On Mutually Nearest and Mutually Furthest Points of Sets in Banach Spaces

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Let $A$ be a nonempty closed bounded subset of a uniformly convex Banach space $E$. Let $C(E)$ denote the space of all nonempty closed convex and bounded subsets of $E$, endowed with the Hausdorff metric. We prove that the set of all $X \in C(E)$ such that the maximization problem $\max(A, X)$ is well posed is a $G_δ$ dense subset of $C(E)$. A similar result is proved for the minimization problem $\min(A, X)$, with $X$ in an appropriate subspace of $C(E)$. © 1992 Academic Press, Inc.

1. INTRODUCTION AND PRELIMINARIES

Let $E$ be a real Banach space. We denote by $B(E)$ the space of all nonempty closed bounded subsets of $E$. For $X, Y \in B(E)$, we set

$$\lambda_{XY} = \inf \{ \|x - y\| : x \in X, y \in Y\},$$

$$\mu_{XY} = \sup \{ \|x - y\| : x \in X, y \in Y\}.$$
Given $X, Y \in \mathcal{B}(E)$, we consider the minimization (resp. maximization) problem, denoted $\min(X, Y)$ (resp. $\max(X, Y)$), which consists in finding points $x_0 \in X$ and $y_0 \in Y$ such that $\|x_0 - y_0\| = \lambda_{XY}$ (resp. $\|x_0 - y_0\| = \mu_{XY}$). Any such pair $(x_0, y_0)$ is called a solution of the corresponding problem. Moreover, any sequence $\{(x_n, y_n)\}$, $x_n \in X$, $y_n \in Y$, such that $\lim_{n \to \infty} \|x_n - y_n\| = \lambda_{XY}$ (resp. $\lim_{n \to \infty} \|x_n - y_n\| = \mu_{XY}$) is called a minimizing (resp. maximizing) sequence. A minimization (resp. maximization) problem is said to be well posed if it has a unique solution $(x_0, y_0)$, and every minimizing (resp. maximizing) sequence converges to $(x_0, y_0)$.

Let $M$ be a metric space with distance $d$. For any $u \in M$ and $r > 0$ we set $B_M(u, r) = \{x \in M \mid d(x, u) < r\}$ and $\overline{B}_M(u, r) = \{x \in M \mid d(x, u) \leq r\}$. If $X \subset M$, by $X$ and $\text{diam } X$ ($X \neq \emptyset$) we mean the closure of $X$ and the diameter of $X$, respectively. As usual, if $X \subset E$, $\overline{c}X$ stands for the closed convex hull of $X$. We put, for short, $B = B_\varepsilon(0, 1)$ and $\overline{B} = \overline{B}_\varepsilon(0, 1)$.

We set

$$\mathcal{G}(E) = \{X \subset E \mid X \text{ is nonempty, convex, closed, bounded}\}.$$ 

In the sequel, we suppose the space $\mathcal{G}(E)$ to be endowed with the Hausdorff distance $h$. As is well known, under such metric, $\mathcal{G}(E)$ is complete.

In this note we consider problems of minimization, $\min(A, X)$, and of maximization, $\max(A, X)$, where $A \in \mathcal{B}(E)$, $X \in \mathcal{G}(E)$, and $E$ is uniformly convex. More precisely, for a fixed $A \in \mathcal{B}(E)$, set $\mathcal{G}_A(E) = \{X \in \mathcal{G}(E) \mid \lambda_{AX} > 0\}$. Then, it is proved (Theorem 3.3) that the set of all $X \in \mathcal{G}_A(E)$, such that the minimization problem $\min(A, X)$ is well posed, is a dense $G_\delta$-subset of $\mathcal{G}_A(E)$. Furthermore, it is shown (Theorem 4.3) that the set of all $X \in \mathcal{G}(E)$, such that the maximization problem $\max(A, X)$ is well posed, is a dense $G_\delta$-subset of $\mathcal{G}(E)$.

The problems considered in this note are in the spirit of Stečkin [22]. Some further developments of Stečkin's ideas, also in other directions, can be found in [4–6, 12, 14–21] and in the monograph [10], by Dontchev and Zolezzi. Recently, a generic theorem on points of single valuedness of the proximity map for convex sets has been established by Beer and Pai [3], in a setting different from ours. Some other generic results in spaces of convex sets can be found in [2, 8].

2. Auxiliary Results

Let $X \in \mathcal{B}(E)$ and $z \in E$ be arbitrary. We set

$$d(z, X) = \inf \{\|z - x\| \mid x \in X\},$$

$$e(z, X) = \sup \{\|z - x\| \mid x \in X\}.$$
For $X, Y \in \mathcal{B}(E)$ and $\sigma > 0$, we set

$$L_{X,Y}(\sigma) = \{ x \in X \mid d(x, Y) \leq \lambda_{X,Y} + \sigma \},$$

$$M_{X,Y}(\sigma) = \{ x \in X \mid e(x, Y) \geq \mu_{X,Y} - \sigma \}.$$  

The sets $L_{X,Y}(\sigma), M_{X,Y}(\sigma)$ are nonempty, closed, and satisfy $L_{X,Y}(\sigma) \subseteq L_{X,Y}(\sigma'), M_{X,Y}(\sigma) \subseteq M_{X,Y}(\sigma')$, if $0 < \sigma < \sigma'$.

**Proposition 2.1.** Let $X, Y \in \mathcal{B}(E)$ and $z \in E$ be arbitrary. Then we have

$$\lambda_{X,Y} \leq d(z, X) + d(z, Y), \quad (2.1)$$

$$\mu_{X,Y} \geq e(z, Y) - d(z, X). \quad (2.2)$$

*Proof.* Both inequalities follow easily from the definitions.

**Proposition 2.2.** Let $X, Y \in \mathcal{B}(E)$ be arbitrary. Then the problem

$$\min(X, Y) \quad \text{(resp. } \max(X, Y) \text{)}$$

is well posed if and only if

$$\inf_{\sigma > 0} \text{diam } L_{X,Y}(\sigma) = 0 \quad \text{and} \quad \inf_{\sigma > 0} \text{diam } L_{Y,X}(\sigma) = 0$$

(resp. $\inf_{\sigma > 0} \text{diam } M_{X,Y}(\sigma) = 0 \quad \text{and} \quad \inf_{\sigma > 0} \text{diam } M_{Y,X}(\sigma) = 0$).

*Proof.* This is an easy adaptation of an argument due to Furi and Vignoli [13].

The following proposition is a variant of a result due to Zabreiko and Krasnošel’skiĭ [23] and Daneš [7] (see also [8]).

**Proposition 2.3.** Let $X \in \mathcal{C}(E), \varepsilon > 0$, and $r > 0$ be arbitrary. Then there exists $0 < \tau_0 < r$ such that for every $x \in E$, with $d(u, X) > r$, and for every $0 < r < \tau$, we have

$$\text{diam } C_{X,u}(\tau) < \varepsilon,$$

where

$$C_{X,u}(\tau) = [\overline{C}(X \cup \{ u \}) \setminus [X + (d(u, X) - \tau)B]]. \quad (2.3)$$

**Proposition 2.4.** Let $E$ be a uniformly convex Banach space. Let $\varepsilon > 0$ and let $r_0, r > 0$, with $r < r_0$, be arbitrary. Then there exists $0 < \sigma_0 < r$ such that for every $x, y \in E$, with $\| y - x \| = r$, and for every $r < r' \leq r_0$ and $0 < \sigma \leq \sigma_0$, we have

$$\text{diam } D(x, y; r', \sigma) < \varepsilon,$$
where
\[ D(x, y; r', \sigma) = \bar{B}_x(\bar{y}, r' - \|y-x\| + \sigma) \setminus B_x(x, r'). \]

**Proof.** Let \( \varepsilon > 0 \) and \( 0 < r < r_0 \) be given. Let \( x, y \in \mathcal{E} \) satisfy \( \|y-x\| = r \). Let \( r < r' \leq r_0 \) be arbitrary and let \( y' = (x + y)/2 \). We have \( D(x, y; r', \sigma) \subseteq D(x, y; r', \sigma), \sigma > 0 \). Moreover, by [9, Lemma 2.1], if \( 0 < \sigma \leq 2 \|y' - x\| \), we have
\[
\text{diam } D(x, y'; r', \sigma) \leq 2\sigma + 2(r' - \|y' - x\|) \delta^*(\frac{\sigma}{2 \|y' - x\|})
\]
\[
\leq 2\sigma + (2r_0 - r) \delta^*(\frac{\sigma}{r}),
\]
where, for \( \eta > 0 \), \( \delta^*(\eta) = \sup\{\varepsilon : 0 < \varepsilon \leq 2 \text{ and } \delta(\varepsilon) \leq \eta\} \) and \( \delta \) denotes the modulus of convexity of \( \mathcal{E} \). Since the last term in the above inequality vanishes as \( \sigma \to 0 \), to complete the proof it suffices to choose \( \sigma_0 > 0 \) such that \( 2\sigma_0 + (2r_0 - r) \delta^*(\sigma_0/r) < \varepsilon \).

3. **MINIMIZATION PROBLEMS**

In this section \( \mathcal{E} \) denotes a uniformly convex Banach space. Let \( A \) be a fixed nonempty closed bounded subset of \( \mathcal{E} \). We put, for short, \( \lambda_X = \lambda_{AX}, \quad X \in \mathcal{A}(\mathcal{E}). \) Define
\[
\mathcal{C}_A(\mathcal{E}) = \{X \in \mathcal{A}(\mathcal{E}) | \lambda_X > 0\}.
\]
Under the Hausdorff distance, \( \mathcal{C}_A(\mathcal{E}) \) is a complete metric space.

For each \( k \in \mathbb{N} \) set \( \varepsilon_k = 1/k \), and define
\[
\mathcal{L}_k = \{X \in \mathcal{C}_A(\mathcal{E}) | \inf_{\delta > 0} \text{diam } L_{XA}(\delta) < \varepsilon_k \text{ and } \inf_{\delta > 0} \text{diam } L_{AX}(\delta) < \varepsilon_k\}.
\]

To prove the main result of this section, Theorem 3.3, we state two lemmas, whose proofs will be given later.

**Lemma 3.1.** \( \mathcal{L}_k \) is dense in \( \mathcal{C}_A(\mathcal{E}) \).

**Lemma 3.2.** \( \mathcal{L}_k \) is open in \( \mathcal{C}_A(\mathcal{E}) \).

**Theorem 3.3.** Let \( \mathcal{E} \) be a uniformly convex Banach space. Let \( A \in \mathcal{A}(\mathcal{E}) \). Then the set
\[
\mathcal{V} = \{X \in \mathcal{C}_A(\mathcal{E}) | \min(A, X) \text{ is well posed}\}
\]
is a dense \( G_\delta \)-subset of \( \mathcal{C}_A(\mathcal{E}) \).
Proof. By Lemmas 3.1 and 3.2, the set
\[ L_0 = \bigcap_{k \in \mathbb{N}} L_k \]
is a dense \( G_\delta \)-subset of \( \mathcal{E}(E) \). Moreover, by Proposition 2.2, we have \( \mathcal{V} = L_0 \). Hence \( \mathcal{V} \) is a dense \( G_\delta \)-subset of \( \mathcal{E}(E) \), completing the proof.

Remark 3.4. If \( A = B \) and \( X_0 = \frac{1}{2} \), then for each \( X \in B_{\mathcal{E}(E)}(X_0, \frac{1}{2}) \) the minimization problem \( \min(A, X) \) is not well posed. This shows that Theorem 3.3 does not hold, in general, if the space \( \mathcal{E}(E) \) is replaced by \( \mathcal{E}(R) \).

Set \( \mathcal{E}_0(E) = \{ X \in \mathcal{E}(E) \mid \lambda_X > 0 \} \) and observe that \( \mathcal{E}_0(E) \) is a Baire space, being completely metrizable by Alexandroff's theorem. Then Theorem 3.3 remains valid with \( \mathcal{E}_0(E) \), in the place of \( \mathcal{E}(E) \).

Remark 3.5. For \( A \in \mathcal{B}(E) \), set \( \mathcal{D}_A(E) = \{ X \in \mathcal{E}(E) \mid X \subset E \setminus A \} \). The space \( \mathcal{D}_A(E) \) endowed with the Hausdorff metric is complete and, clearly, \( \mathcal{E}_A(E) \subset \mathcal{D}_A(E) \). Also in the space \( \mathcal{D}_A(E) \) Theorem 3.3 is, in general, false. To see that, set \( A = Q \setminus C \), where \( Q = \{ (x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 3\pi, -1 \leq y \leq 1 \} \) and \( C = \{ (x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 3\pi, -|\sin x| \leq y \leq |\sin x| \} \), and let \( X_0 = \{ (x, 0) \in \mathbb{R}^2 \mid \pi/2 \leq x \leq 5\pi/2 \} \). Clearly, \( X_0 \in \mathcal{D}_A(E) \). Moreover, if \( r > 0 \) is sufficiently small, for every \( X \in B_{\mathcal{D}_A(E)}(X_0, r) \) the minimization problem \( \min(A, X) \) is not well posed.

Remark 3.6. Theorem 3.3 remains valid if \( A \) is a nonempty closed subset of \( E \), \( A \neq \emptyset \). In this case, Theorem 3.3 is a multivalued version of a theorem due to Steckin [22]. If \( E \) is an arbitrary Banach space, then Theorem 3.3 is, in general, not true. Take, for example, \( E = \mathbb{R}^2 \) with the norm \( \max \{ |x|, |y| \} \), \( (x, y) \in \mathbb{R}^2 \), and set \( A = \overline{B} \), \( X_0 = \{ (0, 2) \} \). Then there exists \( r > 0 \) such that, for every \( X \in B_{\mathcal{E}_A(E)}(X_0, r) \), the minimization problem \( \min(A, X) \) is not well posed.

Proof of Lemma 3.1. Let \( X \in \mathcal{E}_A(E) \) and let \( r > 0 \). We want to show that there exists \( Y \in L_k \) such that \( h(Y, X) \leq r \). Without loss of generality we suppose \( \lambda_X > 0 \) and \( 0 < r < \lambda_X \).

By Proposition 2.4, there exists \( 0 < \sigma_0 < r \) such that for every \( x, y \in E \) with \( \|x - y\| = r \), and for every \( 0 < \sigma \leq \sigma_0 \), we have
\[
\text{diam } D(x, y; \lambda_X, \sigma) < \sigma_k ,
\]
where
\[
D(x, y; \lambda_X, \sigma) = \overline{B}_E(y, \lambda_X - \|y - x\| + \sigma) \setminus B_E(x, \lambda_X).
\]
Set
\[ \sigma = \min\{\sigma_0, \kappa \}. \tag{3.2} \]

By Proposition 2.3, there exists \( 0 < \tau_0 < r/2 \) such that for every \( u \in \mathbb{E} \) with \( d(u, X) \geq r/2 \), and for every \( 0 < \tau \leq \tau_0 \), we have
\[ \text{diam } C_{X,u}(\tau) < \frac{\sigma}{2}, \tag{3.3} \]
where \( C_{X,u}(\tau) \) is given by (2.3). Set
\[ \tau = \min \left\{ \tau_0, \frac{\sigma}{2} \right\}. \tag{3.4} \]

Now, pick \( \tilde{x} \in X \) and \( \tilde{a} \in A \) such that
\[ \|\tilde{x} - \tilde{a}\| \leq \lambda_X + \frac{\tau}{2}. \tag{3.5} \]
Since \( \|\tilde{x} - \tilde{a}\| \geq \lambda_X > r \), in the interval with end points \( \tilde{x} \) and \( \tilde{a} \) there is a point \( u \), say, such that
\[ \|\tilde{x} - u\| = r. \tag{3.6} \]
Define \( Y = \overline{d}(X \cup \{u\}) \). Since \( Y \subset \overline{X + rB} \) and \( A \cap (X + \lambda_X B) = \emptyset \), we have \( \lambda_Y \geq \lambda_X - r > 0 \), and so \( Y \in \mathcal{G}_k(\mathbb{E}) \). Clearly \( h(Y, X) \leq r \). Thus, to complete the proof, it suffices to show that \( Y \in \mathcal{L}_k \).

To this end, we start by proving the following inequalities:
\[ \lambda_Y \leq \lambda_X + \frac{\tau}{2} - r, \tag{3.7} \]
\[ \frac{r}{2} < d(u, X) \leq r. \tag{3.8} \]

Indeed, by virtue of (3.5) and (3.6), we have
\[ \|u - \tilde{a}\| = \|\tilde{x} - \tilde{a}\| - \|\tilde{x} - u\| \leq \lambda_X + \frac{\tau}{2} - r, \tag{3.9} \]
from which (3.7) follows, since \( u \in Y \) and \( \tilde{a} \in A \). Furthermore, by virtue of (2.1) and (3.9), we have
\[ d(u, X) \geq \lambda_X - d(u, A) \geq \lambda_X - \left( \lambda_X + \frac{\tau}{2} - r \right) = r - \frac{\tau}{2}. \]
and thus \( d(u, X) > r/2 \), for \( \bar{\tau} \leq \tau_0 < r/2 \). Since the right inequality in (3.8) is trivially satisfied, the proof of (3.8) is complete.

**Claim 1.** We have

\[
L_{Y,A} \left( \frac{\bar{\tau}}{2} \right) \subseteq C_{X,u}(\bar{\tau}). \tag{3.10}
\]

Indeed, suppose (3.10) not true, and let \( y \in L_{Y,A}(\bar{\tau}/2) \setminus C_{X,u}(\bar{\tau}) \) be arbitrary. We have

\[
\lambda_X \leq d(y, \Lambda) + d(y, X) \tag{by (2.1)}
\]

\[
\leq \lambda_y + \frac{\bar{\tau}}{2} + d(y, X) \tag{as \( y \in L_{Y,A}(\bar{\tau}/2) \)}
\]

\[
< \lambda_y + \frac{\bar{\tau}}{2} + d(u, X) - \bar{\tau} \tag{as \( y \notin C_{X,u}(\bar{\tau}) \)}
\]

\[
< \left( \lambda_X + \frac{\bar{\tau}}{2} - r \right) + \frac{\bar{\tau}}{2} + r - \bar{\tau} \tag{by (3.7) and (3.8)}
\]

\[
= \lambda_X.
\]

From the contradiction, (3.10) follows and Claim 1 is proved.

**Claim 2.** We have

\[
L_{A,Y} \left( \frac{\bar{\tau}}{4} \right) \subseteq D(\bar{x}, u; \lambda_X, \bar{\sigma}). \tag{3.11}
\]

Indeed, let \( a \in L_{A,Y}(\bar{\tau}/4) \) be arbitrary. Evidently, \( a \in \Lambda \) and \( d(a, Y) \leq \lambda_Y + \bar{\tau}/4 \). Now, pick \( y \in Y \) such that \( \|a - y\| \leq \lambda_Y + \bar{\tau}/2 \). This and (3.7) imply

\[
\|a - y\| \leq \lambda_X - r + \bar{\tau}, \tag{3.12}
\]

and thus

\[
d(y, \Lambda) \leq \lambda_X - r + \bar{\tau}. \tag{3.13}
\]

By virtue of (2.1), (3.13), and (3.8), we have

\[
d(y, X) \geq \lambda_X - d(y, \Lambda) \geq \lambda_X - (\lambda_X - r + \bar{\tau}) \geq d(u, X) - \bar{\tau},
\]

which shows that \( y \in C_{X,u}(\bar{\tau}) \). From (3.8) and (3.4), \( d(u, X) > r/2 \) and \( \bar{\tau} \leq \tau_0 \). Thus (3.3) gives \( \text{diam } C_{X,u}(\bar{\tau}) < \bar{\sigma}/2 \), and so

\[
\|y - u\| < \frac{\bar{\sigma}}{2}. \tag{3.14}
\]
Now we have
\[ \|a - u\| \leq \|a - y\| + \|y - u\| \]
\[ < (\lambda_x - r + \delta) + \frac{\delta}{2} \quad \text{(by (3.12), (3.14))} \]
\[ \leq \lambda_x - \|\tilde{x} - u\| + \delta \quad \text{(by (3.6), (3.4)),} \]
which shows that \( a \in \tilde{B}_e(u, \lambda_x - \|\tilde{x} - u\| + \delta) \). Clearly \( \|a - \tilde{x}\| \geq \lambda_x \), that is, \( a \notin \tilde{B}_e(\tilde{x}, \lambda_x) \). Hence \( a \in D(\tilde{x}, u; \lambda_x, \delta) \). As \( a \in L_{\sigma Y}(\delta/4) \) is arbitrary, (3.11) is proved, completing the proof of Claim 2.

As \( \text{diam } C_{X,u}(\tilde{r}) < \delta/2 \) and, by (3.2), \( \delta \leq \varepsilon_k \), from Claim 1 we have
\[ \text{diam } L_{\lambda Y} \left( \frac{\tilde{r}}{2} \right) < \varepsilon_k. \quad (3.15) \]

Furthermore, from (3.6) and (3.2), \( \|\tilde{x} - u\| = r \) and \( \tilde{\delta} \leq \sigma_0 \). Thus (3.1) gives \( \text{diam } D(\tilde{x}, u; \lambda_x, \tilde{\delta}) < \varepsilon_k \). Hence, by Claim 2, we have
\[ \text{diam } L_{\Lambda Y} \left( \frac{\tilde{r}}{4} \right) < \varepsilon_k. \quad (3.16) \]

From (3.15) and (3.16), it follows that \( Y \in \mathcal{L}_k \), which completes the proof of Lemma 3.1.

**Proof of Lemma 3.2.** Indeed, let \( X \in \mathcal{L}_k \) be arbitrary. Let \( \eta > 0 \) be such that
\[ \theta + 2\eta < \varepsilon_k, \quad \text{where} \quad \theta = \min \{ \inf_{\sigma > 0} \text{diam } L_{X\sigma}(\sigma), \inf_{\sigma > 0} \text{diam } L_{\Lambda X}(\sigma) \}. \quad (3.17) \]

Furthermore, let \( \sigma_1 > 0 \) be such that
\[ \text{diam } L_{X\sigma}(\sigma_1) < \theta + \eta, \quad \text{diam } L_{\Lambda X}(\sigma_1) < \theta + \eta. \quad (3.18) \]

Fix \( \sigma_2, 0 < \sigma_2 < \sigma_1 \), and set
\[ \delta = \min \left\{ \frac{\sigma_1 - \sigma_2}{2}, \frac{\eta}{2} \right\}. \quad (3.19) \]

We claim that \( B_{\sigma_2 E}(X, \delta) \subseteq \mathcal{L}_k \). To prove that, let \( Y \in B_{\sigma_2 E}(X, \delta) \) be
arbitrary. Let \( y \in L_{Y \Delta}(\sigma_2) \) be arbitrary. As \( h(Y, X) < \delta \), there exists an \( x \in X \) such that \( \|y - x\| < \delta \). We have

\[
d(x, A) < d(y, A) + \delta \\
\leq \lambda_Y + \sigma_2 + \delta \quad \text{(as } y \in L_{Y \Delta}(\sigma_2))
\]

\[
< (\lambda_X + \delta) + \sigma_2 + \delta \quad \text{(as } h(Y, X) < \delta)
\]

\[
\leq \lambda_X + \sigma_1 \quad \text{(by (3.19))},
\]

and so \( x \in L_{X \Delta}(\sigma_1) \). Hence \( y = x + (y - x) \in L_{X \Delta}(\sigma_1) + \delta B \), from which, since \( y \) is arbitrary in \( L_{Y \Delta}(\sigma_2) \), we have \( L_{Y \Delta}(\sigma_2) \subseteq L_{X \Delta}(\sigma_1) + \delta B \). From this, by virtue of (3.18), (3.19), and (3.17), we have

\[
diam L_{Y \Delta}(\sigma_2) \leq diam L_{X \Delta}(\sigma_1) + 2\delta < \theta + 2\eta < \varepsilon_k. \quad (3.20)
\]

Now, let \( a \in L_{A \Delta}(\sigma_2) \) be arbitrary. We have \( d(a, X) \leq d(a, Y) + h(Y, X) \). From this, it follows that

\[
d(a, X) < d(a, Y) + \delta \\
\leq (\lambda_Y + \sigma_2 + \delta) \quad \text{(as } a \in L_{A \Delta}(\sigma_2))
\]

\[
< (\lambda_X + \delta) + \sigma_2 + \delta \quad \text{(as } h(Y, X) < \delta)
\]

\[
\leq \lambda_X + \sigma_1 \quad \text{(by (3.19))},
\]

which shows that \( a \in L_{A \Delta}(\sigma_1) \). As \( a \in L_{A \Delta}(\sigma_2) \) is arbitrary, we have \( L_{A \Delta}(\sigma_2) \subseteq L_{A \Delta}(\sigma_1) \). From this, by virtue of (3.18) and (3.17), we have

\[
diam L_{A \Delta}(\sigma_2) \leq diam L_{A \Delta}(\sigma_1) < \theta + \eta < \varepsilon_k. \quad (3.21)
\]

From (3.20) and (3.21) it follows that \( Y \in \mathcal{L}_k \). As \( Y \in B_{\varepsilon, \delta}(X, \delta) \) is arbitrary, the proof of Lemma 3.2 is complete.

4. Maximization Problems

Also in this section \( \mathcal{E} \) denotes a uniformly convex Banach space. Let \( A \) be a fixed nonempty closed bounded subset of \( \mathcal{E} \). We put, for short, \( \mu_X = \mu_{A \Delta}, X \in \mathcal{B}(\mathcal{E}) \).

For each \( k \in \mathbb{N} \), set \( \varepsilon_k = 1/k \), and define

\[ M_k = \{ X \in \mathcal{B}(\mathcal{E}) \mid \inf_{\sigma > 0} \text{diam } M_{X \Delta}(\sigma) < \varepsilon_k \} \quad \text{and} \quad \inf_{\sigma > 0} \text{diam } M_{A \Delta}(\sigma) < \varepsilon_k \}. \]

To prove the main result of this section, Theorem 4.3, we state two lemmas whose proofs will be given later.
**Lemma 4.1.** \( \mathcal{M}_k \) is dense in \( C(\mathcal{E}) \).

**Lemma 4.2.** \( \mathcal{M}_k \) is open in \( C(\mathcal{E}) \).

**Theorem 4.3.** Let \( \mathcal{E} \) be a uniformly convex Banach space. Let \( A \in \mathcal{B}(\mathcal{E}) \). Then the set
\[
\mathcal{V}^* = \{ X \in C(\mathcal{E}) \mid \max(A, X) \text{ is well posed} \}
\]
is a dense \( G_\delta \)-subset of \( C(\mathcal{E}) \).

**Proof.** By Lemmas 4.1 and 4.2, the set
\[
\mathcal{M}_0 = \bigcap_{k \in \mathbb{N}} \mathcal{M}_k
\]
is a dense \( G_\delta \)-subset of \( C(\mathcal{E}) \). Moreover, by Proposition 2.2, we have \( \mathcal{V} = \mathcal{M}_0 \). Hence \( \mathcal{V} \) is a dense \( G_\delta \)-subset of \( C(\mathcal{E}) \), completing the proof.

**Remark 4.4.** Theorem 4.3 is a multivalued version of results due to Asplund [1] and Edelstein [11]. Note also that with the notation of the example given in Remark 3.6, there exists \( r > 0 \) such that, for every \( X \in B_{\mathcal{E}}(X_0, r) \), the maximization problem \( \max(A, X) \) is not well posed. This shows that, if \( \mathcal{E} \) is an arbitrary Banach space, then Theorem 4.3 is, in general, not true.

**Proof of Lemma 4.1.** Let \( X \in C(\mathcal{E}) \) and let \( r > 0 \). We want to show that there exists \( Y \in \mathcal{M}_k \) such that \( h(Y, X) < r \). The case \( \mu_y - 0 \) is trivial. Thus, without loss of generality, we suppose \( \mu_y > 0 \) and take \( r \) such that \( 0 < r < \mu_y \).

By Proposition 2.4, there exists \( 0 < \sigma_0 < r \) such that for every \( x, y \in \mathbb{E} \), with \( \| y - x \| = r \), and for every \( 0 < \sigma \leq \sigma_0 \), we have
\[
\text{diam } D(x, y; \mu_x + \| y - x \| - \sigma, \sigma) < \varepsilon_k, \tag{4.1}
\]
where
\[
D(x, y; \mu_x + \| y - x \| - \sigma, \sigma) = \bar{B}_\varepsilon(y, \mu_x) \setminus \bar{B}_\varepsilon(x, \mu_x + \| y - x \| - \sigma).
\]
Set
\[
\tilde{\sigma} = \min\{\sigma_0, \varepsilon_k\}. \tag{4.2}
\]
By Proposition 2.3, there exists \( 0 < \tau_0 < r/2 \) such that for every \( u \in \mathbb{E} \), with \( d(u, X) \geq r/2 \), and for every \( 0 < \tau \leq \tau_0 \), we have
\[
\text{diam } C_{X,u}(\tau) < \frac{\tilde{\sigma}}{2}, \tag{4.3}
\]
where $C_{X,u}(\tau)$ is given by (2.3). Set

$$\tau = \min \left\{ \tau_0, \frac{\sigma}{2} \right\}.$$  \hfill (4.4)

Now, pick $\tilde{x} \in X$ and $\tilde{a} \in A$ such that $\|\tilde{x} - \tilde{a}\| \geq \mu_X - \bar{\tau}/4$, and observe that $\tilde{x} \neq \tilde{a}$, for $\mu_X > r > \sigma_0 \geq \sigma > \bar{\tau}$. Set

$$u = \tilde{x} + r \frac{\tilde{x} - \tilde{a}}{\|\tilde{x} - \tilde{a}\|}, \quad Y = \overline{cO}(X \cup \{u\}).$$

Clearly $Y \in \mathcal{S}(\mathbb{E})$, and $h(Y, X) \leq r$. Thus, to complete the proof it suffices to show that $Y \in \mathcal{M}_k$.

To this end, we start by proving the following inequalities:

$$\mu_Y \geq \mu_X + r - \frac{\tau}{4}, \tag{4.5}$$

$$d(u, X) \geq r - \frac{\tau}{4}. \tag{4.6}$$

Indeed, $\|u - \tilde{a}\| = \|u - \tilde{x}\| + \|\tilde{x} - \tilde{a}\| \geq r + (\mu_X - \bar{\tau}/4)$, from which (4.5) follows, for $u \in Y$ and $\tilde{a} \in A$. Furthermore, from (2.2) we have

$$d(u, X) \geq e(u, A) - \mu_X \geq \left( \mu_X + r - \frac{\tau}{4} \right) - \mu_X = r - \frac{\tau}{4},$$

for $e(u, A) \geq \mu_X + r - \bar{\tau}/4$, and so also (4.6) is proved.

Claim 1. We have

$$M_{Y_A} \left( \frac{\bar{\tau}}{2} \right) \subset C_{X,u}(\bar{\tau}). \tag{4.7}$$

Indeed, suppose (4.7) false, and let $y \in M_{Y_A}(\bar{\tau}/2) \setminus C_{X,u}(\bar{\tau})$ be arbitrary. From the definition of $M_{Y_A}(\bar{\tau}/2)$ and from (4.5), we have

$$e(y, A) \geq \mu_Y - \frac{\bar{\tau}}{2} \geq \mu_X + r - \frac{3}{4} \bar{\tau}. \tag{4.8}$$

On the other hand, we have

$$e(y, A) \leq \mu_X + d(y, Y) \tag{by (2.2)}$$

$$< \mu_X + (d(u, X) - \bar{\tau}) \quad \text{as } y \in Y \setminus C_{X,u}(\bar{\tau}))$$

$$\leq \mu_X + r - \bar{\tau} \quad \text{as } d(u, X) \leq r).$$

Since the latter inequality contradicts (4.8), Claim 1 is true.
Claim 2. We have

\[ M_{\gamma Y}(\frac{\tilde{\gamma}}{4}) \subseteq D(u, \tilde{x}; \mu_X + \|\tilde{x} - u\| - \hat{\sigma}, \hat{\sigma}). \]  
(4.9)

Indeed, let \( a \in M_{\gamma Y}(\frac{\tilde{\gamma}}{4}) \) be arbitrary. As \( e(a, Y) \geq \mu_Y - \frac{\tilde{\gamma}}{4} \), there exists \( y \in Y \) such that

\[ \|y - a\| \geq \mu_Y - \frac{\tilde{\gamma}}{2}. \]  
(4.10)

By (2.2) we have \( d(y, X) \geq e(y, A) - \mu_X \), from which, by using (4.10) and (4.52), we get

\[ d(y, X) \geq \left( \mu_Y - \frac{\tilde{\gamma}}{2} \right) - \mu_Y \geq \left( \mu_X + \frac{r - \tilde{\gamma}}{4} \right) - \frac{\tilde{\gamma}}{2} - \mu_X > r - \frac{\tilde{\gamma}}{2}. \]

From this, since \( r \geq d(u, X) \), we have \( d(y, X) > d(u, X) - \frac{\tilde{\gamma}}{2} \), and so \( y \in C_{X, u}(\tilde{\gamma}) \). From (4.4) and (4.6) we have \( \tilde{\gamma} \leq \tau_0 \) and \( d(u, X) > r/2 \). But, by (4.3), \( \text{diam} \ C_{X, u}(\tilde{\gamma}) < \hat{\sigma}/2 \), which implies

\[ \|y - u\| < \frac{\hat{\sigma}}{2}. \]  
(4.11)

Now, we have

\[ \|a - u\| \geq \|a - y\| - \|y - u\| \]
\[ > \left( \mu_Y - \frac{\tilde{\gamma}}{2} \right) - \frac{\hat{\sigma}}{2} \]  
(by (4.10) and (4.11))
\[ \geq \left( \mu_X + \frac{r - \tilde{\gamma}}{4} \right) - \frac{\tilde{\gamma}}{2} - \frac{\hat{\sigma}}{2} \]  
(by (4.5))
\[ > \mu_X + r - \frac{\hat{\sigma}}{2} \]  
(by (4.4)).

Hence \( a \notin B_L(u, \mu_X + \|\tilde{x} - u\| - \hat{\sigma}) \), for \( \|\tilde{x} - u\| = r \). Clearly, \( a \in \tilde{B}_L(\tilde{x}, \mu_x) \).

Hence \( a \in D(u, \tilde{x}; \mu_X + \|\tilde{x} - u\| - \hat{\sigma}, \hat{\sigma}) \). As \( a \in M_{\gamma Y}(\frac{\tilde{\gamma}}{4}) \) is arbitrary, (4.9) is proved, completing the proof of Claim 2.

As \( \text{diam} \ C_{X, u}(\tilde{\gamma}) < \hat{\sigma}/2 \) and, by (4.2), \( \hat{\sigma} \leq \epsilon_k \), Claim 1 gives

\[ \text{diam} \ M_{\gamma A}(\frac{\tilde{\gamma}}{2}) < \epsilon_k. \]  
(4.12)
Furthermore, from (4.1) we have $\text{diam} \, D(u, \tilde{x}; \mu_{r} + \| \tilde{x} - u \| - \tilde{\sigma}, \tilde{\sigma}) < \varepsilon_{k}$, since $\| \tilde{x} - u \| = r$ and, by (4.2), $\tilde{\sigma} \leq \sigma_{0}$. Hence, by Claim 2,

$$\text{diam} \, M_{A^{2}} \left( \frac{\tilde{x}}{4} \right) < \varepsilon_{k}.$$ 

From (4.12) and the latter inequality it follows that $Y \in M_{k}$, which completes the proof of Lemma 4.1.

**Proof of Lemma 4.2.** This is similar to the proof of Lemma 3.2, and so it is omitted.

**References**


