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A pointwise gradient estimate in possibly unbounded domains with nonnegative mean curvature

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Abstract

We prove a pointwise gradient bound for bounded solutions of $\Delta u + F'(u) = 0$ in possibly unbounded proper domains whose boundary has nonnegative mean curvature.

We also obtain some rigidity results when equality in the bound holds at some point.

Keywords: Semilinear elliptic PDEs; Pointwise a priori estimates in proper domains; Rigidity and classification of solutions

Given $F \in C^{1,1}_{\text{loc}}(\mathbb{R})$, we study bounded solutions of the problem

$$\begin{cases} \Delta u + F'(u) = 0 & \text{in } \Omega, \\ u \geqslant 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

We suppose that Ω is a connected open subset of $\mathbb{R}^n$ (say, a "domain", with $n \geqslant 2$), with nonempty $C^{2,\alpha}_{\text{loc}}$ boundary: in particular, Ω is a proper subset of $\mathbb{R}^n$ (that is, $\Omega \neq \emptyset$ and $\Omega \neq \mathbb{R}^n$), and $\partial\Omega$ is sufficiently smooth to define its normal and to compute its variation, that is to define the mean curvature of $\partial\Omega$ (notice that neither Ω or $\partial\Omega$ need to be bounded).

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If $u \in C^2(\overline{\Omega}) \cap L^\infty(\Omega)$ is a solution of (1), we define

$$c_u := \sup_{r \in [0, \|u\|_{L^\infty(\Omega)}]} F(r). \tag{2}$$

Then, the pointwise gradient estimate that we would like to obtain is the following one:

$$\frac{1}{2}\left|\nabla u(x)\right|^2 \leqslant c_u - F\big(u(x)\big), \quad \text{for any } x \in \Omega. \tag{3}$$

We prove that (3) holds true in cylinders and epigraphs, according to the following results:

Theorem 1. (i) *Let $\Omega = \Omega_o \times \mathbb{R}^{n-n_o}$, where $\Omega_o \subset \mathbb{R}^{n_o}$ is a bounded domain and $1 \leqslant n_o \leqslant n$. Let $u \in C^2(\overline{\Omega}) \cap L^\infty(\Omega)$ be a solution of* (1).
Assume that

$$\textit{the mean curvature of } \partial\Omega \textit{ is nonnegative.} \tag{4}$$

Then, (3) *holds true.*
(ii) *Suppose that Ω is an epigraph, viz. there exists $\Phi : \mathbb{R}^{n-1} \to \mathbb{R}$ such that*

$$\Omega = \left\{\left(x', x_n\right) \in \mathbb{R}^{n-1} \times \mathbb{R} \text{ s.t. } x_n > \Phi\left(x'\right)\right\}.$$

Assume that $\Phi \in C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}^{n-1})$, with

$$\|\nabla \Phi\|_{C^{1,\alpha}(\mathbb{R}^{n-1})} < +\infty. \tag{5}$$

Suppose also that (4) *is satisfied. Let $u \in C^2(\overline{\Omega}) \cap L^\infty(\Omega)$ be a solution of* (1). *Then,* (3) *holds true.*

The pointwise estimate of Theorem 1, in its versions 1(i) and 1(ii), may be seen as an extension of the one obtained in [15], where a similar result was proven in the case of solutions in the entire space $\mathbb{R}^n$. See also [10] for similar results in the halfspace.

We remark that Φ in Theorem 1(ii) does not need to be bounded.

The possibility $n = n_o$ is also allowed in Theorem 1(i): in fact, when $n = n_o$, Theorem 1(i) boils down to the case of bounded domains, for which several pointwise gradient estimates were collected in [18] (see, for instance, Corollary 5.1 there).

An important strengthening of the work of [15] was also performed in [5], where singular and degenerate operators were considered (still in the whole space $\mathbb{R}^n$).

With respect to [15,5], here we need to take into account the presence of the boundary terms, and this will provide some technical complications in the proof. Also, differently from [10], here the boundary is not necessarily flat, and so we will have to take into account the role played by its curvature (see also Remark 2(i) at the end of this introduction).

Besides working in proper domains, we have here some technical improvements with respect to the setting of [15,5]. Indeed, in [15,5], F was assumed to be $C^2(\mathbb{R})$ and nonpositive (while, here, F may be only $C^{1,1}_{\mathrm{loc}}(\mathbb{R})$ and no sign assumption is needed). Also, in [15,5] the term c_u was not present in (3) and so we have a somewhat sharper estimate here (in particular, such c_u may often be simply evaluated, see (6) and (9) below).

Indeed, solutions satisfying (3) enjoy some extra properties as stated in the following results:

Theorem 2. *Let $\Omega \subset \mathbb{R}^n$ be a domain and let $u \in C^2(\overline{\Omega}) \cap L^\infty(\Omega)$ be a solution of* (1) *for which* (3) *holds true.*

Then,

$$c_u = \max\big\{F(0), F\big(\|u\|_{L^\infty(\Omega)}\big)\big\} \quad \text{and} \quad c_u > F(t) \quad \text{for any } t \in \big(0, \|u\|_{L^\infty(\Omega)}\big). \tag{6}$$

Moreover, if

$$\Omega \text{ has } C^1 \text{ boundary} \tag{7}$$

and

$$F'(0) \geqslant 0, \tag{8}$$

then

$$c_u = F\big(\|u\|_{L^\infty(\Omega)}\big) \quad \text{and} \quad c_u > F(t) \quad \text{for any } t \in \big[0, \|u\|_{L^\infty(\Omega)}\big). \tag{9}$$

Of course, Theorem 2 is meaningful when u does not vanish identically (however, there is no need to take this as an assumption, since (6) and (9) are true even in this case, since there is nothing to check for the emptyset).

Moreover, condition (7) may be weakened: for instance, it is enough to assume that any two points in $\overline{\Omega}$ may be connected with C^1 curves, which is enough to obtain (56).

Theorem 3. *Let $\Omega \subset \mathbb{R}^n$ be a domain and $u \in C^2(\overline{\Omega}) \cap L^\infty(\Omega)$ be a solution of* (1) *for which* (3) *holds true.*

Suppose that

$$\text{equality in (3) holds at some point } p \in \Omega \cap \{\nabla u \neq 0\}. \tag{10}$$

Then it holds in the whole Ω and there exists $\omega \in \mathrm{S}^{n-1}$ and $u_o : \mathbb{R} \to \mathbb{R}$ such that $u(x) = u_o(\omega \cdot x)$.

Moreover, the following alternative holds:

(a) *either Ω is a halfspace, say $\{\omega \cdot x > c\}$ for some $c \in \mathbb{R}$,*
(b) *or Ω is a strip, say $\{\omega \cdot x \in (c_-, c_+)\}$, for some $c_+ > c_-$.*

Furthermore, if

$$u > 0 \quad \text{in } \Omega, \tag{11}$$

then the above alternative can be better clarified as follows:

(a⋆) *if $\Omega = \{\omega \cdot x > c\}$ for some $c \in \mathbb{R}$, then u is strictly monotone in direction ω and with nonvanishing gradient,*

(b⋆) *if $\Omega = \{\omega \cdot x \in (c_-, c_+)\}$, for some $c_+ > c_-$, then u is strictly monotone in direction ω and with nonvanishing gradient in both $\{\omega \cdot x \in (c_-, c_\star)\}$ and $\{\omega \cdot x \in (c_\star, c_+)\}$, where $c_\star := (c_+ + c_-)/2$, and it is even with respect to $\{\omega \cdot x = c_\star\}$.*

If, on the other hand,

$$\{u = 0\} \cap \Omega \neq \emptyset, \tag{12}$$

then u_o is periodic.

Remark 1. The symmetry and monotonicity results in the above Theorem 3 are obtained directly and without the moving plane methods used in [7,6,1–3].

Also, as a consequence of Theorem 3, we have that, when Ω is neither a strip or a halfspace, then either (3) is a strict inequality or the equality in (3) is attained at points p which have to be critical for u. In such case, $u(p)$ is equal to either 0 or $\|u\|_{L^\infty(\Omega)}$, thanks to (6) – and, in fact, $u(p) = \|u\|_{L^\infty(\Omega)}$ if (8) holds, due to (9).

Remark 2. The conditions assumed for our results are somewhat natural and they cannot be dropped.

More precisely:

(i) Condition (4) cannot be dropped in Theorem 1. Indeed, it is easy to see that, when $n \geqslant 3$, the function

$$u(x) = 1 - \frac{1}{|x|^{n-2}}$$

is a bounded positive solution of $\Delta u = 0$ in $\mathbb{R}^n \setminus B_1$ with $u = 0$ on ∂B_1: this solution does not satisfy (3), since, in this case, F may be taken as 0 (notice that the mean curvature of $\partial(\mathbb{R}^n \setminus B_1)$ is negative and compare with (4)).

(ii) The assumption that the solution is bounded cannot be removed as well: for instance the function $u(x_1, x_2) = x_2 e^{x_1}$ on the halfspace $\{x_2 > 0\}$ is positive and it solves $\Delta u - u = 0$, so we can take $F(u) = -u^2/2$, but it is easily checked that there exists no $c \in \mathbb{R}$ for which

$$\frac{|\nabla u(x)|^2}{2} \leqslant c - F\big(u(x)\big) \quad \text{for any } x \in \mathbb{R}^2.$$

Analogously, the function $u(x) = x_1$ is an unbounded solution of $\Delta u = 0$ on $\{x_1 > 0\}$ with $u = 0$ on $\partial\{x_1 > 0\}$: in this case, F may be taken as 0, and so $c_u = 0$ and (3) does not hold.

(iii) Estimate (3) is, of course, sharp in the sense that the solutions of the ODE attain the equality in (3) (which can be easily checked, for instance, by integrating the ODE).

(iv) If we want (9) to hold, condition (8) cannot be dropped. For instance, the function $u(x_1) = 1 - \cos(x_1)$ solves (1) with $F(u) = (u^2/2) - u$ in $\{x_1 > 0\}$ but it does not satisfy (9), since, in this case

$$c_u = \sup_{r\in[0,2]} F(r) = 0 = F(2) = F(0).$$

Notice that in this case $F'(0) = -1$, so (8) does not hold.

(v) If condition (11) does not hold, in general it is not possible to obtain the monotonicity results given in (a⋆) and (b⋆) of Theorem 3: a counterexample is again given by $u(x_1) = 1 - \cos x_1$.

(vi) If F is nonnegative and bounded, (3) implies the following universal estimate:

$$\frac{1}{2}\left|\nabla u(x)\right|^2 \leqslant \frac{1}{2}\left|\nabla u(x)\right|^2 + F\big(u(x)\big) \leqslant c_u \leqslant \sup_{r\in\mathbb{R}} F(r).$$

We now prove the results stated above.

Proof of Theorem 1. The method of proof is inspired by the work done in [15,5], as modified in [10].

We prove (3) in case Ω is an epigraph, that is Theorem 1(ii).

The case of Theorem 1(i) is even simpler, and we will comment on this at the end of the proof. Let

$$G(t) := c_u - F(t).$$

Note that

$$G(t) \geqslant 0 \tag{13}$$

for any $t \in [0, \|u\|_{L^\infty(\Omega)}]$.

Given any v, which is C^2 in its domain, and any x in the domain of v, we define

$$P(v,x) := \left|\nabla v(x)\right|^2 - 2G\big(v(x)\big). \tag{14}$$

These type of P-functions have been extensively investigated in the PDE literature, after the pioneering work of [16] (see, e.g., Chapter 5 in [18] and references therein).

By formula (2.7) of [5], we know that, if $\Delta v = G'(v)$,

$$\left|\nabla v(x)\right|^2 \Delta P(v,x) - 2G'\big(v(x)\big)\nabla v(x)\cdot\nabla P(v,x) \geqslant \frac{|\nabla P(v,x)|^2}{2} \tag{15}$$

weakly in $\{\nabla v \neq 0\}$.

Thus, if we set $a(x) := \left|\nabla v(x)\right|^2$, for any $\phi \in C_0^1(\{\nabla v \neq 0\}, [0,+\infty))$, the weak formulation of (15) gives that

$$\int_\Omega -\nabla P\cdot\nabla(a\phi) - 2G'(v)\phi\nabla v\cdot\nabla P\,dx \geqslant \int_\Omega \frac{|\nabla P|^2\phi}{2}\,dx. \tag{16}$$

Accordingly, for any $\varphi \in C_0^1(\{\nabla v \neq 0\}, [0,+\infty))$, taking $\phi := \varphi/a$ in (16), we have that

$$\int_{\Omega} -\nabla P\cdot\nabla\varphi - \frac{2G'(v)\varphi\nabla v\cdot\nabla P}{|\nabla v|^2}\,dx \geqslant \int_{\Omega} \frac{|\nabla P|^2\varphi}{2|\nabla v|^2}\,dx.$$

That is,

$$\Delta P(v,x) - \frac{2G'(v(x))\nabla v(x)\cdot\nabla P(v,x)}{|\nabla v(x)|^2} \geqslant \frac{|\nabla P(v,x)|^2}{2|\nabla v(x)|^2} \geqslant 0 \tag{17}$$

weakly in $\{\nabla v \neq 0\}$.

Now, recalling (5), we also denote by $\mathfrak{S}$ the set of all the functions $\Psi \in C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}^{n-1})$ such that the surface $\{x_n = \Psi(x')\}$ has nonnegative mean curvature and

$$\|\nabla\Psi\|_{C^{1,\alpha}(\mathbb{R}^{n-1})} \leqslant \|\nabla\Phi\|_{C^{1,\alpha}(\mathbb{R}^{n-1})}. \tag{18}$$

For any $\Psi \in \mathfrak{S}$, we define its epigraph

$$\Omega_\Psi := \{x_n > \Psi(x')\}.$$

Note that, by construction, if $\Psi \in \mathfrak{S}$ we have that

$$\text{the mean curvature of } \partial\Omega_\Psi \text{ is nonnegative.} \tag{19}$$

We set

$$\begin{aligned}\mathfrak{F} := \big\{ &v \in C^2(\overline{\Omega_\Psi}) \text{ solutions of } \Delta v = G'(v) \text{ in } \Omega_\Psi,\\ &\text{with } 0 \leqslant v \leqslant \|u\|_{L^\infty(\Omega)},\ v = 0 \text{ on } \partial\Omega_\Psi \text{ and } \Psi \in \mathfrak{S}\big\}.\end{aligned} \tag{20}$$

Note that if $v \in \mathfrak{F}$ then there exists $\Psi^{(v)} \in \mathfrak{S}$ such that v is C^2 in $\overline{\Omega_{\Psi^{(v)}}}$, it solves $\Delta v = G'(v)$ in Ω_Ψ, with $0 \leqslant v \leqslant \|u\|_{L^\infty(\Omega)}$ and $v = 0$ on $\partial\Omega_\Psi$.

Thus, with a slight abuse of notation, we will write

$$\Omega_v := \Omega_{\Psi^{(v)}},$$

that is, Ω_v will denote the domain of any $v \in \mathfrak{F}$.

Now, we consider

$$P_o := \sup_{\substack{v\in\mathfrak{F}\\ x\in\Omega_v}} P(v,x). \tag{21}$$

By elliptic regularity (see, e.g., Theorems 6.6 and 6.19 of [11]), we have that

$$\|v\|_{C^{2,\alpha}(\overline{\Omega_v})} \leqslant C, \tag{22}$$

for any $v \in \mathfrak{F}$, where $C > 0$ only depends[1] on n, $\|u\|_{L^\infty(\Omega)}$ and $\|\nabla\Phi\|_{C^{1,\alpha}(\mathbb{R}^{n-1})}$.

Thus, an obvious consequence of (22) is that the sup in (21) is finite.

Also, recalling (5) and (4), we observe that $u \in \mathfrak{F}$, $\Omega_u = \Omega$, and then (3) is proved if we show that

$$P_o \leqslant 0. \tag{23}$$

Therefore, we focus on the proof of (23).

Its proof is by contradiction, namely we assume that

$$P_o > 0 \tag{24}$$

and we take $v_k \in \mathfrak{F}$ and $x_k \in \Omega_{v_k}$ in such a way that

$$P_o - \frac{1}{k} \leqslant P(v_k, x_k) \leqslant P_o. \tag{25}$$

Let

$$u_k(x) := v_k(x + x_k).$$

Notice that $u_k \in \mathfrak{F}$ and

$$0 \in \Omega_{u_k}, \tag{26}$$

because $x_k \in \Omega_{v_k}$.

Note also that $P(u_k, 0) = P(v_k, x_k)$, hence (25) reads

$$P_o - \frac{1}{k} \leqslant P(u_k, 0) \leqslant P_o. \tag{27}$$

From (22) and Lemma 6.37 of [11], we know that we can take $\tilde{u}_k \in C^{2,\alpha}(\mathbb{R}^n)$ to be a smooth extension of u_k, that is $\tilde{u}_k = u_k$ on Ω_{u_k}, such that[2]

$$\|\tilde{u}_k\|_{C^{2,\alpha}(\mathbb{R}^n)} \leqslant C\|u_k\|_{C^{2,\alpha}(\overline{\Omega_v})} \leqslant C. \tag{28}$$

Now, we denote by $\Psi_k \in \mathfrak{S}$ the function describing $\partial\Omega_{u_k}$. We know from (26) that

$$\Psi_k(0) \leqslant 0. \tag{29}$$

We claim that

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[1] The reason for which C in (22) depends only on n, $\|u\|_{L^\infty(\Omega)}$ and $\|\nabla\Phi\|_{C^{1,\alpha}(\mathbb{R}^{n-1})}$ lies in the remark on page 98 of [11]. Namely, the constant K on page 98 of [11] is bounded here by $\|\nabla\Phi\|_{C^{1,\alpha}(\mathbb{R}^{n-1})}$, which is finite, thanks to (5).

As a notational remark, we also notice that, here below, we will call C different constants, which only depend on n, $\|u\|_{L^\infty(\Omega)}$ and $\|\nabla\Phi\|_{C^{1,\alpha}(\mathbb{R}^{n-1})}$. No confusion should arise from this slight abuse of notation.

[2] Notice that we apply here Lemma 6.37 of [11] to a domain with an unbounded boundary. To do this, one has to take a locally finite partition of unity in the argument of page 137 of [11]. See also Footnote 1.

$$\Psi_k(0) \text{ is bounded.} \tag{30}$$

To prove (30), we suppose the converse. Then, by (29), we would have that $\Psi_k(0) \to -\infty$, up to a subsequence.

Hence, by (18), we would have that $\Psi_k(x') \to -\infty$ locally uniformly.

Therefore, by (28), $\tilde{u}_k$ would converge, up to a subsequence, in $C^2_{\mathrm{loc}}(\mathbb{R}^n)$ to some $u_\infty \in C^2(\mathbb{R}^n)$ which solves

$$\Delta u_\infty = G'(u_\infty) \quad \text{in } \mathbb{R}^n. \tag{31}$$

By taking the limit in (27), we would also have that $P_o = P(u_\infty, 0)$, and so, by (24), that

$$\left|\nabla u_\infty(0)\right|^2 - 2G\big(u_\infty(0)\big) > 0.$$

This and (31) are in contradiction with Lemma 4.11 in [9], so (30) is proved.

As a consequence of (18) and (30), we conclude that Ψ_k converges in $C^2_{\mathrm{loc}}(\mathbb{R}^{n-1})$, up to a subsequence, to a function $\Psi_\infty \in \mathfrak{S}$.

Thus we write

$$\Omega_\infty := \big\{x_n > \Psi_\infty\big(x'\big)\big\}.$$

Also, recalling (28), we have that $\tilde{u}_k$ converges, up to a subsequence, in $C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$ to some u_∞, with

$$\|u_\infty\|_{C^{2,\alpha}(\mathbb{R}^n)} \leqslant C. \tag{32}$$

We have that u_∞ is a solution of

$$\Delta u_\infty = G'(u_\infty) \quad \text{in } \Omega_\infty. \tag{33}$$

Indeed, if $x = (x', x_n) \in \Omega_\infty$, then $x' > \Psi_\infty(x')$ and so $x' > \Psi_k(x')$ for large k, that gives

$$\Delta u_\infty(x) = \lim_{k\to+\infty} \Delta u_k(x) = \lim_{k\to+\infty} G'\big(u_k(x)\big) = G'\big(u_\infty(x)\big),$$

that is (33).

Moreover, for any $x' \in \mathbb{R}^{n-1}$,

$$\begin{aligned}
\big|u_\infty\big(x', \Psi_\infty\big(x'\big)\big)\big| &\leqslant \big|u_\infty\big(x', \Psi_\infty\big(x'\big)\big) - u_k\big(x', \Psi_\infty\big(x'\big)\big)\big| \\
&\quad + \big|u_k\big(x', \Psi_\infty\big(x'\big)\big) - u_k\big(x', \Psi_k\big(x'\big)\big)\big| + \big|u_k\big(x', \Psi_k\big(x'\big)\big)\big| \\
&\leqslant \sup_{B_1(x', \Psi_\infty(x'))} |u_\infty - u_k| + C\big|\Psi_\infty\big(x'\big) - \Psi_k\big(x'\big)\big| + 0.
\end{aligned}$$

Hence, by sending $k \to +\infty$, we obtain that $u_\infty(x', \Psi_\infty(x')) = 0$, that is that

$$u_\infty \text{ vanishes on } \partial\Omega_\infty. \tag{34}$$

In particular, $u_\infty \in \mathfrak{F}$.

Passing to the limit in (27) and recalling (24), we also obtain that

$$\left|\nabla u_\infty(0)\right|^2 - 2G\big(u_\infty(0)\big) = P(u_\infty, 0) = P_o > 0. \tag{35}$$

Also, by (34),

$$\partial_\nu u_\infty(x) \geqslant 0 \quad \text{for any } x \in \partial\Omega_\infty. \tag{36}$$

Here above and in the sequel, ν denotes the interior normal of $\partial\Omega_\infty$.

We now claim that

$$\inf_{\Omega_\infty} |\nabla u_\infty| = 0. \tag{37}$$

To prove (37), we argue by contradiction once more. Namely, we suppose that

$$\inf_{\Omega_\infty} |\nabla u_\infty| \geqslant c \tag{38}$$

for some $c > 0$, we fix Q inside Ω_∞ and we consider the solution $\gamma \in C^1(\mathbb{R}, \Omega_\infty)$ of the ODE

$$\begin{cases} \gamma'(t) = \dfrac{\nabla u_\infty(\gamma(t))}{|\nabla u_\infty(\gamma(t))|}, \\ \gamma(0) = Q. \end{cases}$$

Note that γ is globally defined and it does not hit $\partial\Omega_\infty$ because of (36) and (38).

Consequently, utilizing (38) once more, we obtain that, for any $t > 0$,

$$2\|u\|_{L^\infty(\Omega)} \geqslant u_\infty\big(\gamma(t)\big) - u_\infty\big(\gamma(0)\big) = \int_0^t \nabla u_\infty\big(\gamma(s)\big) \cdot \gamma'(s)\, ds = \int_0^t \left|\nabla u_\infty\big(\gamma(s)\big)\right| ds \geqslant ct.$$

Since this is a contradiction for large t, we conclude that (37) holds true.

Now, we claim that

$$\text{if } P(u_\infty, y) = P_o \quad \text{for some } y \in \Omega_\infty, \text{ then } y \in \partial\Omega_\infty. \tag{39}$$

To prove this, we argue again by contradiction, and we suppose that y lies inside Ω_∞.

Then,

$$\text{the set } U := \big\{x \in \Omega_\infty \text{ s.t. } P(u_\infty, x) = P_o\big\} \text{ is nonempty.} \tag{40}$$

Hence, for any $x \in U$,

$$0 < P_o = P(u_\infty, x) = \left|\nabla u_\infty(x)\right|^2 - G\big(v(x)\big) \leqslant \left|\nabla u_\infty(x)\right|^2,$$

thanks to (13) and (24).

This shows that

$$\text{if } x \in U, \quad \text{then } \left|\nabla u_\infty(x)\right| > 0. \tag{41}$$

Moreover, since $u_\infty \in C^2(\mathbb{R}^n)$, we have that

$$U \text{ is closed in } \Omega_\infty. \tag{42}$$

We plan to prove that

$$U \text{ is also open.} \tag{43}$$

For this, we take x in U and employ (41) to deduce that

$$\inf_{B_{r_x}(x)} |\nabla u_\infty| > 0$$

for some small $r_x > 0$. This, (17) and the Strong Maximum Principle (see Theorem 8.19 of [11]) imply that $P(y, u_\infty) = P_o$ for any $y \in B_{r_x}(x)$.

This proves (43).

From (40), (42), (43) and the fact that Ω_∞ is an epigraph (hence, it is connected), we conclude that

$$U = \Omega_\infty. \tag{44}$$

We now recall (37) and we take $x_j \in \Omega_\infty$ such that

$$\lim_{j\to+\infty} \left|\nabla u_\infty(x_j)\right| = 0. \tag{45}$$

Then, from (13), (40) and (44)

$$P_o = P(u_\infty, x_j) = \left|\nabla u_\infty(x_j)\right|^2 - 2G\big(u_\infty(x_j)\big) \leqslant \left|\nabla u_\infty(x_j)\right|^2$$

for any $j \in \mathbb{N}$.

Therefore, by (45), we obtain $P_o \leqslant 0$, in contradiction with (24).

This proves (39).

Accordingly, from (35) and (39), we deduce that

$$0 \in \partial\Omega_\infty \tag{46}$$

and that

$$P(u_\infty, 0) = P_o > P(u_\infty, x) \quad \text{for any } x \in \Omega_\infty. \tag{47}$$

We now observe that

$$\partial_\nu u_\infty(0) > 0. \tag{48}$$

Indeed, if (48) did not hold, recalling (36), we would have that $\partial_\nu u_\infty(0) = 0$ and so $\nabla u_\infty(0) = 0$, which, together with (35), gives that $G(u_\infty(0)) < 0$, in contradiction with (13). Hence, (48) holds true.

Consequently, (17), (48), (47), and the Hopf Principle (see Theorem 5.5.1 on page 120 of [17]) give that[3]

$$\partial_\nu P(u_\infty, 0) < 0. \tag{49}$$

Moreover, from (14),

$$\frac{1}{2}\partial_\nu P(u_\infty, x) = \nabla u_\infty(x) \cdot \nabla\big(\partial_\nu u_\infty(x)\big) - G'\big(u_\infty(x)\big)\partial_\nu u_\infty(x) \tag{50}$$

for any $x \in \Omega_\infty$.

Also, from (34), $\nabla u_\infty = (\partial_\nu u_\infty)\nu$ on $\partial\Omega_\infty$, and so (46) and (50) give that

$$\frac{1}{2}\partial_\nu P(u_\infty, 0) = \partial_\nu u_\infty(0)\big(\partial^2_{\nu\nu} u_\infty(0) - G'\big(u_\infty(0)\big)\big). \tag{51}$$

We now take normal coordinates for $\partial\Omega_\infty$ at 0, that is endow $\mathbb{R}^n$ with coordinates $(X_1, \dots, X_n)$ in such a way that the last coordinate is parallel to ν and we write, near 0, $\partial\Omega_\infty$ as the graph of a smooth function Θ in the ν-direction.

Namely, we say that, for some $\epsilon_o > 0$,

$$\Omega_\infty \cap B_{\epsilon_o}(0) = \big\{\big(X', X_n\big) \in \mathbb{R}^n \text{ s.t. } X_n > \Theta\big(X'\big)\big\},$$

with $\Theta(0) = 0$ and

$$\partial_i \Theta(0) = 0 \quad \text{for any } i = 1, \dots, n-1. \tag{52}$$

With a slight abuse of notation we write $u_\infty(X)$ when we compute u_∞ with respect to this new system of coordinates.

Indeed, due to the rotational invariance of the Laplacian, it is convenient to compute Δu_∞ in this coordinate system: by (34),

$$u_\infty\big(X', \Theta\big(X'\big)\big) = 0$$

when X' is near 0, therefore, for any $i = 1, \dots, n-1$,

$$\partial_i u_\infty\big(X', \Theta\big(X'\big)\big) + \partial_n u_\infty\big(X', \Theta\big(X'\big)\big)\Theta_i\big(X'\big) = 0.$$

Differentiating once more,

---

[3] We would like to stress that the use of the Hopf Principle here is somewhat a delicate issue. Indeed, it requires to observe first that $|\nabla u_\infty| > 0$ near 0, due to (48), which gives that $G'(u_\infty)\nabla u_\infty \cdot \nabla P/|\nabla u_\infty|^2$ is bounded near 0. Then, we look at (17), which holds in the weak sense, and apply to it the general Hopf Principle in Theorem 5.5.1 on page 120 of [17].

$$\partial^2_{ii} u_\infty\big(X', \Theta\big(X'\big)\big) + 2\partial^2_{in} u_\infty\big(X', \Theta\big(X'\big)\big)\Theta_i\big(X'\big) + \partial^2_{nn} u_\infty\big(X', \Theta\big(X'\big)\big)\Theta_i^2\big(X'\big)$$
$$+ \partial_n u_\infty\big(X', \Theta\big(X'\big)\big)\Theta_{ii}\big(X'\big) = 0$$

for any $i = 1, \dots, n-1$, when X' is near 0.

Then, by (52),

$$\partial^2_{ii} u_\infty(0) + \partial_n u_\infty(0)\Theta_{ii}(0) = 0$$

for any $i = 1, \dots, n-1$ and so, by summing up in i,

$$\Delta u_\infty(0) - \partial^2_{\nu\nu} u_\infty(0) + \partial_n u_\infty(0)\Delta\Theta(0) = 0. \tag{53}$$

Notice that, by (52), the mean curvature of Ω_∞ at 0 is exactly $\Delta\Theta$ (or it is proportional to it, depending on the convention chosen to define the mean curvature).

Hence, recalling (19), we have that

$$\Delta\Theta \geqslant 0. \tag{54}$$

Also, from (48),

$$\partial_n u(0) = \partial_\nu u(0) > 0. \tag{55}$$

Therefore, (53), (54) and (55) give that

$$\Delta u_\infty(0) \leqslant \partial^2_{\nu\nu} u_\infty(0).$$

So, by (33),

$$G'\big(u_\infty(0)\big) = \Delta u_\infty(0) \leqslant \partial^2_{\nu\nu} u_\infty(0).$$

As a consequence of this, of (48) and of (51), we see that $\partial_\nu P(u_\infty, 0) \geqslant 0$.

Since this is in contradiction with (49), the proof of (23) – and therefore of (3) – is finished.

We recall that the proof above was performed under the assumption that Ω is an epigraph. If, on the other hand, $\Omega = \Omega_o \times \mathbb{R}^{n-n_o}$, where $\Omega_o \subset \mathbb{R}^{n_o}$ is a bounded domain, the above proof also goes through, with the following minor modifications:

- The definition in (20) is replaced by

$$\mathfrak{F} := \big\{v \in C^2(\mathcal{U}) \text{ solutions of } \Delta v = G'(v) \text{ in } \mathcal{U},$$
$$\text{with } 0 \leqslant v \leqslant \|u\|_{L^\infty(\mathcal{U})},\ v = 0 \text{ on } \partial\mathcal{U},\ \text{and } \mathcal{U} \in \mathfrak{D}\big\},$$

  with $\mathfrak{D}$ being the collection of all the translation of Ω, namely

$$\mathfrak{D} := \big\{p + \Omega,\ \text{with } p \in \mathbb{R}^n\big\}.$$

- The sets Ω_{u_k} would now belong to $\mathfrak{D}$ and so they are of the form $p_k + \Omega$. Of course, since Ω_{u_k} is a cylinder, we may suppose that $p_k = (p'_k, 0) \in \mathbb{R}^{n_o} \times \mathbb{R}^{n-n_o}$.
  From (26), there must exist $\omega_o \in \Omega_o$ such that $\mathbb{R}^{n_o} \ni 0 = p'_k + \omega_o$.
  Accordingly, p'_k is bounded and so

$$p_k \text{ is bounded.}$$

  This replaces (30).

Keeping in mind these modifications, the above proof shows that (3) also holds for Theorem 1(i). □

Proof of Theorem 2. First of all, we prove the following result.

$$\begin{aligned} &\text{Let } p \in \overline{\Omega}. \\ &\text{Suppose that either } p \in \Omega \text{ or that (7) holds true.} \\ &\text{Assume that } F\big(u(p)\big) = c_u \text{ and } F'\big(u(p)\big) = 0. \\ &\text{Then } u \text{ is constant.} \end{aligned} \tag{56}$$

To prove this, we set $r := u(p)$, and we fix any $q \in \Omega$.

We prove that $u(q) = r$. Since q is an arbitrary point of Ω, this would show that (56) holds true.

For this, by connectedness, we take $T > 0$ and a smooth curve $\gamma : [0, T] \to \overline{\Omega}$, parameterized by arclength, such that $\gamma(0) = p$ and $\gamma(T) = q$. Notice that we can take $\gamma(t) \in \Omega$ for any $t \in (0, T]$, and, in fact, $\gamma(t) \in \Omega$ for any $t \in [0, T]$ if $p \in \Omega$. Also, when $p \in \partial\Omega$, we have that γ is smooth up to $t = 0$ thanks to (7).

For any $t \in [0, T]$, we define

$$\phi(t) := u\big(\gamma(t)\big) - r.$$

From (3), and noticing that $F'(r) = F'(u(p)) = 0$, we have that

$$\begin{aligned} \big(\dot\phi(t)\big)^2 &= \big|\nabla u\big(\gamma(t)\big) \cdot \dot\gamma(t)\big|^2 \leqslant \big|\nabla u\big(\gamma(t)\big)\big|^2 \leqslant 2\big(c_u - F\big(u\big(\gamma(t)\big)\big)\big) \\ &= 2\big(F(r) - F\big(u\big(\gamma(t)\big)\big)\big) = -2 \int_r^{u(\gamma(t))} F'(\tau)\, d\tau \\ &\leqslant 2\|F\|_{C^{1,1}(0, \|u\|_{L^\infty(\Omega)})} \big|u\big(\gamma(t)\big) - r\big|^2 = \big(C\phi(t)\big)^2, \end{aligned}$$

with $C := \sqrt{2\|F\|_{C^{1,1}(0, \|u\|_{L^\infty(\Omega)})}}$.

As a consequence, the function $[0, T] \ni t \mapsto (\phi(t))^2 e^{-2Ct}$ is decreasing, and so

$$\big(u(q) - r\big)^2 e^{-2CT} = \big(\phi(T)\big)^2 e^{-2CT} \leqslant \big(\phi(0)\big)^2 = \big(u(p) - r\big)^2 = 0.$$

This shows that $u(q) = r$, proving (56).

Let us now prove (6), by assuming, with no loss of generality, that u is not constant, and by arguing by contradiction. If (6) were false, we would obtain from (2) that

$$c_u = \max_{r \in [0, \|u\|_{L^\infty(\Omega)}]} F(r) = F(r_o), \tag{57}$$

with

$$r_o \in \big(0, \|u\|_{L^\infty(\Omega)}\big).$$

This says that r_o is an interior maximum of F and so

$$F'(r_o) = 0. \tag{58}$$

Now, recalling that $u = 0$ on $\partial\Omega$ and using the continuity of u, we can take $p \in \Omega$ such that $u(p) = r_o$. Then, (57) and (58) say that $F(u(p)) = c_u$ and $F'(u(p)) = 0$.

Consequently, (56) gives that u is constant, against our assumption.

This shows that (6) holds.

Let us now suppose that (8) holds true. Then, in this case, we claim that

$$\text{if } c_u = F(0), \quad \text{then } u \text{ vanishes identically in } \Omega. \tag{59}$$

To prove this, suppose that

$$c_u = \max_{r \in [0, \|u\|_{L^\infty(\Omega)}]} F(r) = F(0).$$

Then 0 is a boundary maximum for F and so $F'(0) \leqslant 0$.

This and (8) give that $F'(0) = 0$ and, since $\partial\Omega \neq \emptyset$, we know that there exists $p \in \partial\Omega$ such that $u(p) = 0$.

Then, $F(u(p)) = F(0) = c_u$ and $F'(u(p)) = F'(0) = 0$, so (56) gives that u is constant and so it vanishes identically.

This proves (59).

Hence, (6) and (59) imply (9) and so the proof of Theorem 2 is completed. □

Proof of Theorem 3. Let us suppose that equality in (3) holds at some point $p \in \Omega \cap \{\nabla u \neq 0\}$, and let C be the open connected component of $\{\nabla u \neq 0\}$ containing p.

First, we prove that

$$\text{equality in (3) holds in the whole } C. \tag{60}$$

For this, we observe that $P(u, x)$ has an interior maximum at $x = p$, thanks to (23). Thus by (17) and the Strong Maximum Principle, $P(u, x)$ vanishes in C, so (60) holds true.

Now, we observe that, by definition of C,

$$\nabla u(x) = 0 \quad \text{for any } x \in \partial C \cap \Omega. \tag{61}$$

Of course, $\partial C \cap \Omega$ may well be empty, but in this case (61) would still be true.

Therefore, by (60),

$$F\big(u(x)\big) = c_u \quad \text{for any } x \in \partial C \cap \Omega. \tag{62}$$

Since $\partial C \subseteq \partial\Omega \cup \Omega$, we obtain from (6) and (62) that

$$\begin{aligned}&\text{if } \beta \text{ is a connected component of } \partial C,\\ &\text{then } u \text{ is constantly equal to either 0 or } \|u\|_{L^\infty(\Omega)} \text{ in } \beta.\end{aligned} \tag{63}$$

Moreover, we observe that (1) and (60) imply that

$$\Delta u + F'(u) = 0 \quad \text{and} \quad |\nabla u| = g(u) \quad \text{in } C \subseteq \{\nabla u \neq 0\},$$

where $g(r) := \sqrt{2(c_u - F(r))}$. We remark that $g \in C^1((0, \|u\|_{L^\infty(\Omega)}))$, due to (6).

This says that the level sets of u in C are a family of isoparametric hypersurfaces (see [14,21] or page 353 in [8]), thence these level sets in C possess planar, spherical or cylindrical symmetry, viz., up to change of variables, only one of the following three cases occur:

(C1) there exists $u_o : \mathbb{R} \to \mathbb{R}$ such that $u(x) = u_o(x_n)$ for any $x = (x', x_n) \in C \subseteq \mathbb{R}^{n-1} \times \mathbb{R}$,
(C2) there exists $u_o : \mathbb{R} \to \mathbb{R}$ such that $u(x) = u_o(|x|)$ for any $x \in C$,
(C3) there exist $m \in \mathbb{N}$, $1 \leqslant m \leqslant n$, and $u_o : \mathbb{R} \times \mathbb{R}^{n-m} \to \mathbb{R}$ such that $u(x) = u_o(|x'|)$ for any $x = (x', x'') \in C \subseteq \mathbb{R}^m \times \mathbb{R}^{n-m}$.

In fact, the case $m = 1$ in (C3) reduces to (C1), so we may assume with no loss of generality that

$$m \geqslant 2 \quad \text{in (C3)}. \tag{64}$$

With this convention, we claim that

$$\text{(C3) cannot occur.} \tag{65}$$

The proof of (65) is by contradiction. Suppose that (C3) holds. Then, by (60), we have that

$$0 = \frac{1}{2}\big|\dot{u}_o\big(|x'|\big)\big|^2 - c_u + F\big(u_o\big(|x'|\big)\big) \quad \text{for any } x = \big(x', x''\big) \in C. \tag{66}$$

Since $p \in C$ and C is open, we can take $p_o = (p_o', p_o'') \in C$ with $p_o' \neq 0$ and we evaluate (66) at $x = p_o + t(p_o'/|p_o'|, 0)$: we conclude that

$$0 = \frac{1}{2}\big|\dot{u}_o\big(|p_o'| + t\big)\big|^2 - c_u + F\big(u_o\big(|p_o'| + t\big)\big) \quad \text{for any } t \text{ sufficiently close to 0}.$$

Therefore, by differentiating the above formula in t, using (1), and writing the Laplacian in cylindrical coordinates,

$$0 = \big(\ddot{u}_o\big(|p_o'| + t\big) + F'\big(u_o\big(|p_o'| + t\big)\big)\big)\dot{u}_o\big(|p_o'| + t\big) = -\frac{m-1}{|p_o'| + t}\big|\dot{u}_o\big(|p_o'| + t\big)\big|^2. \tag{67}$$

From (64) and the fact that $C \subseteq \{\nabla u \neq 0\}$, we deduce that the last quantity in (67) is strictly negative: this contradiction proves (65).

An analogous argument (just writing x instead of x', n instead of m and dropping x'') shows that

$$\text{(C2) cannot occur.} \tag{68}$$

So, thanks to (65) and (68), we know that case (C1) holds[4] and so we can write

$$u(x) = u_o(x_n) \quad \text{for any } x \in C.$$

Therefore, by the Unique Continuation Principle (see e.g. [13,12] for a general statement),

$$u(x) = u_o(x_n) \quad \text{for any } x \in \Omega. \tag{69}$$

We now observe that

$$\partial C \cap \{u = 0\} \neq \emptyset. \tag{70}$$

Indeed, if (70) were false, we would have that u is constantly equal to $\|u\|_{L^\infty(\Omega)}$ on ∂C, thanks to (63), and so ∇u would vanish somewhere in C, due to Rolle Theorem. Since this is in contradiction with the fact that $C \subseteq \{\nabla u \neq 0\}$, the proof of (70) is finished.

Also, ∇u vanishes at points of $\{u = 0\} \cap \Omega$, since u attains its minimum there, therefore, since $C \subseteq \{\nabla u \neq 0\}$, we have that

$$u > 0 \quad \text{in } C. \tag{71}$$

Our goal is now to reconstruct Ω from the connected component $C \subseteq \{\nabla u \neq 0\}$.

From (70) and (71), we have that there exists $t_o \in \mathbb{R}$ such that

$$u_o(t_o) = 0 \tag{72}$$

and (possibly reverting the orientation of e_n) $T > 0$ such that

$$u_o(t) > 0 \quad \text{and} \quad u_o'(t) > 0 \quad \text{for any } t \in (t_o, t_o + T). \tag{73}$$

We now take T as large as possible in (73) above.

If we can take T up to $+\infty$, viz. if (73) holds for any $t > t_o$, we have that case (a$\star$), and so in particular (a), holds in Theorem 3.

Therefore, we may and do suppose that

$$\text{either } u_o(t_o + T) = 0 \quad \text{or} \quad u_o'(t_o + T) = 0. \tag{74}$$

Notice however that it cannot be that $u_o(t_o + T) = 0$, otherwise (72) and Rolle Theorem would give that $u_o'(\tau_\star) = 0$ for some $\tau_\star \in (t_o, t_o + T)$, in contradiction with (73).

[4] For a different proof of this, see the comments at the end of this proof.

Accordingly, (74) says that

$$u_o'(t_o + T) = 0. \tag{75}$$

Thus, in order to complete the proof of Theorem 3, we classify the solutions $u_o \in C^2(\mathbb{R})$ of the ODE

$$\begin{cases} \ddot{u}_o(t) = -F'\big(u_o(t)\big) & \text{for any } t \in \mathbb{R}, \\ u_o(t_o) = 0, \end{cases} \tag{76}$$

under condition (75).

Classifying solutions of (76) is, of course, a widely studied topic: we perform here a detailed and self-contained analysis, for the reader's facility.

For this, we first observe that u_o is even with respect to its critical points, that is

$$\text{if } \dot{u}_o(\bar{t}) = 0 \quad \text{for some } \bar{t} \in \mathbb{R}, \quad \text{then } u_o(\bar{t} - t) = u_o(\bar{t} + t) \quad \text{for any } t \in \mathbb{R}. \tag{77}$$

Indeed, the functions $\gamma_\pm(t) := u_o(\bar{t} \pm t)$ are solutions of the ODE $\ddot{\gamma}_\pm(t) = -F'(\gamma_\pm(t))$ for any $t \in \mathbb{R}$, with $\gamma_\pm(0) = u_o(\bar{t})$ and $\dot{\gamma}_\pm(0) = \pm\dot{u}_o(\bar{t}) = 0$, and so, by Cauchy Uniqueness Theorem, $\gamma_+(t) = \gamma_-(t)$ for any $t \in \mathbb{R}$, which proves (77).

Moreover,

$$\begin{aligned} &\text{if } u_o \text{ has two critical points, say } \dot{u}_o(t_1) = 0 = \dot{u}_o(t_2) \text{ for some } t_2 > t_1, \\ &\text{then it is } 2(t_2 - t_1)\text{-periodic}. \end{aligned} \tag{78}$$

Indeed, (77) gives that $u_o(t_j - t) = u_o(t_j + t)$ for any $t \in \mathbb{R}$ and $j = 1, 2$, and so

$$\begin{aligned} u_o\big(t + (t_2 - t_1)\big) &= u_o\big(t_2 + (t - t_1)\big) = u_o\big(t_2 - (t - t_1)\big) \\ &= u_o\big(t_1 - (t - t_2)\big) = u_o\big(t_1 + (t - t_2)\big) = u_o\big(t - (t_2 - t_1)\big), \end{aligned}$$

which proves (78).

From (75) and (77), we obtain that

$$u_o \text{ is even with respect to } t_o + T. \tag{79}$$

In particular, recalling (72),

$$u_o(t_o + 2T) = u_o(t_o) = 0 \tag{80}$$

and

$$u_o'(t_o + 2T) = -u_o'(t_o). \tag{81}$$

We now distinguish two cases: either $u_o'(t_o) = 0$ or $u_o'(t_o) \neq 0$.

If $u_o'(t_o) = 0$, then u_o is $2T$-periodic, due to (75) and (78). Also, u_o is even with respect to $t_o + T + 2kT$, for any $k \in \mathbb{Z}$, due to (77), and it is positive outside $\{t_o + 2kT,\ k \in \mathbb{Z}\}$, due to (73).

This says that we are either in case (a) or (b) of Theorem 3.

More precisely, if (11) holds, we are in case (b⋆), while if (12) holds, the periodicity of u_o is given by the observations above.

On the other hand, if $u_o'(t_o) \neq 0$, we have that $u_o'(t_o + 2T) \neq 0$, because of (81). Hence, u_o is negative in $(t_o - \epsilon, t_o) \cup (t_o + 2T, t_o + 2T + \epsilon)$ for some $\epsilon > 0$, due to (80).

Since $u \geqslant 0$ in Ω, the latter observation and (73) give that Ω is a strip of width $2T$ on which $u > 0$, that is case (a) in Theorem 3 holds (notice that in this case (12) cannot hold). In fact, alternative (a⋆) must hold in this case, thanks to (79).

This completes the proof of Theorem 3. □

We would like to give now a different argument which shows that once equality holds in (3), then the solution is one dimensional (that is, case (C1) holds).

This argument does not use the notion of isoparametric hypersurfaces, but it employs a geometric identity found in [19,20].

The alternative proof goes like that. Suppose that equality in (3) holds at some point $p \in \Omega \cap \{\nabla u \neq 0\}$. Then, from (60), it holds in the whole connected component C of $\{\nabla u \neq 0\}$.

Therefore, the function

$$x \mapsto \frac{|\nabla u(x)|^2}{2} + F\big(u(x)\big)$$

is constant for any $x \in C$.

Thus, by taking the gradient and the Laplacian of such a function, we obtain that

$$0 = \nabla\left(\frac{|\nabla u|^2}{2} + F(u)\right) = |\nabla u|\nabla|\nabla u| + F'(u)\nabla u = |\nabla u|\nabla|\nabla u| - \Delta u \nabla u \tag{82}$$

and

$$\begin{aligned} 0 = \Delta\left(\frac{|\nabla u|^2}{2} + F(u)\right) &= \left|D^2 u\right|^2 + \nabla(\Delta u)\cdot\nabla u + F''(u)|\nabla u|^2 + F'(u)\Delta u \\ &= \left|D^2 u\right|^2 - (\Delta u)^2 \end{aligned} \tag{83}$$

in the weak sense and thus almost everywhere (indeed, we remark that, by standard elliptic regularity, all the first derivatives of u are in $H^2_{\mathrm{loc}}(\Omega)$).

In fact, by the continuity of the second derivatives of u, we obtain that the above equality holds everywhere in Ω.

Here above, we have used the PDE in (1) and the Bochner–Weitzenböck formula (see, for instance, [4] and references therein) to compute $\Delta|\nabla u|^2$.

Also, the standard notation

$$\left|D^2 u\right| := \sqrt{\sum_{1 \leqslant i,j \leqslant n} \left(\partial^2_{ij} u\right)^2}$$

was employed.

From (82),

$$\Delta u|\nabla u|^2 = |\nabla u|\nabla|\nabla u| \cdot \nabla u,$$

and plugging it into (83), we obtain

$$0 = \left|D^2 u\right|^2 - \left|\nabla|\nabla u| \cdot \frac{\nabla u}{|\nabla u|}\right|^2 \geqslant \left|D^2 u\right|^2 - \left|\nabla|\nabla u|\right|^2 = \mathcal{K}^2|\nabla u|^2 + \left|\nabla_{\mathfrak{L}}|\nabla u|\right|^2, \tag{84}$$

in C, where in the last step we have used formula (2.1) of [19].

In the above notation, $\mathcal{K}$ is the length of the second fundamental form of the level sets of u and $\nabla_{\mathfrak{L}}$ is the tangential gradient along such level sets in C (see [19,20,9] for further details).

Therefore, from (84), we see that both $\mathcal{K}$ and $|\nabla_{\mathfrak{L}}|\nabla u||$ vanish identically on C.

Accordingly, from Lemma 2.3 of [10], we have that $u(x) = u_o(x_n)$, up to rotation, for any $x \in C$. This gives an alternative proof of (C1).

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