Explicit/implicit domain decomposition method with modified upwind differences for convection-diffusion equations

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Abstract

Based on domain decomposition and multi-layer explicit computation technique, one new explicit/implicit algorithm is presented for convection-diffusion equations, which has high parallelism and applies modified upwind differences to discretize diffusion term. By the analysis of auxiliary difference equations and the maximum principle, the resulting procedure is shown to be of second-order global accuracy in space. Numerical experiments illustrating the accuracy, efficiency and parallelism are shown.

Keywords: Decomposition; Difference equations; Modified upwind differences; Multi-layer explicit/implicit schemes; Error estimates

1. Introduction

Domain decomposition [1,2] is one powerful tool to solve the large-scale scientific computational problems. Explicit schemes are often naturally parallel and easy to implement. They usually require small time steps (denoted by \(\Delta t\)) to ensure the stability of these schemes, but this increases much computational work. Implicit schemes can proceed with any large time steps (denoted by \(\Delta t\)), but they are not inherently parallel. Thus, Dawson, Du, and Dupont [1,2] present one type of explicit/implicit alternating parallel schemes. They solve the heat equation by one explicit/implicit algorithm which is easy to implement and is of high efficiency in [1]. The values at inter-boundaries are calculated by explicit schemes with the larger space step \(H_D\), while those in subdomains are obtained by implicit computation with the space step \(h \ll H_D\). Then the stability condition, \(\Delta t \leq CH_D^2\), is much weaker than that caused by the whole-domain explicit scheme. But the larger step \(H_D\) affects the order of accuracy. To increase the order of accuracy and parallel efficiency, Du et al. [2] propose one modified explicit/implicit algorithm for the heat equation applying multi-step second-order explicit computation at interface boundaries with an intermediate mesh size \(H\) lying inside \((h, H_D)\) and implicit schemes in subdomains with the time step \(\Delta t\). The explicit computation is carried out

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with the time step $\Delta \tau$, where the number of time levels is $O(\Delta t/\Delta \tau)$, until the interface value at the distance of $\Delta t$ is obtained. The analysis of parallelism and advantages are proved in [2].

The central difference method for convection-dominated diffusion equations has second-order accuracy $O(h^2)$, but it usually introduces numerical dispersion and nonphysical oscillations into the numerical solutions. Characteristic difference methods [3] can reflect the hyperbolic properties of the solutions and efficiently solve convection-dominated problems. To ensure that mesh points along characteristics are all inside the whole domain, variant time-step technique [4] is introduced and more complicated computation is expended. In the articles [5,6], Ewing and the other authors offer the upwind difference methods to approximate the convection-diffusion problems. Modified upwind difference methods are developed in displacement problems and multi-layer porous composite systems by Yuan [7,8].

For convection-diffusion equations, based on the properties of upwind methods [5–8] and combined with domain decomposition and parallel efficiency analysis [2], one new explicit/implicit domain decomposition algorithm is presented here, which applies modified upwind differences to approximate the diffusion term. To improve the parallelism and efficiency, we discretize the equation in subdomains by implicit scheme, and compute the interface value by multi-level explicit scheme with the space step $h \in (h, H_D)$. We derive the optimal convergence analysis in $l^\infty$ norm by analyzing some difference equations based on the convection-diffusion operator.

The rest of this paper is organized as follows. The one-dimensional model, the algorithm and some notations are given in Section 2. In Section 3 the error analysis in $l^\infty$ norm is discussed. Some numerical experiments are given in the last section.

2. The model, notations and procedures

Consider the following convection-diffusion problem:

$$
\frac{\partial u}{\partial t} = Lu + d(x, t), \quad x \in \Omega = (0, 1), \quad t \in (0, T],
$$

$$
u(x, 0) = u^0(x), \quad x \in \Omega,
$$

$$u(0, t) = u(1, t) = 0, \quad t \in (0, T],
$$

where $Lu = \frac{\partial}{\partial x}(a(x) \frac{\partial u}{\partial x}) - b(x) \frac{\partial u}{\partial x} - c(x) u$. Assume that the coefficient $a(x)$ satisfies

$$0 < a_0 \leq a(x) \leq a_1,$$

and $b(x), \ c(x)$ are bounded functions.

For simplicity, let $\Omega$ be discretized uniformly by $\Omega_h$, $x_i = ih$, with the space step $h = 1/N_1$. We decompose $\Omega$ into only two subdomains (see Fig. 1.) $\Omega_1 = (0, \bar{x})$ and $\Omega_2 = (\bar{x}, 1)$. Suppose that $\bar{x} = x_l, 0 < \bar{x} < 1, 0 < H \leq \min(\bar{x}, 1 - \bar{x})$, and $H = M_H h$ for some integer $M_H$. Let $\Delta t = T/N_2$, $\Delta \tau = \Delta t/m$ for some integer $m$ be time steps for implicit schemes and multi-step explicit schemes, respectively. Let $t^n = n \Delta t$, and $t^k = k \Delta \tau$. From $t^n$ to the next level $t^{n+1}$, the value at the interface point $\bar{x}$ can be obtained by $m$ levels explicit computation. During the progress, we redefined the mesh points by two sets $\Omega_h^E$ and $\Omega_h^I$. At the time level $t^n + \tau^1$, the points involved in explicit computation are denoted by $X_J = x_i + JH, \ -m+1 \leq J \leq m-1$. Take $\Omega_h^E = \{(X_J, t^n + \tau^k)|J = 0, \pm 1, \ldots, \pm (m-k), \ n \geq 0\}$ for $k = 1, 2, \ldots, m$, then $\Omega_h^E = \bigcup_{k=1}^{m} \Omega_h^{E_k}$ and $\Omega_h^I = \{(x_i, t^n)|i \neq l, \ n > 0\}$. In Fig. 2, “o” points correspond to where the implicit scheme is applied, “x” points correspond to where the explicit scheme is applied, and “-“ points correspond to where the boundary conditions are applied.

For any grid function $u(x, t)$, let $u_i^n = u(x_i, t^n)$, $u_i^{n,k} = u(x_i, t^n + k \Delta \tau)$, and let $u_i^{n+1} = u_i^{n,m}, \ u_i^n = u_i^{n,0}$ for simplicity. Denote the difference operators by $\delta_t, \delta_{\Delta \tau} u_i^n = \frac{u(x_i, t^n + \Delta \tau) - u(x_i, t^n)}{\Delta \tau}$, $\delta_x, \delta_{h} u_i^n = \frac{u_i^{n+1} - u_i^n}{h}, \ \delta_{\bar{x},h} u_i^n = \frac{u_i^n - u_i^{n-1}}{h}, \ \delta_{\bar{x},\bar{h}} u_i^n = \frac{u_i^n - u_i^{n-1}}{\bar{h}}$, and $u_i^{n+1/2} = u(ih + h/2, t^n)$. Let

$$
\delta_{\bar{x},\bar{h}}(a \delta_{x,h} u_i^n) = h^{-2}[a_{i+1/2} (u_i^{n+1} - u_i^{n}) - a_{i-1/2}(u_i^n - u_i^{n-1})].
$$

Fig. 1. Partition of two nonoverlapping subdomains.

Next the multi-step explicit/implicit domain decomposition procedures with modified upwind differences are discussed. Define $\lambda(x)$ by

$$\lambda(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

Take $L_h u$, $L_H u$ as the discrete operators of $Lu$ in subdomains and at interface point, respectively. Then the procedures of (2.1)–(2.3) are described as follows:

$$U_i^{n+1} = u_i^{n+1}, \quad i = 0, N_1 \quad (2.6)$$

$$\delta_t \Delta \tau \delta_{j,k}^n - L_H u_j^{n,k-1} = a_j^{n,k}, \quad (X_j, t^{n,k}) \in \Omega_h^E \quad (2.7)$$

$$\delta_t \Delta \tau U_i^{n+1} - L_H u_i^{n+1} = d_i^{n+1}. \quad (x_i, t^{n+1}) \in \Omega_h^I \quad (2.8)$$

The operators $L_H$ and $L_h$, applying second-order upwind differences to the diffusion term, are defined by

$$L_H U_j^{n,k-1} = \left( 1 + \frac{H}{2a_j} |b_j| \right)^{-1} \delta_{x,H}(a \delta_{x,H} U)_j^{n,k-1} - \delta_{b,x,H} U_j^{n,k-1} - c_j U_j^{n,k-1}. \quad (2.9)$$

$$L_h U_i^{n+1} = \left( 1 + \frac{h}{2a_i} |b_i| \right)^{-1} \delta_{x,h}(a \delta_{x,h} U)_i^{n+1} - \delta_{b,x,h} U_i^{n+1} - c_i U_i^{n+1}. \quad (2.10)$$

where

$$\delta_{b,x,h} U_i^{n,k-1} = b_i [\lambda(b_i) a_{i+1/2} / a_i \delta_{x,h} U_i^{n,k-1} + (1 - \lambda(b_i)) a_{i+1/2} / a_i \delta_{x,h} U_i^{n,k-1}].$$

The values at initial level are computed by

$$U_i^0 = u_i^0 \quad (0 \leq i \leq N_1).$$

The algorithm works in the following order. For a typical time step, starting with the solution at time level $t^n$, we can obtain the values at explicit points $U_j^{n,k}$ ($J = 0, \pm 1, \ldots, \pm (m-k)$) for any time level $t^{n+k\tau}$ by (2.7) with time step $\tau$, then we obtain the value at interface point $U_i^{n+1}$ ($k = m$) easily. Noting that two subdomains are nonoverlapping, we can compute the values of $\Omega_1$ and $\Omega_2$ in parallel by the implicit scheme (2.8).

3. Error estimates

Because of the explicit nature of (2.7) the following stability conditions are necessary

$$2a(\bar{x}) \left( 1 + \frac{|b| H}{2a(\bar{x})} \right)^{-1} \frac{\Delta \tau}{H^2} - \frac{|b| a_0 \Delta \tau}{a_1 H} - c \Delta \tau \leq 1, \quad (3.1)$$

$$a(\bar{x}) - H |b(\bar{x})| \geq 0. \quad (3.2)$$
and
\[ 1 + \Delta tc(x) \geq 0. \] (3.3)

The conditions (3.2) and (3.3) hold obviously for \( H, \Delta t \) sufficiently small. In the remainder \( M \) means some generic positive constant, which denotes different meanings at different places.

To discuss the convergence analysis, we introduce the maximum principle.

**Lemma 1.** Suppose that (3.1) holds, and that \( z_i^{n+1} \) satisfies the following relations
\[
\begin{align*}
    z_i^{n+1} &\leq 0, \quad i = 0, N_1 \\
    \delta_{t_i, \Delta t} z_{i,j}^{n,k} - L_H z_{j,k-1}^{n,i} &\leq 0, \quad (X_J, t_{n,k}) \in \Omega_h^E \quad (3.5) \\
    \delta_{t_i, \Delta t} z_i^{n+1} - L_h z_i^{n+1} &\leq 0, \quad (x_i, t^{n+1}) \in \Omega_h^I \quad (3.6)
\end{align*}
\]

Then
\[ z_i^{n+1} \leq 0. \] (3.7)

**Proof.** The main task of the proof is to verify the positivity of the coefficients of the values at the points involved in the procedures, which is ensured by the stability conditions (3.1)–(3.3).

To obtain the error analysis with the maximum principle, we define comparison functions \( \theta(x) \) and \( \beta^J(x) \) on \( \Omega_h^I \) and \( \Omega_h^E \) corresponding to the implicit part and explicit part of the scheme.

\[
\begin{align*}
    \theta_i &= 0, \quad i = 0, N_1 \quad (3.8) \\
    \theta_i &= \left( 1 + \frac{h|b_i|}{2a_i} \right)^{-1} \delta_{x_i} (a \delta_{x_i} \theta_i) + \delta_{b_i, x_i} \theta_i + c_i \theta_i = 1, \quad 1 \leq i \leq N_1 - 1, \quad i \neq l \quad (3.9) \\
    \theta_i &= \left( 1 + \frac{H|b_i|}{2a_i} \right)^{-1} \delta_{x_i} (a \delta_{x_i} \theta_i) + \delta_{b_i, x_i} \theta_i + c_i \theta_i = 1, \quad i = l \quad (3.10)
\end{align*}
\]

and
\[
\begin{align*}
    \beta_i^J &= 0, \quad i = 0, N_1 \quad (3.11) \\
    \beta_i^J &= \left( 1 + \frac{h|b_i|}{2a_i} \right)^{-1} \delta_{x_i} (a \delta_{x_i} \beta_i^J) + \delta_{b_i, x_i} \beta_i^J + c_i \beta_i^J = 0, \quad x_i \neq X_J \quad (3.12) \\
    \beta_i^J &= \left( 1 + \frac{H|b_i|}{2a_i} \right)^{-1} \delta_{x_i} (a \delta_{x_i} \beta_i^J) + \delta_{b_i, x_i} \beta_i^J + c_i \beta_i^J = 1, \quad x_i = X_J. \quad (3.13)
\end{align*}
\]

Now we give the advantages of the functions \( \theta(x) \), \( \beta(x)^J \) in the following lemma.

**Lemma 2.** The solutions \( \theta \) and \( \beta^J \) of (3.8)–(3.10), (3.11)–(3.13), respectively, exist and are nonnegative, unique, and the following estimates hold:
\[ |\theta(x)| \leq M, \] (3.14)

and
\[ |\beta^J(x)| \leq MH. \] (3.15)

**Proof.** By the maximum principle [6], we have obviously
\[ \theta_i \geq 0, \quad \beta_i^J \geq 0. \] (3.16)
and the existence and uniqueness hold. Let

$$A_i = \left( 1 + \frac{|b_i| h}{2a_i} \right)^{-1} a_i - 1 + \lambda(b_i) a_i^{-1/2} b_i h, \quad B_i = \left( 1 + \frac{|b_i| h}{2a_i} \right)^{-1} a_i + 1/2 - (1 - \lambda(b_i)) a_i^{-1/2} b_i h. $$

There exists an integer $d_0$ such that for $h$ sufficiently small,

$$0 < d_0 \leq \left( 1 + \frac{|b_i| h}{2a_i} \right)^{-1} \leq 1. \quad (3.17)$$

Then

$$0 < d_0 a_0 \leq A_i \leq a_1 (1 + Mh), \quad 0 < d_0 a_0 \leq B_i \leq a_1 (1 + Mh). \quad (3.18)$$

We express (3.9) in the following way

$$A_i (\theta_i - \theta_{i-1}) - B_i (\theta_{i+1} - \theta_i) + c_i h^2 \theta_i = h^2. \quad (3.19)$$

Let $g(n) = (1 + \frac{1}{n})^n$, and the following result is to be used later

$$\lim_{n \to \infty} g(n) = e, \quad (3.20)$$

where $g(n)$ is monotone increasing. Denote the coefficients of involved points by

$$\hat{A}_i = \prod_{j=1}^{i} \frac{A_j + c_j h^2}{B_j}, \quad \hat{B}_i = \frac{1}{B_i} + \sum_{j=1}^{i-1} \left( \frac{1}{B_j} \sum_{k=j+1}^{i} \frac{A_k + c_k h^2}{B_k} \right), \quad i = 1, \ldots, l - 1 \quad (3.21)$$

and

$$\tilde{A}_i = \prod_{j=i}^{N_i-1} \frac{B_j + c_j h^2}{A_j}, \quad \tilde{B}_i = \frac{1}{A_i} + \sum_{j=i+1}^{N_i-1} \left( \frac{1}{A_j} \sum_{k=j}^{i} \frac{B_k + c_k h^2}{A_k} \right), \quad i = l + 1, \ldots, N_1 - 1 \quad (3.22)$$

then

$$\tilde{A}_i \leq \frac{a_1}{d_0 a_0} (1 + Mh)^i, \quad \tilde{A}_i \leq \frac{a_1}{d_0 a_0} (1 + Mh)^{N_1 - i}. \quad (3.23)$$

Note that

$$\theta_2 - \theta_1 = \frac{A_1 + c_1 h^2}{B_1} \theta_1 - \frac{1}{B_1} h^2 = \tilde{A}_i \theta_1 - \tilde{B}_i h^2, $$

and

$$\theta_3 - \theta_2 = \left( \frac{A_2 + c_2 h^2}{B_2} \frac{A_1 + c_1 h^2}{B_1} + \frac{c_2}{B_2} h^2 \right) \theta_1 - \left( \frac{1}{B_2} + \frac{1}{B_1} \frac{A_2 + c_2 h^2}{B_2} \right) h^2 = (\tilde{A}_2 + \tilde{M}_2 h^2) \theta_1 - \tilde{B}_2 h^2, $$

where $\tilde{M}_1 = \tilde{M}_1^* = \tilde{M}_2^* = 0$, $\tilde{M}_2 = \frac{c_2}{B_2} h^2$.

Then by (3.8) and (3.19) and recursion relations, we have for $1 \leq i \leq l - 1$

$$\theta_{i+1} - \theta_i = (\tilde{A}_i + \tilde{M}_i h^2) \theta_1 - (\tilde{B}_i + \tilde{M}^*_i h^2) h^2, \quad (3.24)$$

where $\tilde{M}_i$ and $\tilde{M}^*_i$ are positive numbers dependent on $A_i$, $B_i$, $c_i$ and bounded by some positive constant $M$.

Similarly, it holds for $l + 1 \leq i \leq N_1 - 1$

$$\theta_{i+1} - \theta_i = (\tilde{A}_i + \tilde{M}_i h^2) \theta_1 - (\tilde{B}_i + \tilde{M}^*_i h^2) h^2, \quad (3.25)$$

where $0 < \tilde{M}_i + \tilde{M}^*_i \leq M$.

By (3.20)–(3.22), (3.24), (3.25) and (3.23), we obtain

$$\theta_i \leq \left( 1 + \frac{a_1}{d_0 a_0} e^M N_1 \right) \theta_1 + M, \quad 1 \leq i \leq l \quad (3.26)$$
and
\[ \theta_l \leq \left( 1 + \frac{a_1}{d_0} e^M N_1 \right) \theta_{N_l - 1} + M, \quad l + 1 \leq i \leq N_1 - 1. \] (3.27)

Replace \( i \) in (3.24) with \( l - M_H, l - M_H + 1, \ldots, l - 1 \), and summarize them together, then we have
\[ \theta_l - \theta_{l - M_H} = \sum_{j=l-M_H}^{l} \left( \tilde{A}_j + \tilde{M}_j h^2 \right) \theta_1 - \sum_{j=l-M_H}^{l} \left( \tilde{B}_j + \tilde{M}_j^* h^2 \right) h^2. \] (3.28)

Substitute \( l + 1, l + 2, \ldots, l - M_H - 1, l - M_H \) for \( i \) in (3.25), then
\[ \theta_l - \theta_{l + M_H} = \sum_{j=l+M_H}^{N_l-1} \left( \tilde{A}_j + \tilde{M}_j h^2 \right) \theta_{N_l - 1} - \sum_{j=l+M_H}^{N_l-1} \left( \tilde{B}_j + \tilde{M}_j^* h^2 \right) h^2. \] (3.29)

Note that
\[ \left\{ \left( 1 + \frac{|b_l| H}{2a_l} \right)^{-1} a_{l-M_H/2} + \lambda(b_l) \frac{a_{l-M_H/2} b_l H}{a_l} \right\} (\theta_l - \theta_{l-M_H}) \]
\[ + \left\{ \left( 1 + \frac{|b_l| H}{2a_l} \right)^{-1} a_{l-M_H/2} - (1 - \lambda(b_l)) \frac{a_{l-M_H/2} b_l H}{a_l} \right\} (\theta_l - \theta_{l+M_H}) + c_l H^2 = H^2, \] (3.30)

and substitute (3.28) and (3.29) in the above expression,
\[ \theta_1 + \theta_{N_l - 1} \leq MH. \] (3.31)

Then it follows from (3.26), (3.27) and (3.31)
\[ \theta_l \leq M, \quad 1 \leq i \leq N_1 - 1. \] (3.32)

Next we consider the comparison function \( \beta^J \).
Suppose that \( X_j = x_{ij} \). As \( i \neq l_j \), (3.12) is turned into
\[ A_i (\beta^J_i - \beta^J_{i-1}) - B_i (\beta^J_{i+1} - \beta^J_i) + c_i h^2 \hat{\beta}_i^J = 0. \] (3.33)

For \( 0 < i \leq l_j \), we obtain
\[ \beta^J_i = \sum_{j=1}^{l-1} \left( \tilde{A}_j + \tilde{M}_j h^2 \right) \beta^J_1, \] (3.34)

and for \( l_j \leq i < N_1 \),
\[ \beta^J_i = \sum_{j=l+1}^{N_l-1} \left( \tilde{A}_j + \tilde{M}_j h^2 \right) \beta^J_{N_l-1}. \] (3.35)

Furthermore,
\[ \beta^J_{l_j} - \beta^J_{l_j-M_H} = \sum_{j=l_j-M_H}^{l_j-1} \left( \tilde{A}_j + \tilde{M}_j h^2 \right) \beta^J_1, \] (3.36)

and
\[ \beta^J_{l_j} - \beta^J_{l_j+M_H} = \sum_{j=l_j+1}^{l_j+M_H} \left( \tilde{A}_j + \tilde{M}_j h^2 \right) \beta^J_{N_l-1}. \] (3.37)

It follows from (3.11) and (3.13),
\[ \beta^J_1 + \beta^J_{N_l-1} \leq MHh. \] (3.38)
Thus, by (3.34) and (3.35),

$$\beta_i^J \leq MH, \quad 1 \leq i \leq N_1 - 1.$$  \hfill (3.39)

The proof is completed. ■

By (3.34), (3.35) and (3.39), it is easy to see that for any $\beta^J$

$$0 < - \left(1 + \frac{h|b_i|}{2a_i}\right)^{-1} \delta_{x,h}(a\delta_{x,h}\beta^J)_i + \delta_{b,x,h}\beta_i^J + c_i\beta_i^J \leq \frac{M}{h}, \quad x_i = X_J.$$  \hfill (3.40)

Let the function $\beta_i = \sum_{|J|\leq m-1} \beta_i^J$, then

$$|\beta(x_i)| \leq MmH.$$  \hfill (3.41)

With the maximum principle, we have the error analysis for the multi-step explicit/implicit scheme.

**Theorem.** Under the Assumptions (3.1)–(3.3) there exists a constant $M$ independent of $\Delta t, h, H, m$ and the coefficients $a, b, c$ such that

$$\max_{i,n} |u^n_i - U^n_i| \leq M(\Delta t + h^2 + mH^3),$$  \hfill (3.42)

where $u$ and $U$ are the true solution of (2.1)–(2.3) and the approximation generated by the multi-step explicit/implicit scheme, respectively.

**Proof.** Let $\epsilon_i^{n+1} = u_i^{n+1} - U_i^{n+1}$, then we obtain the error equations from the models and difference scheme

$$\epsilon_i^{n+1} = 0, \quad i = 0, N_1$$  \hfill (3.43)

$$\delta_{t,\Delta \tau} \epsilon_i^{n,k} - L_H \epsilon_i^{n,k-1} = K_i^{n,k}(\Delta \tau + H^2), \quad (x_i, t^{n,k}) \in \Omega_h^E$$  \hfill (3.44)

$$\delta_{t,\Delta \tau} \epsilon_i^{n+1} - L_H \epsilon_i^{n+1} = K_i^{n+1}(\Delta t + h^2), \quad (x_i, t^{n+1}) \in \Omega_h^I,$$  \hfill (3.45)

where for all $i, n, k$, there exists a constant $C_0 > 0$ such that

$$|K_i^{n,k}| \leq C_0.$$  \hfill (3.46)

Define a comparison function $\xi_i = C_0[\theta_1(\Delta t + h^2) + \beta_1(\Delta \tau + H^2)].$ Let $z_i^{n+1} = \epsilon_i^{n+1} - \xi_i$, taking the place of $\epsilon$ in (3.43)–(3.45), then from the definition of $\theta, \beta$ and (3.46) we have

$$z_i^{n+1} = 0, \quad i = 0, N_1$$  \hfill (3.47)

$$\delta_{t,\Delta \tau} z_i^{n,k} - L_H z_i^{n,k-1} = (K_i^{n,k} - C_0)(\Delta \tau + H^2) \leq 0, \quad (x_i, t^{n,k}) \in \Omega_h^E$$  \hfill (3.48)

$$\delta_{t,\Delta \tau} z_i^{n+1} - L_H z_i^{n+1} = (K_i^{n+1} - C_0)(\Delta t + h^2) \leq 0 \quad (x_i, t^{n+1}) \in \Omega_h^I.$$  \hfill (3.49)

Applying Lemma 1, we obtain $z_i^{n+1} \leq 0$, which implies that $\epsilon_i^{n+1} \leq \xi_i$.

Replacing $z_i^{n+1}$ by $-z_i^{n+1}$ in the above argument leads to

$$|\epsilon_i^{n+1}| \leq \xi_i.$$  \hfill (3.50)

Thus it follows from (3.32) and (3.41)

$$0 \leq \xi_i \leq C_0 M(\Delta t + h^2 + mH^3).$$  \hfill (3.51)

This completes the proof. ■
4. Numerical experiments

Some experimental results are given to show the accuracy and parallelism of the procedures presented in Section 3. The true solution is

\[ u(x, t) = \exp^{-\pi^2 t} \sin \pi x, \]  

satisfying (2.1), where the coefficients are defined by

\[ a(x) = x^2 + 10^{-2}, \quad b(x) = 10^2(x - 0.5), \quad c(x) = 1.0, \]

\[ d(x, t) = (1 + (a(x) - 1)\pi^2) \exp^{-\pi^2 t} \sin \pi x + (b(x) - 2\pi) \exp^{-\pi^2 t} \cos \pi x. \]  

The data obtained by the full-implicit upwind difference scheme (FIUDS), multi-level domain decomposition with upwind difference scheme (MDDUDS) and the exact solution at \( T = 0.1 \) are shown in Figs. 3 and 4 and Table 1. During the whole runs, take the parameters \( h = \frac{1}{N_1}, \Delta t = 0.01, \Delta \tau = \frac{\Delta t}{M_H} \), and denote the interface point, the space step of explicit, the number of explicit level needed in every interval of the large step size, error estimates in \( l^\infty \)-norm and \( l^2 \)-norm by \( I, M_H, m, E_{l^\infty} \) and \( E_{l^2} \), respectively. Two grids used in Figs. 3 and 4 are defined by \( \Omega_{dh1} : h = \frac{1}{200}, l = 100, M_H = 30 \) and \( \Omega_{dh2} : h = \frac{1}{200}, l = 100, M_H = 15, m = 2. \)

In Table 1 MDDUDS is the scheme presented by Daswone, Du, Dupont in [1] as \( m = 1 \). From the figures and Table 1 we can conclude that the following results are consistent with the theoretical analysis.

1. MDDUDS can simulate the convection-diffusion problems well;
2. MDDUDS has higher order of accuracy than FIUDS;
3. The procedures presented here not only effectively reduce numerical diffusion and nonphysical oscillations, but also can reduce much work by working on the massive parallel computer without accuracy cost.
Fig. 4. Discrete solution \((m = 2)\) and exact solution, \(\Omega_{h^2}\).

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