desired sum. But since \( f(d) = d \mod \beta(V_G, V_G) \) there is no summand \( r \beta(u, y) \)
in the linear representation of \( f(d) \). By linear independence
\[ \sum_{y_1 < y} r_{y_1} t_{y_1} \beta(u, y) - \sum_{y < y_1} r_{y} t_{y_1} \beta(u, y) = -r_{xy} \beta(u, y). \]

By (9.5) \( \beta(u, y) \neq 0 \) for \( y \in YY \). Hence we have
\[ \sum_{y_1 < y} r_{y_1} t_{y_1} - \sum_{y < y_1} r_{y} t_{y_1} = -r_{xy} \tag{9.6} \]
for all \( y \). Now, we replace every \( y \in YY \) by \( y^* = y + t_x \) if \( f(y) = y + t_x u + w_x \); this yields a substitution for \( y \in D \) as well. Eqs. (9.6) and (9.2) imply
\[ d = \sum_{x' \in X \setminus \{x\}} r_{x', y} \beta(x', y^*) \]
\[ + \sum_{y_1 < y_2 \atop y \in YY} r_{y_{12}} \beta(y_1^*, y_2^*) \mod \langle \beta(V_G, V_G) \rangle. \]

Moving commutators from \( \beta(V_G, V_G) \) to \( c \) we obtain a contradiction to (9.4) if we can define a support \((X \setminus \{x\}, X^* \amalg Y^*, Y^* \amalg D^*)\) for this new “\( d \)”. Let us consider \( y \in YY \) or \( y \in D \) with \( y \neq y^* \). By (b) \( y \in Y_{p,e} Y_p^* \) for \( e \in X \) or \( y \in Y_1 \) or \( y \in Y \).

If \( y \neq y^* \) only for \( y \in Y_1 \) or \( y \in Y \), then there is no problem. We obtain \( X^* \amalg Y^* \amalg D^* \) by replacing \( y \) by \( y^* \).

For \( y \in Y_{p,e} Y_p^* \) with \( y \neq y^* \) we have to be more careful. By (b) we know \( f(e) = e \). We assume that there is some \( y_0 \in Y_p \) with \( y_0^* \neq y \). To define \((X \setminus \{x\}, X^* \amalg Y^*, Y^* \amalg D^*)\) we distinguish the three cases of the proof of (9.5). In Case 3 we have \( x \in X \).

In this case we define
\[ D^* = \{ y^* : y \in D \text{ or } (y \in Y_p \text{ and } y \neq y^*) \}, \]
\[ Y^* = \{ y^* : y \in (Y \setminus Y_p) \text{ or } (y \in Y_p \text{ and } y = y^*) \}, \]
\[ Y^* \amalg D^* = Y^* \amalg \{ y \in Y_p : y \neq y^* \}, \]
\[ X^* = X^* \amalg \{ x \} \amalg \{ e \in X : \text{there is some } y \neq y^* \text{ in } Y_{p,e} \}. \]
and obtain a support as desired. In the other cases we define

\[ D^{\square} = \{ y^* : y \in D^{\square} \}, \quad Y^{\square} = \{ y^* : y \in Y \}, \]
\[ Y^{\square} = \{ y^* : y \in Y^{\square} \} \quad \text{and} \quad X^{\square} = X^{\square}. \]

In these cases we have to show that \( \beta(e, x) = 0 \) for all \( e \) that are handles of some \( y \in Y_{p,e}^{\square} \) with \( y \neq y^* \). Then \( y^* = y + t_j x \) is again of type \( p \) with handle \( e \). Since \( y^* \neq y \) we have \( f(y) = y + tu + w \) with \( t \neq 0 \) and \( w \in \langle X \{u\} \rangle \). \( \beta(y, e) = 0 \) implies \( \beta(f(y), f(e)) = 0 \). Since \( e = f(e) \) we get \( \beta(u, e) = 0 \). In Cases 1 and 2.1 \( u = f(x) \) and \( \beta(u, f(e)) = 0 \) implies \( \beta(x, e) = 0 \), as desired.

In Case 2.2 we have \( x \in X_{p,e}^{\square} \) and therefore \( \beta(x, e') = 0 \), \( u \in \tilde{X}_f \) and \( f(e') = e' \). Hence \( \beta(u, e') = 0 \). Above we have shown \( \beta(u, e) = 0 \). \( e \neq e' \) would imply that \( u \) is a connection of \( e \) and \( e' \). Then \( u \) would be an element of \( X \), a contradiction. Hence \( e' \approx e \) and \( \beta(x, e') = 0 \), as desired. \( \square \)

**Proof of (e)**. Our assumptions have been the following: \( d, (X, X^{\square}, Y, Y^{\square}, D^{\square}), \), \( \mathfrak{K} \), and \( \mathfrak{K}^H \) are chosen such that \( (X, X^{\square}, Y, Y^{\square}, D^{\square}) \) is a support of \( d, (a), (a)^H, (b), (b)^H, (9.2), (9.3), (9.4) \) and also (c) are true.

Using acl\((aG) \cap H = G, \) and \( Y^{\square} \subseteq acl(aG) \) we obtain that \( Y^{\square} \) is linearly independent modulo \( V_H \). Hence \( Y^{\square} \) is linearly independent modulo \( V_H \). Now we choose again a platform \( (X^H, X^{\square}, Y^H, Y^{\square}, D^{\square}) \) for \( \langle XY \rangle \) with respect to \( H \). We can choose this platform such that \( X \subseteq X^H \) and \( Y \cup \bigcup_{e \in Y^H} Y_{p,e} \subseteq Y^H \). To get this we apply first Lemma 6.7 to \( \langle X \rangle \subseteq \langle X^H \rangle \). Since \( \langle X^H, X^{\square}, Y \cup \bigcup_{e \in Y^H} Y_{p,e} \rangle \) is a good subspace of \( \langle X^H, X^{\square}, Y \rangle \) we can apply Lemma 6.7 again.

Furthermore, we can assume that

\[ (+) \ Y^{\square} \subseteq \langle (X^H \setminus X) X^{\square} Y^H Y^H \rangle. \]

Elements of \( Y^H \) are w.l.o.g elements of \( Y \). All elements of \( Y^H \cup Y^H \) occur in a representation of elements of \( Y^{\square} \) as a linear combination over \( Y^H \cup Y^H \) modulo \( V_H \) or as handles or connections as described in Lemma 6.10. Hence by (a)^H and that lemma we have w.l.o.g.:

\( (ab)^H \) Let \( f \) be an element of \( \mathfrak{K}^H \). \( f \) fixes all elements of \( Y^H \cup Y^H \) modulo \( V_H \). Furthermore \( f \) fixes \( Y^H \cup \bigcup_{e \in Y^H} Y^H_{p,e} \) pointwise and also all handles in \( X^H \) of elements of \( Y^H \cup Y^H \) and all roots of elements of \( Y^H \).

Similarly as above let \( Q^H \) be a part of a \( 1^\prime \)-transversal for the subspace of roots of \( Y^H \) in \( V_H \) such that \( X^H \cap Q^H \) is a basis for \( \langle X^H \cap Q^H \rangle \).

Now we choose \( \mathfrak{K}^H \) such that in addition \( (e)^H \) \( \text{weak} \) is true:

\( (e^H) \text{weak} \) Let \( x \) be an element of \( X \). If \( f(x) \in \langle X^H \cap Q^H \rangle \) for all \( f \in \mathfrak{K}^H \), then \( f(x) = x \).

By (b)^H we already know that the automorphisms of \( \mathfrak{K}^H \) fix the elements of \( Y \cup \bigcup_{e \in Y^H} Y_{p,e}, \) the handles \( x \) of some \( y \in Y_{p,x} \cup Y^H_{p,x} \) and the elements of \( Q^H \).
We have to show \( f(x) = x \) for \( x \in X^e \) and \( f \in \mathbb{K}^H \). Our main argument is \( f(d) = d \) modulo \( \beta(V_{f1}, V_{f2}) \) for \( f \in \mathbb{K}^H \). We assume that there are \( f \in \mathbb{K}^H, x \in X^e \) such that \( f(x) \notin \langle X^H X^H \square Q^H \rangle \) and obtain a contradiction. Similarly as in the proof of (c) we show:

There are a transversal \( \tilde{X}^H \tilde{y}^H \) of \( V \) where \( \tilde{X}^H \) is a transversal of \( V_H \), \( X^H X^H \square Q^H \subseteq \tilde{X}^H \), and \( Y^H Y^H \square \subseteq \tilde{Y}^H \) and some \( u \in \tilde{X}^H \backslash \tilde{X}^H X^H \square Q^H \) such that \( f(x) = u + v \) with \( v \in \langle \tilde{X}^H \backslash \{u\} \rangle \), \( f(x') \in \langle \tilde{X}^H \backslash \{u\} \rangle \) for \( x' \in XX \square \backslash \{x\} \), and for \( y \in Y^H Y^H \square \) we have \( \beta(u, y) \neq 0 \). (9.5^H)

Note \( \beta(u, y) \neq 0 \) for \( y \in Y^H Y^H \square \) is clear since \( u \notin \langle X^H X^H \square Q^H \rangle \) and all roots and handles of \( Y^H Y^H \square \) are in this subspace.

**Case 1:** We can choose \( x \in X_{\triangledown} \). W.l.o.g. \( f(x) \in \tilde{X}^H \). We define \( u = f(x) \) and replace \( x' \in XX \square \backslash \{x\} \) by \( x' - tx \) if \( tu \) occurs in the representation of \( f(x') \). Again \( t \neq 0 \) implies \( x' \in X_{p,e} \) or \( x' \in X_t X_{\triangledown} \). For \( x' \in X_{p,e} \) we have to show \( \beta(x, e) = 0 \). But \( t \neq 0 \) implies \( \beta(u, f(e)) = 0 \) and therefore \( \beta(x, e) = 0 \) as desired.

**Case 2:** \( x \in X_{p,e} \) and \( f(x') = x' \) for all \( x' \in X^e \).

**Case 2.1:** \( f(x) \notin \langle X^H X^H \square V_{\triangledown} \rangle \).

We define \( u = f(x) \) and choose \( \tilde{X}^H \) such that \( u \in \tilde{X}^H \). \( u \) is a proper element of type \( p \) with handle \( f(e) \). \( u \) cannot occur in the representation of \( f(x') \) for \( x' \in X_{\triangledown} \). Let \( x' \in XX \square \) be such that \( u \) occurs in the representation of \( f(x') \). Assume \( x' \in X_{p,e'} \). Then \( \beta(u, f(e')) = 0 \). We have \( \beta(u, f(e)) = 0 \). Hence \( f(e) \equiv f(e') \) since \( u \) was a proper element of type \( p \). We get \( e \approx e' \) and \( x' \in X_{p,e} \). A suitable substitution \( x' - tx \) is possible. If \( x' \in X_{\triangledown} X_{\triangledown} \), then we have no difficulties with such a substitution.

**Case 2.2:** Condition \( (c)^H \) is true for all \( x' \in X_{\triangledown} \) and there is some \( x \in X_{p,e} \) with \( f(x) \notin \langle X^H X^H \square Q^H \rangle \) but \( f(x) \notin \langle X^H X^H \square V_{\triangledown} \rangle \).

By assumption we have \( f(e) = e \). We choose \( u \) in the representation of \( f(x) \) such that \( u \notin \tilde{X}^H \backslash \tilde{X}^H X^H \square Q^H \). Then \( u \notin \{X_{\triangledown}\} \). If \( u \) occurs in the representation of \( f(x') \) for some \( x' \in X_{p,e'} \) with \( e' \neq e \) then \( \beta(u, e) = 0 \) and \( \beta(u, e') = 0 \) since \( f(e) = e \) and \( f(e') = e' \). Since \( u \) is a connection of \( e \) and \( e' \), we get \( u \in X \), a contradiction. Therefore, the occurrence of \( u \) in the representation of \( f(x') \) implies \( x' \in X_{p,e} \cup X_{\triangledown} \). A suitable substitution \( x' \rightarrow x' - tx \) gives the desired conclusion.

**Case 3:** \( x \in X_{\triangledown} \) and \( f(x') = x' \) for all \( x' \in X_{\triangledown,p} \). As above in Case 3 we choose \( u \) in the representation of \( f(x) \) in \( \langle X^H \rangle \) such that \( u \notin X^H X^H \square Q^H \). Possible substitutions \( x' - tx \) occur only for \( x' \in X_{\triangledown} X_{\triangledown} \). Hence there are no problems.

Now we choose \( f, x, \) and \( u \) according to (9.5^H). We have \( \beta(x, y) \neq 0 \) for all \( y \in Y^H Y^H \square \) by \((ab)^H \) since \( \beta(x, y) = 0 \) would imply that \( x \) is a root or a handle for \( y \). We use (9.2)^\square to get a representation of \( f(d) \) as a linear combination of commutators \( \beta(v_1, v_1) \) modulo \( \langle \beta(V_{f1}, V_{f2}) \rangle \) with \( v_1, v_2 \in \tilde{X}^H \tilde{y}^H \). We consider the summand \( r_s \beta(f(x), f(y)) \). If we want to write down (9.2)^\square with respect to \( \tilde{X}^H \tilde{y}^H \), then
we have to replace the $y \in YY$ by their representations as linear combinations over $(X^H \setminus X)X^HYY$. As mentioned above $YY$ is linearly independent modulo $V_H$. We get

\[
d = \sum_{x' \in X^H \setminus X} r^*_{x', y} \beta(x', y) + \sum_{y_1 < y_2} r^*_{y_1, y_2} \beta(y_1, y_2) \text{ modulo } \langle \beta(V_H, V_H) \rangle. \tag{9.2} \]

By (ab)$^H$ we have $f(y) = y$ modulo $V_H$ for $y \in Y^HYY$. Furthermore $d = f(d)$ modulo $\langle \beta(V_H, V_H) \rangle$. Since $u \in X^H \setminus X^HQ^H$, it does not occur in the unique representation as a linear combination of any $y \in YY$ over $(XH \setminus X)XHYH$. Hence in the representation of $d$ and therefore of $f(d)$ over $X^HYY$ any commutator $\beta(u, y)$ with $y \in Y^HYY$ does not occur.

On the other side in the representation of $f(d)$ as a linear combination of basic commutators over $X^HYY$ we get $r^*_{x,y} \beta(u, y)$ from $r^*_{x,y} \beta(f(x), f(y))$ in (9.2)$^H$. By (9.5)$^H$ the other $r^*_{x,y} \beta(f(x'), f(y))$ do not produce any $\beta(u, y)$ for $x' \in X^H \setminus X^H\{x\}$.

Let $f(y) = y + t_yu + w_y$ for $y \in Y^HYY$ where $w_y \in (X \setminus \{u\})$. Then the second sum in (9.2)$^H$ produces

\[
\sum_{y_1 < y} r^*_{y_1, y} \beta(u, y) - \sum_{y_1 < y} r^*_{y_1, y} \beta(u, y) = \left( \sum_{y_1 < y} r^*_{y_1, y} - \sum_{y_1 < y} r^*_{y_1, y} \right) \beta(u, y). \]

Note that $\beta(u, y) \neq 0$ for all $y \in Y^HYY$. $\beta(u, y) = 0$ would imply that $u$ is a handle or root of $y$ and therefore an element of $\langle X^H \setminus X^HQ^H \rangle$, a contradiction. Hence we have

\[
\sum_{y_1 < y} r^*_{y_1, y} - \sum_{y_1 < y} r^*_{y_1, y} = -r^*_{x,y} \tag{9.6} \]

for $y \in Y^HYY$. Now we replace $y \in Y^HYY$ according to this by $y^* = y + t_yx$ if $f(y) = y + t_yx + w_y$. If we use (9.6)$^H$ in (9.2)$^H$, then we have

\[
d = \sum_{x' \in X^H \setminus X} r^*_{x', y} \beta(x', y^*) + \sum_{y_1 < y_2} r^*_{y_1, y_2} \beta(y_1^*, y_2^*) \text{ modulo } \langle \beta(V_H, V_H) \rangle. \tag{9.2}^H \]
If $v \in YY$ and
\[
v = \sum_{y \in YY} h_y y + \sum_{z \in XH} h_z z,
\]
then we define
\[
v^* = v + t_v x \quad \text{where} \quad t_v = \sum_{y \in YY} h_y t_y.
\]
Hence
\[
v^* = \sum_{y \in YY} h_y y^* + \sum_{z \in XH} h_z z.
\]
If we replace $v \in YY$ by $v^*$ in (9.2), then we obtain
\[
e = \sum_{x \in XX} r_{xv} \beta(x, v^*) + \sum_{v_1 < v_2} r_{v_1 v_2} \beta(v_1^*, v_2^*) \mod \langle \beta(V_H, V_H) \rangle.
\]
In the computation of (9.2) from (9.2) we replaced $v \in YY$ by its linear representation over $Y^H X^H X^H X^H \setminus X$. Now we do the same formal steps with $e$ where $y^*$ plays the role of $y \in Y^H Y^H$. That means we use
\[
v^* = \sum_{y \in YY} h_y y^* + \sum_{z \in XH} h_z z.
\]
We obtain
\[
e = \sum_{x' \in H} r_{x'y} \beta(x', y^*) \mod \langle \beta(V_H, V_H) \rangle
+ \sum_{y_1 < y_2} r_{y_1 y_2} \beta(y_1^*, y_2^*) \mod \langle \beta(V_H, V_H) \rangle.
\]
By (9.2)
\[
e = d + \sum_{y \in YY} r_{xy} \beta(x, y^*) \mod \langle \beta(V_H, V_H) \rangle.
\]
If we consider (9.2) \( Y \mod \langle Xuu(\mathcal{VH}, \mathcal{VH}) \rangle \) and compare it with (9.2) \( Y_H \), then we see

\[
\sum_{y \in YHY_H} r^*_x \beta(x, y^*) = \sum_{y \in YHY_H} r^*_x \beta(x, y) = \sum_{v \in YY} r_x \beta(x, v) = \sum_{v \in YY} r_x \beta(x, v^*) \mod \langle \beta(V_H, V_H) \rangle.
\]

We use \( YY \subseteq \langle YHY_H \rangle \) and that any set of commutators \( \beta(a, b) \neq 0 \), where \( a < b \) are elements of a given transversal, is linearly independent. Then we have

\[
d = \sum_{x' \in X \setminus \{x\}} r_{x'y} \beta(x', y^*) + \sum_{y_1 < y_2} r_{y_1y_2} \beta(y_1^*, y_2^*) \mod \langle \beta(V_H, V_H) \rangle.
\]

As in the proof of (c) we find a support \((X \setminus \{x\}, X^*, Y^*, Y^*, D^*)\) for \( d \), such that \( \{y^* : y \in YY\} = Y^* Y^*\). We get the desired contradiction to (9.4). \( \square \)

We repeat our assumptions:

\((X, X^*, Y, Y^*, D^*)\) is a support of \( d \) and \( K \subseteq \text{Aut}_{\{G\}}(\mathbb{C}) \) and \( K_H \subseteq \text{Aut}_{\{H\}}(\mathbb{C}) \) both of finite index are given such that (9.2), (9.3), (9.4), (a), (b), (c), (a)\( H \), (b)\( H \), and (c)\( H \) are true. All roots of \( Y \) are in \( X \). (9.7)

Furthermore, remember that \( YD^* \) is linearly independent modulo \( V_H \). We use again

\[
d = \sum_{x \in X} s_{xy} \beta(x, y) + \sum_{y_1 < y_2} s_{y_1y_2} \beta(y_1, y_2).
\]

We split \( Y \cup D^* \) into \( U^* \) and \( P^* \) where

\[P^* = Y_1 \cup \bigcup_{e \in Y_1} Y_{pe} \cup P\]

with

\[P = \{y \in Y \cup D^* : s_{yy'} \beta(y, y') \neq 0 \text{ or } s_{y'y} \beta(y', y) \neq 0\}

for some \( y' \in Y \cup D^* \) \( \setminus \left( Y_1 \cup \bigcup_{e \in Y_1} Y_{pe} \right) \)

and

\[U^* = (Y \cup D^*) \setminus P^*\]
Now, we add to our assumptions that we have chosen a situation where \( |P| \) is minimal. Then we have

\[
(d) \quad f(y) = y \text{ for } y \in P^* \text{ and } f \in \mathbb{K} \text{ and } \quad (d)^{\mathcal{H}} \quad g(y) = y \text{ for } y \in P^* \text{ and } g \in \mathbb{K}^{\mathcal{H}}.
\]

**Proof of (d).** If \( f(y) \neq y \) for some \( f \in \mathbb{K} \) and \( y \in P^* \), then \( y \in P \) by (b). By (a) we have \( f(y) = y \) modulo \( V_0 \) for \( y \in P \) and \( f \in \mathbb{K} \). If some \( y \in P \) satisfies \( f(y) \in \langle XX \mathcal{Q} YY \mathcal{Q} \rangle \) for all \( f \in \mathbb{K} \), then there is a subgroup \( \mathbb{K}_0 \subseteq \mathbb{K} \) of finite index such that \( f(y) = y \) for all \( f \in \mathbb{K}_0 \) and the considered \( y \). Hence we can assume w.l.o.g.

\[
(d)^{\text{weak}} \quad \text{ for all } y \in P \text{ we have: If } f(y) \in \langle XX \mathcal{Q} YY \mathcal{Q} \rangle \text{ for all } f \in \mathbb{K}, \text{ then } f(y) = y \text{ for all } f \in \mathbb{K}.
\]

To show (d) we assume that there are some \( y_0 \in P \) and some \( f \in \mathbb{K} \) such that \( f(y_0) \notin \langle XX \mathcal{Q} YY \mathcal{Q} \rangle \). Let \( \tilde{X} \) be a transversal of \( V \) where \( \tilde{X} \) is a transversal of \( V_0 \), \( XX \mathcal{Q} \subseteq \tilde{X} \), and \( YY \mathcal{Q} \subseteq \tilde{Y} \). Then w.l.o.g. \( f(y_0) = y_0 + u + w \) by (a) where \( u \in \tilde{X} \backslash XX \mathcal{Q} \) and \( w \in \langle \tilde{X} \backslash \{u\} \rangle \). We show that we can modify \( y \), \( Y \mathcal{Q} \), and \( D \mathcal{Q} \) such that \( f(y) \in \langle \tilde{X} \tilde{Y} \backslash \{u\} \rangle \) for \( y \in YD \) and \( y \neq y_0 \). We distinguish three cases.

**Case 1:** We can choose \( y_0 \) in \( Y \mathcal{Q} \cap P \). Since \( y_0 \notin P \) we have \( e \in X \). By (c) \( f(e) = e \). Then \( \beta(f(e), f(y_0)) = 0 \) implies \( \beta(e, u) = 0 \) and \( \beta(e, w) = 0 \).

For \( y \in YY \mathcal{Q} \) we write \( f(y) = y + t_y u + w_y \) where \( w_y \in \langle \tilde{X} \backslash \{u\} \rangle \). Note that \( t_y \neq 0 \) implies \( y \in Y \mathcal{Q} \) and \( y \neq y_0 \)

\[
y' = y - t_y y_0 \quad \text{ and } \quad y'_0 = y_0.
\]

We denote by \( Y' \mathcal{Q} \) the new basis of \( \langle Y, Y \mathcal{Q} \rangle \). According to this we obtain \( D \mathcal{Q} \). We write \( (YD \mathcal{Q})' \) for \( Y'D \mathcal{Q} \). Then \( (X, X \mathcal{Q}, Y', Y \mathcal{Q}, D \mathcal{Q}) \) is again a support for \( d \), for which (9.7) remains true.

**Case 2:** For all \( y \in \mathbb{K} \) and \( y \in Y \mathcal{Q} \), we have \( f(y) \in \langle XX \mathcal{Q} YY \mathcal{Q} \rangle \), but there is some \( y_0 \in D \mathcal{Q} \) and \( f \in \mathbb{K} \) such that w.l.o.g.

\[
f(y_0) = y_0 + u + w \quad \text{where } u \in \tilde{X} \backslash XX \mathcal{Q}, \ w \in \langle \tilde{X} \backslash \{u\} \rangle.
\]

By \( (d)^{\text{weak}} \) we can assume \( f(y) = y \) for \( y \in \mathbb{K} \). For \( y \in YD \) let \( f(y) = y + t_y u + w_y \). \( t_y \neq 0 \) implies \( y \in D \mathcal{Q} Y \mathcal{Q} \). For these \( y \neq y_0 \) we define \( y' = y - t_y y_0 \). For all other \( y \in YD \) we define \( y' = y \). Let \( Y'D \mathcal{Q} \) be the new sets. We do not change \( Y \mathcal{Q} \). But we have a new \( X \mathcal{Q} \). Then \( (X, X \mathcal{Q}, Y', Y \mathcal{Q}, D \mathcal{Q}) \) is a support of \( d \) and (9.11) is true.

By \( (d)^{\text{weak}} \) again we have the following remaining case:

**Case 3:** For all \( f \in \mathbb{K} \) and \( y \in Y \mathcal{Q} \cup D \mathcal{Q} \) we have \( f(y) = y \) but there is some \( y_0 \in Y \mathcal{Q} \) and \( f \in \mathbb{K} \) such that \( f(y_0) \notin \langle XX \mathcal{Q} YY \mathcal{Q} \rangle \). We do the same substitution as in Case 2. It concerns only elements of \( Y \mathcal{Q} \). Therefore, it is easy to see that (9.7) remains true for the new support.
Now we assume w.l.o.g. that $f(y) \in \langle \tilde{X} \tilde{Y} \setminus \{u\} \rangle$ for all $y \in YD$ with $y \neq y_0$. We use

$$
\begin{align*}
    d &= \sum_{x \in X} s_{xy} \beta(x, y) + \sum_{y_1 < y_2, y_i \in P^*} s_{y_1 y_2} \beta(y_1, y_2) 
\end{align*}
$$

(9.2)

and

$$
\begin{align*}
    f(d) &= \sum_{x \in X} s_{xy} \beta(x, f(y)) + \sum_{y_1 < y_2, y_i \in P^*} s_{y_1 y_2} \beta(f(y_1), f(y_2)). 
\end{align*}
$$

(9.2)

If we rewrite (9.2) with respect to $\tilde{X} \tilde{Y}$, then this representation contains no commutator $\beta(u, w)$ with $w \in YY$. But $f(9.2)$ produces

$$
\sum_{y < y_0, y \in P^*} s_{y y_0} \beta(u, y) - \sum_{y < y_0, y \in P^*} s_{y y_0} \beta(u, y)
$$

in the representation of $f(d)$ and this sum contains all commutators $\beta(u, w)$ with $w \in YY$ in the representation of $f(d)$ over $\tilde{X} \tilde{Y}$. Since $u \notin \langle XX \rangle$ the element $u$ is not a root or handle of $YY$ and therefore $\beta(u, w) \neq 0$. Since $P^*$ is linearly independent over $VH$ and $s_{y_0 y} \neq 0$ or $s_{y y_0} \neq 0$ for some $y$ by minimality of $|P|$, we have a non-vanishing commutator $\beta(u, w)$ with $w \in YY$ in the representation of $f(d)$ over $\tilde{X} \tilde{Y}$. This contradicts $d = f(d)$ modulo $\langle \beta(VH, VH) \rangle$. □

**Proof of (d)**. By $(b)^H$ $g(y) \neq y$ for some $g \in K^H$ implies $y \in P$. By $(a)^H$ $g(y) = y$ modulo $VH$ for $y \in P$ and $g \in K^H$. We choose a platform $(X^H, X^H \sqcap, Y^H, Y^H \sqcap, D^H \sqcap)$ for $\langle YD \rangle$ with respect to $VH$. We have w.l.o.g.

$(d)^{H\text{weak}}$ For all $y \in P$ we have: If $g(y) \in \langle X^H X^H \sqcap Y^H Y^H \rangle$ for all $g \in K^H$, then $g(y) = y$ for all $g \in K^H$.

As in the proof of $(c)^H$ we can assume that $X \subseteq X^H$ and

$$
Y_1 \cup \bigcup_{e \in Y_1} Y_{p,e} \subseteq Y^H.
$$

We use a transversal $\tilde{X}^H \tilde{Y}^H$ of $V$ where $\tilde{X}^H$ is a transversal of $VH$, $X^H X^H \sqcap \subseteq \tilde{X}^H$ and $Y^H Y^H \sqcap \subseteq \tilde{Y}^H$. As above we choose $y_0 \in YD$ and $g \in K^H$ with $g(y_0) = y_0 + u + w$ where $u \in \tilde{X}^H \setminus X^H X^H \sqcap$, $w \in \tilde{Y}^H \setminus \{u\}$. Using the argumentation as in the proof of (d) with the three cases we can w.l.o.g. assume that for $y \in YD$ with $y \neq y_0$ the image $g(y)$ does not contain $u$ in its representation over $\tilde{X}^H \tilde{Y}^H$. If we rewrite (9.2) with respect to $\tilde{X}^H \tilde{Y}^H$ then commutators $\beta(u, w)$ with $w \in \tilde{Y}^H$ do not occur. On the other
hand, we have
\[ g(d) = \sum_{x \in X, \ y \in Y} s_{x,y}(x, g(y)) + \sum_{y_1 < y_2, y \in P} s_{y_1,y_2}(y_1, g(y_2)). \tag{g.2} \]

Here we produce
\[ \beta \left( u, \sum_{y_0 < y, y \in P^*} s_{y_0,y}y - \sum_{y < y_0, y \in P^*} s_{y_0,y}y \right) \]
as the sum of all commutators of the form \( \beta(u, w) \) with \( w \in Y^H \). As above \( \beta(u, w) \neq 0 \). By the minimality of \( |P| \) the linear combination on the right side is non-trivial. Since \( P^* \) is linearly independent over \( V_H \) there are commutators \( \beta(u, w) \) with \( w \in Y^H \) in the representation of \( g(d) \) over \( X^H \). We have a contradiction to \( d = g(d) \) modulo \( \langle Xuu(V_h, V_H) \rangle \). \( \square \)

Now it is easy to finish the proof of Theorem 9.1. We look at (9.2) and define
\[ a' = \sum_{x \in X, y \in P^*} s_{x,y}(x, y) + \sum_{y_1 < y_2, y \in P} s_{y_1,y_2}(y_1, y_2). \]
By (c) and (d) we have \( f(a') = a' \) for \( f \in K \). By (c) and (d) we have \( g(a') = a' \) for \( g \in \mathbb{K} \). If \( y \in U^* \cap D^\Box \), then \( y = u_x + x_y \) where \( u_x \in \langle Y^\Box \rangle \) and \( x_y \in X^\Box \). In this case
\[ s_{x,y}(x, y) = s_{x,y}(x, u_x) + s_{x,y}(x, x_y). \]
We move \( s_{x,y}(x, x_y) \) into \( c \). Let \( U \) be \( U^* \setminus D^\Box \) joint with all elements of \( Y^\Box \) that occur in some \( u_x \) above. Using \( a = cd \) and (9.2) we have
\[ a = a^{G} \prod_{x \in X, u \in U} [x, u]^{\nu u} a', \]
where \( a^{G} = c \). To check the remaining conditions in Theorem 9.1 we use (a), (a), (c), (c) and the fact that \( YY^\Box \) is linearly independent modulo \( V_H \). \( \square \)

10. Finale

We show Theorem 8.1. By Corollary 8.7 we can assume that \( A \cap G = A \cap G' \) and \( A \cap (G \cdot Z(G)) = A \cap (G \cdot C') \). We compute \( \text{Cb}(\text{tp}(A/G)) \) up to the algebraic closure.
and show that it is included in acl(Cb(tp(A/H))). Let $A_0$ be a basis of $A \cap C'$. Choose $A_1 \subseteq A$ such that $A_1/Z(C)$ is a basis of $A/Z(C)$ modulo $G/Z(C)$. By our assumption acl($AG \cap H = G$ we have that $A_1/Z(C)$ is also linearly independent modulo $H/Z(C)$. Consider $A \cap G = A \cap G' \subseteq A \cap C' \subseteq A \cap (G \cdot C') = A \cap (G \cdot Z(C))$. Then $A = \langle A_0 \cup A_1 \cup \{a_1, \ldots, a_m\} \rangle$ where $a_1, \ldots, a_m$ are mixed elements. We can choose $a_1, \ldots, a_m$ in such a way that there is some $\ell$ with $0 \leq \ell \leq m$, $a_1, \ldots, a_\ell$ are linearly independent modulo $Z(C)$, and $a_{\ell+1}, \ldots, a_m \subseteq Z(C)$ are linearly independent modulo $C'$. By Theorem 9.1 there are $a_i^G \in G$, $a_i^G \in C'$, parts of a transversal $X^i \cup i \ (1 \leq i \leq m)$ that respect $V_G$ with $\langle X^i \cup i \rangle \cap V_G = \langle X^i \rangle$ and $U^i \cup \bigcup_{e \in \mathbb{T}_e} U^i_{p,e} = \emptyset$, and subgroups $\mathbb{K} \subseteq Aut_{\{G\} \cdot (A)}(C)$ and $\mathbb{K}^H \subseteq Aut_{\{H\} \cdot (A)}(C)$ both of finite index such that

$$a_i = a_i^G \prod_{x \in X^i} [x, u]^i \cdot a_i'(10.1)$$

and every $f \in \mathbb{K}$ fixes all $a_i'$ and $X^i$ pointwise, and fixes $U^i$ pointwise modulo $V_G$, and every $g \in \mathbb{K}^H$ fixes all $a_i'$ and $X^i$ pointwise, and fixes $U^i$ pointwise modulo $V_H$.

Since $f(a_i') = a_i'$ for $f \in \mathbb{K}$ and $g(a_i') = a_i'$ for $g \in \mathbb{K}^H$ we have

$$acl(Cb(tp(A \cup \{a_1', \ldots, a_m'\}/G))) = acl(Cb(tp(A/G)))$$

and

$$acl(Cb(tp(A \cup \{a_1', \ldots, a_m'\}/H))) = acl(Cb(tp(A/H))).$$

Hence we can w.l.o.g. assume that $a_i' \in \langle A_0 \rangle \subseteq A$ for $1 \leq i \leq m$. We can choose $a_1, \ldots, a_m$ in such a way that

$$a_i = a_i^G \prod_{x \in X^i} [x, u]^i \cdot (10.2)$$

Using Theorem 6.9 and Corollary 6.4 there is a good subspace $D$ such that $A \cap C' \subseteq \beta(D, D)$, $A/Z(C) \subseteq D$, $X^i \subseteq D$ for $1 \leq i \leq m$ and conversely $D \subseteq acl(A \cap C') \cup A/Z(C) \cup \bigcup_{1 \leq i \leq m} X_i$. W.l.o.g. we can assume that $D$ is fixed pointwise by all automorphisms in $\mathbb{K}$ and in $\mathbb{K}^H$.

By Theorem 7.4 there is a platform $(X, X^\square, Y, Y^\square, D^\square)$ for $D$ with respect to $V_G$. By Theorem 7.4(iii) $Aut_{\{G\} \cdot (D)}(C)$ and $Aut_{\{H\} \cdot (XYD)}(C)$ are commensurable, as are analogously $Aut_{\{H\} \cdot (D)}(C)$ and $Aut_{\{H\} \cdot (XYD)}(C)$ by Theorem 7.4(iv). Hence $XYD^\square$ is w.l.o.g. fixed pointwise by all automorphisms $f \in \mathbb{K}$ and $g \in \mathbb{K}^H$. Furthermore, we have w.l.o.g. $f(y) = y$ modulo $V_G$ for $y \in Y^\square$ and $f \in \mathbb{K}$, and similarly $g(y) = y$ modulo $V_H$ for $y \in Y^\square$ and $g \in \mathbb{K}^H$.

To unify (10.2) we will enlarge $D$. Let $D^*$ be the smallest good subspace that contains $XX^\square Y Y^\square$ and $\bigcup_{1 \leq i \leq m} U^i$. Again by Theorem 7.4 there is a platform $(X^*, X^* \square, Y^*, \ldots$,
\(Y^\square, D^\square)\) for \(D^+\) with respect to \(V_G\). W.l.o.g. \(X^\square \subseteq X^+\) and \(Y^\square \subseteq Y^+.\) \(\langle V_G Y^*Y^\square \rangle\) is the smallest good subspace that contains \(V_G, D\) and \(\bigcup_{i \leq m} U^i\).

Therefore this space and also \(V_G\) are fixed setwise by all \(f \in K\). We can replace \(K\) in such a way that furthermore \(f(y) = y\) modulo \(V_G\) for all \(y \in Y^*Y^\square\) and \(f \in K\). \(\langle V_H Y^*Y^\square \rangle\) is contained in the smallest good subspace that contains \(V_H, D\) and \(\bigcup_{i \leq m} U^i\). This subspace is finite modulo \(V_H\). Again we can w.l.o.g. assume that \(g(y) = y\) modulo \(V_H\) for \(y \in Y^*Y^\square\) and \(g \in K^H\). Let \(X^+\) be the extension of \(X\) by handles of \(Y^*Y^\square\) in \(V_G\). Let \(Y^+\) be \(Y \cup Y^\square \cup \bigcup_{e \in X^+} Y^+_p,e\) and \(D^+ = \langle D \cup X^+ \cup Y^+ \rangle\). Note \(Y^\square \cap Y^+ = \emptyset\). \((X^+, X^\square, Y^+, Y^\square, D^\square)\) is a platform for \(D^+\). Again w.l.o.g. \(D^+, X^+, Y^+, D^\square\) are fixed pointwise by automorphisms of \(K\) and \(K^H\). Therefore \(C_b(tp(D^+/G)) \subseteq acl(C_b(tp(A/G)))\) and \(C_b(tp(D^+/H)) \subseteq acl(C_b(tp(A/H)))\). By Corollary 7.5 we have

\[
acl(C_b(tp(D^+/G))) = acl(C_b(tp_Y^+/\Gamma(G) \cup X^+ \cup \square))
\]

and by Corollary 7.6 we have

\[
C_b(tp(D^+/G)) \subseteq acl(C_b(tp(D^+/H))).
\]

We will show that we obtain \(acl(C_b(tp(A/G)))\) from \(acl(C_b(tp(D^+/G)))\) if we replace \(\square\) by another imaginary element over \(X^+\) and that this imaginary element is also contained in \(acl(C_b(tp(A/H)))\). Then CM-triviality is shown. Note \(Y^+ \subseteq Y^*\) and \(Y^\square \subseteq (Y^*)_Y^+\). Let \(U\) be \((Y^*Y^\square) \setminus Y^+.\) We can write \(u \in U^j\) as a linear combination over \(X^*X^\square Y^*Y^\square\). If \(u = u_0 + u_1 + u_2\) with \(u_0 \in \langle X^*X^\square \rangle, u_1 \in Y^+\) and \(u_2 \in U,\) then for \(x \in X\)

\[
[x, u] = [x, u_0][x, u_1][x, u_2].
\]

Let us consider (10.1) again. Since \([x, u_1]\) is fixed by all automorphisms in \(K\) and in \(K^H\) we can assume w.l.o.g. that \([x, u_1]\) is an element of \(A \cap C'.\) \([x, u_0]\) is in \(G\) and can be moved to \(a_i^G.\) Hence, we can rewrite (10.2)

\[
a_i = a_i^G \prod_{x \in X} [x, u]^{v_u}.
\]

As assumed above \(a_1^G, \ldots, a^G\) are linearly independent modulo \(Z(A)\) and \(a_i^G, \ldots, a_m^G \subseteq Z(G)\) are linearly independent modulo \(G'.\)

By definition of \(D\) all \(a_i^G/Z(A)\) for \(1 \leq i \leq \ell\) are in \(D \cap V_G\) and therefore in \(\langle X \rangle.\) Let \(X^u \subseteq X^+\) be the set of handles for those elements in \(U\) that are of type \(p\) and have a handle in \(V_G.\) According to the special triple \((X^\square, Y^\square, D^\square)\) let \(d = x_d + y_d\) for \(d \in D^\square,\) where \(x_d \in X^\square\) and \(y_d \in Y^\square\). Using (10.3) we extend the definition of \(\theta^d.\) Note \(Y^\square \subseteq U.\)

**Definition.** Let \(\bar{v}^\prime = (v_{d}^\prime, d \in D^\square)^{v^\prime} (v_{d}^\prime, \ldots, v_{d}^\prime) (i = 0, 1)\) be two sequences in \(V^{|D^\square|} \times C^m,\)

where \(v_d^\prime \in V\) and \(v_d^\prime \in C.\) We say \(\theta^d(\bar{v}^\prime, \bar{v}^\prime),\) if there is a \(B_p\)-homomorphism \(h\) of \(\langle X^u \rangle\)
into $\mathcal{F}(C)$ with $h(x) = x$ for $x \in X^u$ such that
\[ v_d^0 = v_d^1 + h(y_d) \quad \text{for } d \in D^\Box \]
and
\[ v_i^0 = v_i^1 \prod_{x \in X} [x, h(u)]^{i_u} \quad \text{for } 1 \leq i \leq m. \]

The intention of this definition is that we are interested in the $\theta_d$-class of $(x_d : d \in D^\Box) \wedge (a_1^G, \ldots, a_m^G)$.

Note:
- In $\langle X^u U \rangle$ $\beta(e, y) = 0$ for $e \in X^u$ and $y \in U_{p,e}$ are the only relations. Then we have $\beta(e, h(y)) = 0$ in this case. Hence in the definition above we can characterize $h$ as an vector space homomorphism with $h(x) = x$ for $x \in X^u$ and $\beta(e, h(y)) = 0$ for $e \in X^u$ and $u \in U_{p,e}$.
- $\theta_1(v^0, v^1)$ defines an equivalence relation.
- $\theta_1$ is definable over $X$.
- If $X$ is fixed pointwise, then we consider the $\theta_d$-classes as elements of $C^{eq}$.
- Under this assumption let $A$ be the $\theta_d$-class of $(x_d : d \in D^\Box) \wedge (a_1^G, \ldots, a_m^G)$.
- $G$ is in $\text{dcl}(AX)$.

For $f \in k$ we have $f(d) = d$ for $d \in D^\Box$ and $f(a_i) = a_i$. For $u \in U$ we have $f(u) = u + v_u$ where $v_u \in V_G$. Let $k$ be the vector space homomorphism of $\langle X^u U \rangle$ into $V_G$ defined by $k(u) = v_u$ for $u \in U$ and $k(x) = x$ for $x \in X^u$. It can be extended to an $B_p$-homomorphism that we call again $k$. Since $f(d) = d$ for $d \in D^\Box$ implies
\[ f(x_d + y_d) = f(x_d) + f(y_d) = x_d + y_d, \]
we obtain
\[ f(x_d) = x_d + (y_d - f(y_d)) = x_d + v_{y_d} = x_d + k(y_d). \]

Furthermore,
\[ a_i = f(a_i) = f(a_i^G) \prod_{x \in X} \beta(x, u)^{i_u} \prod_{x \in X} \beta(x, v_u)^{i_u} \]
and therefore
\[ a_i^G = f(a_i^G) \prod_{x \in X} \beta(x, k(u))^{i_u}. \]
We have shown that
\[ \theta_A((x_d : d \in D^\Box)^\wedge (a^G_1, \ldots, a^G_m), (f(x_d) : d \in D^\Box)^\wedge (f(a^G_1), \ldots, f(a^G_m))) \]
holds. Hence \( A \) is preserved by all \( f \in \mathcal{K} \). Therefore \( \mathcal{A} \in \text{acl}(\text{Cb}(\text{tp}(A/G))) \). Since all \( g \in \mathcal{K}^H \) fix \( D^\Box \), \( X \) and all \( a_i \) pointwise, we can use the same steps to prove \( \mathcal{A} \in \text{acl}(\text{Cb}(\text{tp}(A/H))) \).

Now, we know that \( \text{acl}[\text{Cb}(\text{tp}(D^+(G))) \cup \mathcal{A}] \) is contained in \( \text{acl}(\text{Cb}(\text{tp}(A/G))) \) and in \( \text{acl}(\text{Cb}(\text{tp}(A/H))) \). Hence it is sufficient to show that
\[ \text{acl}(\text{Cb}(\text{tp}(D^+/G)) \cup \mathcal{A}) = \text{acl}(\text{Cb}(\text{tp}(A/G))). \]

Consider \( f^G \in \text{Aut}(G) \) fixing \( \text{Cb}(\text{tp}(D^+/G)) \cup \mathcal{A} \) pointwise. We will extend \( f^G \) to \( f \in \text{Aut}(G)(\mathbb{C}) \) that fixes \( A \) pointwise. By the assumption we can extend \( \mathcal{F}(f^G) \) to \( (g, h) \in \text{Aut}(G)(\mathbb{C}) \) with \( g(d) = d \) for all \( d \in D^+ \). Since \( f^G \) fixes \( \mathcal{A} \) we have a \( \mathbb{B}_p \)-homomorphism \( k \) of \( \langle X^u U \rangle \) into \( V_G \) such that \( g^G(x_d) = x_d + k(y_d) \) for \( x_d \in X^\Box \), and \( f^G(a^G_i) = a^G_i \prod_{u \in U} [x, k(u)]^\mu \) for \( 1 \leq i \leq m \).

Now we change \( (g, h) \) into \( (g^*, h^*) \). Let \( \bar{X} \bar{Y} \) be a transversal of \( V \) such that \( \bar{X} \) is a transversal of \( V_G, X^+ X^\Box \subseteq \bar{X} \) and \( Y^+ U \subseteq \bar{Y} \). Since \( g(d) = d \) for \( d \in D^\Box \) we have \( d = g(d) = x_d + k(y_d) + g(y_d) \) and therefore \( g(y_d) = y_d - k(y_d) \) for \( y_d \in (Y^\Box) \) with \( d = x_d + y_d \). We obtain \( g^* \) from \( g \) if we define \( g^*(v) = g(v) \) for \( v \in \bar{X} \bar{Y} \backslash U \) and \( g^*(u) = u - k(u) \) for \( u \in U \). Note \( g^*(u) = g(u) \) for \( u \in Y^\Box \). Therefore \( g^*(y) = g(y) \) for \( y \in D^\Box \). Since \( k \) is a \( \mathbb{B}_p \)-homomorphism of \( \langle X^u U \rangle \) into \( V_G \) with \( k(x) = x \) for \( x \in X^u \) the map \( g^* \) induces an automorphism \( (g^*, h^{**}) \) of \( \langle V, \beta(V, V), \beta \rangle \).

Then \( g^*(v) = g(v) = v \) for \( v \in \langle X^+ Y^+ D^\Box \rangle \) and therefore for \( v \in D^+ \). \( (g^*, h^{**}) \) extends \( \mathcal{F}(f^G) \) and we can extend it to an automorphism \( (g^*, h^*) \) of \( \langle V, W, \beta \rangle \). Since \( A_1 \) is linearly independent modulo \( \langle Z(\mathbb{C}) \cup G \rangle \), by Corollary 3.3 there is an automorphism \( f \) in \( \text{Aut}(G)(\mathbb{C}) \) that extends \( f^G \) with \( \mathcal{F}(f) = (g^*, h^*) \) and \( f(a) = a \) for \( a \in A_1 \).

Furthermore \( f \) fixes \( A \cap C' \) pointwise since \( (A \cap C') \subseteq \beta(D, D) \). Finally for \( 1 \leq i \leq m \)
\[
f(a^G_i) = f(a^G_i \prod_{x \in X} [x, f(u)]^\mu) = a^G_i \prod_{x \in X} [x, k(u)]^\mu = a^G_i \prod_{x, v \in U} [x, u]^\mu = a^G_i.
\]
We have shown $CM$-triviality and

$$\text{acl}(\text{Cb}(\text{tp}(A/G))) = \text{acl}(\text{Cb}(\text{tp}(Y_{1}^{\infty}/I(G)))) \cup X^{+} \cup A).$$

11. $CM$-triviality of $\text{Th}(F_{2}(p, \omega))$

Let $p$ be a prime greater than 2. Let $F_{2}(p, \kappa)$ be the free group in the variety of all nilpotent groups of class 2 and exponent $p$ with $\kappa$ free generators. The elementary theory of $F_{2}(p, \omega)$ is $\omega$-stable [1]. The models $G$ of $\text{Th}(F_{2}(p, \omega))$ could be considered as “Mekler-groups, where $V_{G}$ contains only elements of type 1”. Hence, the proof of the preservation of $CM$-triviality provides us a proof of the following result. We will give some details in this chapter.

**Theorem 11.1.** $\text{Th}(F_{2}(p, \omega))$ is $\omega$-stable and $CM$-trivial.

**Proof.** Again we consider the following situation. Assume $G \preceq H \preceq C$ are saturated models of $\text{Th}(F_{2}(p, \omega))$ where $C$ is big. Let $A$ be a finite subgroup of $C$ such that $\text{acl}(AG) \cap H = G$. We have to show

$$\text{Cb}(\text{tp}(A/G)) \subseteq \text{acl}(\text{Cb}(\text{tp}(A/H))).$$

As above we use $\mathcal{F}(C) = \langle V, W, \beta \rangle$ and its substructures. If $U$ is a subspace of $V$, then a transversal of $U$ is just a basis of this vectorspace. (Every subspace is good!) It is not a problem to find a transversal $X$ of $U$ that respects $V_{G}$. That means $X \cap V_{G}$ is a basis of $U \cap V_{G}$.

Analogously to Theorem 6.9 it is much easier to show:

**Lemma 11.2.** For every $a \in \beta(V, V)$ there is a smallest finite subspace $U(a) \subseteq V$ such that $a \in \beta(U(a), U(a))$. $U(a)$ is definable over $a$.

By Corollary 8.7 we can w.l.o.g. assume that $A \cap G = A \cap G'$ and $A \cap (G \cdot Z(C)) = A \cap (G \cdot C')$. The critical case are the mixed elements $a$ in $A$, where $a \in G \cap C'$, $a \notin G$, and $a \notin C'$.

We can show the following version of Theorem 9.1.

**Lemma 11.3.** Let $a$ be a mixed element of $A$. Then there are $a^{G} \in G$, $a' \in C$ and a transversal $XU$ with $\langle XU \rangle \cap V_{G} = \langle X \rangle$ such that

$$a = a^{G} \prod_{x \in X, \ u \in U} [x, u]^{y_{u}} = a'. $$
Furthermore, there are subgroups $\mathcal{K}$ of $\text{Aut}_{\{G\}_{(a)}}(\mathbb{C})$ and $\mathcal{K}^H$ of $\text{Aut}_{\{H\}_{(a)}}(\mathbb{C})$ both of finite index such that all $f \in \mathcal{K}$ and $g \in \mathcal{K}^H$ fix $a'$ and $X$ pointwise, and $U$ pointwise modulo $V_G$ and $V_H$, respectively.

**Proof (Sketch).** Assume $a = cd$ with $c \in G$ and $d \in C'$. Then $d$ is uniquely determined modulo $G'$. By Lemma 11.2 there is a smallest finite good subspace $D \subseteq V$ such that $d \in \beta(D, D)$ and $D$ is definable over $a$. Let $XY$ be a transversal for $D$ with $D \cap V_G = \langle X \rangle$. Then $V_G$ and $\langle V_G \cup D \rangle$ are fixed setwise by all $f \in \text{Aut}_{\{G\}_{(a)}}(\mathbb{C})$. Hence there is a subgroup $\mathcal{K}$ of $\text{Aut}_{\{G\}_{(a)}}(\mathbb{C})$ of finite index such that

1. $f(y) \equiv y$ modulo $V_G$ for every $y \in Y$ and $f \in \mathcal{K}$.

$V_H$ and $\langle V_H \cup D \rangle$ are fixed setwise by all $g \in \text{Aut}_{\{H\}_{(a)}}(\mathbb{C})$. Since $cd = g(c)g(d)$, we have $c^{-1}g(c) = dg(d)^{-1} \in C' \cap H = H'$. We get a subgroup $\mathcal{K}^H$ of $\text{Aut}_{\{H\}_{(a)}}(\mathbb{C})$ of finite index such that

2. $g(y) \equiv y$ modulo $V_H$ for every $y \in Y$ and $g \in \mathcal{K}^H$.

We can assume

$$d = \sum_{x \in X} s_{xy} \beta(x, y) + \sum_{y \in Y} \sum_{y_1, y_2} s_{y_1, y_2} \beta(y_1, y_2)$$

(11.1)

and for every $x \in X$ there is some $y \in Y$ with $s_{xy} \neq 0$.

We choose $a = cd$, $D$, and $XY$ in such a way that (11.1) is true and $|X|$ is minimal. We show that we can choose $\mathcal{K} \subseteq \text{Aut}_{\{G\}_{(a)}}(\mathbb{C})$ of finite index in such a way that furthermore

3. $f(x) = x$ for all $x \in X$ and $f \in \mathcal{K}$.

Suppose not. Then there are some $x_0 \in X$ and $f \in \mathcal{K}$ such that $f(x_0) \neq u \in \langle X \rangle$. Let $X \supseteq X \cup \{u\}$ be a transversal for $V_G$ and $Y \supseteq Y$ such that $X \bar{Y}$ is a transversal for $V$. We can change $X$ and therefore $\bar{X}$ in such a way that $f(x) \in \langle \bar{X} \setminus \{u\} \rangle$ for all $x \in X \setminus \{x_0\}$.

Assume $f(y) = y + t_x + w_y$ for $y \in Y$. Using (11.1) and $f(d) = d$ modulo $V_G$ we obtain for every $y \in Y$

$$s_{xy} \beta(u, y) + \left[ \sum_{y_1 \in Y} s_{y_1, y} \beta(t_{y_1} u, y) - \sum_{y \in Y} s_{y, y_1}(t_{y_1} u, y) \right] = 0.$$ 

Since $\beta(u, y) \neq 0$

$$s_{xy} + \left[ \sum_{y_1 \in Y} s_{y_1, y} t_{y_1} - \sum_{y \in Y} s_{y, y_1} t_{y_1} \right] = 0$$

for all $y \in Y$. 

If we set \( y' = y + t,x_0 \), then we obtain (11.1) using only \( X \setminus \{x_0\} \) a contradiction to the minimality of \(|X|\). As \( acl(AG) \cap H = G \) we have \( \langle XY \rangle \cap V_H = \langle X \rangle \). We can do the same steps and choose \( \mathbb{K}^H \) in such a way that furthermore
\[
(c)^H \ g(x) = x \text{ for } x \in X \text{ and } g \in \mathbb{K}^H.
\]
Let \( Y \) be the disjoint union of \( P \) and \( U \), where \( P = \{ y \in Y \text{ there is some } y' \text{ with } s_{y,y'} \neq 0 \text{ or } s_{y',y} \neq 0 \} \).

Now we choose \( a = cd, D, \) and \( X, U, P \) such that (11.1), (a), and (c) are true and \(|P| \) is minimal. We show that we can choose \( \mathbb{K} \) such that
\[
(d) f(y) = y \text{ for } y \in P \text{ and } f \in \mathbb{K}.
\]
By (a) \( f(y) = y \) modulo \( V_G \). If (d) is not true, then there are \( y_0 \in P \) and \( u \notin \langle X \rangle \) such that \( f(y_0) = y_0 + u \).

We can choose another basis of \( \langle P \rangle \) that we call again \( P \) such that for \( y \in P \setminus \{y_0\} \)
\[
f(y) = y + w \text{ where } w \in \langle X \setminus \{u\} \rangle.
\]
But then we have a contradiction to the minimality of \(|P|\), since summands \( s_{y,y_0} \beta(y,u) \) or \( s_{y_0,y} \beta(u,y_0) \) cannot occur in the representation of \( f(d) = d \) modulo \( G' \) as linear combination of basic commutators over \( \tilde{X} \tilde{Y} \). Note that \( s_{y,y_0} \neq 0 \) or \( s_{y_0,y} \neq 0 \) for some \( y \) since \( y_0 \in P \). Similarly, we can assume
\[
(d)^H \ g(y) = y \text{ for } y \in P \text{ and } g \in \mathbb{K}^H.
\]
(a), (a)^H, (c), (c)^H, (d), (d)^H give the desired result. □

Now we can prove the theorem.

By Corollary 8.7 we have assumed that \( A \cap G = A \cap G' \) and \( A \cap (G \cdot Z(\mathbb{C})) = A \cap (G \cdot \mathbb{C}') \). Then \( A = \langle (A \cap \mathbb{C}') \cup A_1 \cup \{a_1, \ldots, a_m\} \rangle \) where \( A_1 \) is a basis of \( A \) modulo \( Z(\mathbb{C}) \) and \( a_1, \ldots, a_m \) are mixed elements. We assume
\[
a_1, \ldots, a_r \text{ are linearly independent modulo } Z(\mathbb{C})
\]
and
\[
a_{r+1}, \ldots, a_m \subseteq Z(G) \text{ are linearly independent modulo } G'.
\]
We apply Lemma 11.3 and obtain a part of a transversal \( X, U \) with \( \langle X \rangle = \langle XU \rangle \cap V_G \), and \( \mathbb{K} \subseteq Aut_{\langle G \rangle}(A)(\mathbb{C}) \) and \( \mathbb{K}^H \subseteq Aut_{\langle H \rangle}(A)(\mathbb{C}) \) both of finite index such that
\[
a_i = a_i^G \prod_{x \in X} [x,u]^j w a_i^U
\]
where \( a_i^G \in G, a_i^U \in \mathbb{C}' \), all \( f \in \mathbb{K} \) fix all \( a_i^U \) and \( X \) pointwise, and \( U \) pointwise modulo \( V_G \), and all \( g \in \mathbb{K}^H \) fix all \( a_i^U \) and \( X \) pointwise, and \( U \) pointwise modulo \( V_H \).

Hence we can assume w.l.o.g. \( a_i^U \in A \cap \mathbb{C}' \). We choose \( D \subseteq V \) definable over \( A \cup X \) such that \( (A \cap \mathbb{C}') \subseteq \beta(D,D) \), \( A/Z(\mathbb{C}) \subseteq D \), and \( X \subseteq D \). W.l.o.g. \( D \) is fixed pointwise by
the automorphisms of $K$ and $K^H$. We choose a basis $X^+Y^+$ of $D$ with $D \cap V_G = \langle X^+ \rangle$. Then we choose $U^+$ such that $X^+Y^+U^+$ is a basis of $\langle D \cup U \rangle$. Note that $U^+$ is linearly independent over $V_GY^+$.

We have that $X^+Y^+$ is fixed pointwise by the automorphism of $K$ and $K^H$ and $f(u) = u$ modulo $V_G$ for $u \in U^+$ and $f \in K$ and $g(u) = u$ modulo $V_H$ for $u \in U^+$ and $g \in K^H$. We can write $u \in U$ as $u = u_0 + u_1 + u_2$ with $u_0 \in \langle X^+ \rangle$, $u_1 \in \langle Y^+ \rangle$, and $u_2 \in \langle U^+ \rangle$. In (11.2) we can multiply $[xu_0]^{v_i}$ with $a_i^G$ and w.l.o.g. consider $[xu_1]$ as an element of $A \cap C^i$. Hence w.l.o.g.

$$a_i = a_i^G \prod_{x \in X \atop u \in U^+} [x, u]^{v_i}.$$  \hspace{1cm} (11.3)

Note that

$$X^+ \subseteq \text{acl}(Cb(tp(A/G))) \quad \text{and} \quad X^+ \subseteq \text{acl}(Cb(tp(A/H))).$$

To describe $\text{acl}(Cb(tp(A/G)))$ completely we need:

**Definition.** Let $\vec{v}^i = (v^i_1, \ldots, v^i_m) \ (i = 0, 1)$ be two sequences in $C$. We say $\theta_3(\vec{v}^0, \vec{v}^1)$ holds if there is an vector space homomorphism $h$ of $\langle U^+ \rangle$ into $V$ such that

$$v^0_i = v^1_i \prod_{x \in X \atop u \in U^+} [x, h(u)]^{v_i} \quad \text{for} \ 1 \leq i \leq m.$$  

This is an equivalence relation definable over $X$. Let $A$ be the $\theta_3$-class of $(a^G_1, \ldots, a^G_m)$. If $X$ is fixed pointwise we can consider $A$ as an imaginary. Since $a_i = f(a_i)$ for $f \in K$ we can show that $A \in \text{acl}(Cb(tp(A/G)))$ by the same proof as in Section 10. Analogously we have $A \in \text{acl}(Cb(tp(A/H)))$. Hence it remains to show that $\text{acl}(Cb(tp(A/G))) = \text{acl}(X^+ \cup A)$.

Let $f^G$ be an automorphism of $G$ that fixes $X^+$ pointwise and $A$. Let $\bar{Y}$ with $Y^+U^+ \subseteq \bar{Y}$ be a basis for $V$ modulo $V_G$. $A$ is the $\theta_3$-class of $(a^G_1, \ldots, a^G_m)$. Since $f^G(A) = A$ there is some linear map $k$ of $U^+$ into $V_G$ such that

$$f(a^G_i) = a^G_i \prod_{x \in X \atop u \in U^+} [x, k(u)]^{v_i}.$$  

Now we define an extension $(g, h) \in \text{Aut}(F(C))$ of $F(f^G)$ by $g(y) = y$ for $y \in \bar{Y} \setminus U^+$ and $g(u) = u - k(u)$ for $u \in U^+$. The automorphism $h$ of $F(V, V)$ is determined by $g$. Then $h$ can be any extension on $W$. By Corollary 3.3 there is an automorphism $f$ in $\text{Aut}_{F(G)}(C)$ that extends $f^G$ with $F(f) = (g, h)$ and $f(a) = a$ for $a \in A_1$. Since $D = \langle X^+Y^+ \rangle$ $D$ is fixed pointwise by $f$. 

Hence \( f \) fixes \( A_1 \) and \( D \) and therefore \( A \cap C' \). Finally we show \( f(a_i) = a_i \). We have

\[
\begin{align*}
\quad\quad f(a_i) &= f(a^G_i) \cdot f \left( \prod_{x \in X, u \in U^+} [x, u]^{v_{u}} \right) \\
&= a^G_i \prod_{x \in X, u \in U^+} [x, k(u)]^{v_{u}} \cdot \prod_{x \in X, u \in U^+} [x, u(k(u))^{-1}]^{v_{u}} \\
&= a^G_i \prod_{x \in X, u \in U^+} [x, u]^{v_{u}} = a_i. \quad \square 
\end{align*}
\]

The \( CM \)-triviality of \( \text{Th}(F_2(p, \omega)) \) is shown. It follows that it is impossible to interpret a field in this theory.

References