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Two-dimensional Gibbsian point processes with continuous spin symmetries

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Abstract

The conservation of continuous symmetries in two-dimensional systems with interaction is a classical subject of statistical mechanics. So far, all results of this sort required some smoothness properties of the interaction. Only recently Ioffe et al. (Comm. Math. Phys. 226 (2002) 433) succeeded to treat the case of lattice systems with continuous, rather than smooth, interaction. Here we establish a similar result for Gibbsian systems of point particles with internal degrees of freedom.

Keywords: Gibbsian point processes; Mermin–Wagner theorem; Continuous symmetries; Stochastic domination; Percolation; Superstability

1. Introduction

Gibbsian processes were introduced by Dobrushin (see [1,2]), Lanford and Ruelle (see [10]) as a model for equilibrium states in statistical physics. (For general results on Gibbs measures on a d-dimensional lattice we refer to the detailed book of Georgii [5], which covers a wide range of phenomena.) The first results concerned existence and uniqueness of Gibbs measures and the structure of the set of Gibbs measures related to a given potential. The question of uniqueness is of special

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importance, as the nonuniqueness of Gibbs measures can be interpreted as a certain type of phase transition occurring within the particle system. A phase transition occurs whenever a symmetry of the potential is broken, so it is natural to ask, under which conditions symmetries are broken or conserved. The answer to this question depends on the type of the symmetry (discrete or continuous), the number of spatial dimensions and smoothness and decay conditions on the potential (see [5, Chapters 6.2, 8, 9 and 20]). It turns out that the case of continuous symmetries in two dimensions is especially interesting. The first progress in this case was achieved by Mermin and Wagner, who showed for special two-dimensional lattice models that symmetries are conserved [12,13]). In [3] Dobrushin and Shlosman established conservation of symmetries for more general potentials which satisfy smoothness and decay conditions, and Pfister improved this result in [14]. The case of marked point particles in the continuum was considered by Shlosman [16], Fröhlich and Pfister [4] and Georgii [6]. All these results rely on the smoothness of the interaction, and only recently Ioffe et al. showed that mere continuity suffices in the lattice model [9] using a perturbation expansion and percolation theory.

Our aim is to generalise the last result from a lattice to a continuous model. Using superstability techniques and percolation arguments, we will show how to combine ideas of Pfister [14], Fröhlich and Pfister [4] and Ioffe et al. [9] in order to obtain this result.

In Section 2 we will describe the situation considered and state the result obtained. The precise setting is then given in Section 3. In Section 4 a proof of a special case of the result is given. The proofs of all lemmas are relegated to Section 5, and in Section 6 we will show how to deal with the general case.

2. The result

We consider infinitely many particles in the plane, where a particle has a position in $\mathbb{R}^2$ and internal degrees of freedom. These can be modeled by assigning to the particle a value from some measurable spin space (or mark space) S. The particles may interact via a pair potential U. So U is a measurable function

$$U : (\mathbb{R}^2 \times S)^2 \to \bar{\mathbb{R}} := \mathbb{R} \cup \{+\infty\},$$

such that $U(y_1, y_2) = U(y_2, y_1)$ for all $(y_1, y_2) \in D$, i.e. U is symmetric. Here we assume U to be of the form

$$U(x_1, \sigma_1; x_2, \sigma_2) = J(x_1 - x_2)\tilde{U}(\sigma_1, \sigma_2) + K(x_1 - x_2), \tag{2.1}$$

such that the functions $\tilde{U} : S^2 \to \mathbb{R}$, $J : \mathbb{R}^2 \to \mathbb{R}$ and $K : \mathbb{R}^2 \to \bar{\mathbb{R}}$ are measurable and symmetric, and J is ψ-dominated, i.e.

$$|J(x)|(1 + \|x\|^2) \leqslant \psi(\|x\|) \quad \forall x \in \mathbb{R}^2,$$

where $\psi : \mathbb{R}_+ := [0, \infty[\; \to \mathbb{R}_+$ is a given decreasing function such that

$$\int_0^\infty \psi(r) r \, \mathrm{d}r := \psi_s < \infty.$$

We will call a potential U of the above form (2.1) a ψ-dominated potential corresponding to $J, K, \tilde{U}$.

Since K may take the value $+\infty$ we need not restrict ourselves to the case of bounded or finite potentials, but are also able to consider hardcore potentials and potentials with a singularity at 0, which are the more interesting cases.

We are only interested in the equilibrium states of a thermodynamical system as described above, and as a model for these we take the concept of Gibbs measures. Supposing that the given potential has some internal symmetry, we would like to know whether the possible equilibrium states inherit this symmetry necessarily. For example, considering a potential which does not change under rotation of spins, under what conditions are the equilibrium states invariant under spin rotation? Here we are concerned with continuous symmetries only, so that we can model the symmetries by a Lie-group G acting on the spin space S. Our result is then the following:

Theorem 1. *Let $(S, \mathscr{B}(S), \lambda_S)$ be a probability space such that S is a compact topological space and $\mathscr{B}(S)$ its Borel-σ-algebra. Let G be a compact connected Lie-group operating on S such that the operation is continuous and the reference measure λ_S is G-invariant. Let U be a superstable, lower regular, ψ-dominated potential corresponding to $J, K, \tilde{U}$ such that $\tilde{U}$ is continuous and G-invariant. Then every tempered Gibbs measure corresponding to U is G-invariant.*

The exact definitions of the objects and properties in the formulation of the above theorem will be given in the next section.

3. The setting

3.1. Configurations of particles

We consider the plane $\mathbb{R}^2$ with maximum norm $\|.\|$. Let

$$\Lambda_t := [-t, t[^2 \quad \text{for } t \in \mathbb{R}_+ \quad \text{and} \quad C_r := r + \left[-\tfrac{1}{2}, \tfrac{1}{2}\right]^2 \quad \text{for } r \in \mathbb{Z}^2$$

be subsets of $\mathbb{R}^2$. On $\mathbb{R}^2$ let $\mathscr{B}^2$ be the Borel-σ-algebra, and $\mathscr{B}^2_{\mathrm{b}} \subset \mathscr{B}^2$ the set of all bounded Borel sets. The Lebesgue measure on $(\mathbb{R}^2, \mathscr{B}^2)$ will be denoted by λ^2.

For describing the marks or spins of the particles let S be a topological space, $\mathscr{B}(S)$ the Borel-σ-algebra on S and λ_S a normed reference measure on $(S, \mathscr{B}(S))$. As λ_S is the only measure to be considered on $(S, \mathscr{B}(S))$, we will simply write $\mathrm{d}\sigma := \mathrm{d}\lambda_S(\sigma)$ when integrating with respect to λ_S.

A configuration Y of marked particles is described by a subset of $\mathbb{R}^2 \times S$ which is locally finite, in that $|Y \cap (\Lambda \times S)| < \infty$ for all $\Lambda \in \mathscr{B}^2_{\mathrm{b}}$, and simple, in that for all $(x_1, \sigma_1) \neq (x_2, \sigma_2) \in Y$ we have $x_1 \neq x_2$. The configuration space $\mathscr{Y}$ is defined to be the set of all locally finite and simple subsets of $\mathbb{R}^2 \times S$. A configuration $Y \in \mathscr{Y}$ is said to be finite if $|Y| < \infty$. Given a particle $y \in \mathbb{R}^2 \times S$, we want to consider the position $y^o \in \mathbb{R}^2$ and the spin $\sigma_y \in S$ of the particle, and given a configuration $Y \in \mathscr{Y}$ let

$Y^o := \{x \in \mathbb{R}^2 : \exists\, \sigma \in S : (x,\sigma) \in Y\}$. For $Y \in \mathscr{Y}$ and $x \in \mathbb{R}^2$ such that $(x,\sigma) \in Y$ let $\sigma_x(Y) := \sigma$ and $\sigma_x := \sigma_x(Y)$ if it is clear which configuration is to be considered.

For $Y \in \mathscr{Y}, \Lambda \in \mathscr{B}^2, B \in \mathscr{B}(S)$ let $Y_{\Lambda,B} := Y \cap (\Lambda \times B)$ and $Y_\Lambda := Y_{\Lambda,S}$ the restriction of Y to $\Lambda \times S$ and Λ respectively, $\mathscr{Y}_\Lambda := \{Y \in \mathscr{Y} : Y \subset \Lambda \times S\}$ the set of all configurations in Λ, $N_{\Lambda,B}(Y) := |Y_{\Lambda,B}|$ the number of particles of Y in Λ with marks in B and $N_\Lambda := N_{\Lambda,S}$. The counting variables $N_{\Lambda,B}$ generate a σ-algebra $\mathscr{F}_{\mathscr{Y}}$ on $\mathscr{Y}$. For $\Lambda \in \mathscr{B}^2$ let $\mathscr{F}'_{\mathscr{Y},\Lambda}$ be the σ-algebra on $\mathscr{Y}_\Lambda$ obtained by restricting $\mathscr{F}_{\mathscr{Y}}$ to $\mathscr{Y}_\Lambda$, and let $\mathscr{F}_{\mathscr{Y},\Lambda} := e_\Lambda^{-1}\mathscr{F}'_{\mathscr{Y},\Lambda}$ be the σ-algebra on $\mathscr{Y}$obtained from $\mathscr{F}'_{\mathscr{Y},\Lambda}$ by the restriction mapping $e_\Lambda : \mathscr{Y} \to \mathscr{Y}_\Lambda$, $Y \mapsto Y_\Lambda$. For disjoint sets $\Lambda_1, \Lambda_2 \in \mathscr{B}^2$ and configurations $Y, \bar{Y} \in \mathscr{Y}$ let $Y_{\Lambda_1}\bar{Y}_{\Lambda_2} := Y_{\Lambda_1} \cup \bar{Y}_{\Lambda_2}$.

The mean quadratic particle density per unit square for $Y \in \mathscr{Y}$ is defined by

$$s_n(Y) := \frac{1}{\lambda^2(\Lambda_{n+1/2})} \sum_{r \in \mathbb{Z}^2 \cap \Lambda_{n+1/2}} N^2_{C_r}(Y).$$

A configuration $Y \in \mathscr{Y}$ is said to be tempered if $s(Y) := \sup_{n \in \mathbb{N}} s_n(Y) < \infty$. Let $\mathscr{Y}_\mathrm{t} \in \mathscr{F}_{\mathscr{Y}}$ be the set of all tempered configurations.

Now similar objects can be considered for particles without marks. Let $\mathscr{X} := \{X \subset \mathbb{R}^2\colon\ |X \cap \Lambda| < \infty\ \forall \Lambda \in \mathscr{B}^2_\mathrm{b}\}$ be the configuration space of particle positions. The restrictions X_Λ, the set of configurations in Λ $\mathscr{X}_\Lambda$, the counting variables N_Λ, the σ-algebras $\mathscr{F}_{\mathscr{X}}$, $\mathscr{F}'_{\mathscr{X},\Lambda}$ and $\mathscr{F}_{\mathscr{X},\Lambda}$ and $X_{\Lambda_1}\bar{X}_{\Lambda_2}$ are then defined analogously to the objects above. The projection $o : \mathscr{Y} \to \mathscr{X}$, $Y \mapsto Y^o$ obviously is measurable, so $\mathscr{F}_{\mathscr{X}}$ can be considered as a subset of $\mathscr{F}_{\mathscr{Y}}$ via the identification of a set $\mathscr{X}_1 \in \mathscr{F}_{\mathscr{X}}$ with $o^{-1}\mathscr{X}_1 \in \mathscr{F}_{\mathscr{Y}}$. For example we have that $\mathscr{Y}_\mathrm{t} \in \mathscr{F}_{\mathscr{X}}$. For any $X \in \mathscr{X}$ and a family of marks $(\sigma_x)_{x \in X}$ let $(X,\sigma) := \{(x,\sigma_x) : x \in X\}$ the configuration determined by X and σ.

Let $z > 0$ be an activity parameter which will be fixed throughout this paper. Let $\nu := \nu_z$ be the distribution of the Poisson point process on $(\mathscr{Y}, \mathscr{F}_{\mathscr{Y}})$ with intensity z and distribution of marks λ_S, and $\nu^o := \nu^o_z$ be the distribution of the Poisson point process on $(\mathscr{X}, \mathscr{F}_{\mathscr{X}})$ with intensity z. So

$$\int f\,\mathrm{d}\nu^o = \mathrm{e}^{-z\lambda^2(\Lambda)} \sum_{k \geqslant 0} \frac{z^k}{k!} \int_{\Lambda^k} \mathrm{d}x_1 \cdots \mathrm{d}x_k f(\{x_i : 1 \leqslant i \leqslant k\}),$$

for any $\mathscr{F}_{\mathscr{X},\Lambda}$-measurable nonnegative function $f : \mathscr{X} \to \mathbb{R}_+$ and

$$\int f\,\mathrm{d}\nu = \int \nu^o(\mathrm{d}X) \int_{S^{X_\Lambda}} \mathrm{d}\sigma_{X_\Lambda} f((X_\Lambda, \sigma))$$

for any $\mathscr{F}_{\mathscr{Y},\Lambda}$-measurable nonnegative function $f : \mathscr{Y} \to \mathbb{R}_+$.

3.2. Configurations of bonds

For any set Z and distinct $z_1, z_2 \in Z$ let $z_1z_2 := \{z_1, z_2\}$ be the bond joining z_1 and z_2. Let $E(Z) := \{z_1z_2\colon\ z_1, z_2 \in Z, z_1 \neq z_2\}$ be the set of all bonds in Z. On $E(\mathbb{R}^2)$ the σ-algebra

$$\mathscr{F}_{E(\mathbb{R}^2)} := \{\{x_1x_2 \in E(\mathbb{R}^2) : (x_1,x_2) \in B\}\colon\ B \in (\mathscr{B}^2)^2\}$$

is given. Let

$$\mathscr{E} := \{E \subset E(\mathbb{R}^2) : |\{xy \in E : xy \subset B\}| < \infty \quad \forall B \in \mathscr{B}_\mathrm{b}^2\}$$

be the configuration space of bonds, i.e. the set of all locally finite bond sets. On $\mathscr{E}$ the σ-algebra $\mathscr{F}_\mathscr{E}$ is defined to be generated by the counting variables $N_{E'} : \mathscr{E} \to \mathbb{N}$, $E \mapsto |E' \cap E|$ $(E' \in \mathscr{F}_{E(\mathbb{R}^2)})$.

For a countable set $E \in \mathscr{E}$ one can also consider the Bernoulli-σ-algebra $\mathscr{B}_E$ on $\mathscr{E}_E := \mathscr{P}(E) \subset \mathscr{E}$, which is defined to be generated by the family of sets $(\{E' \subset E : e \in E'\})_{e \in E}$. It is easy to check that the inclusion $(\mathscr{E}_E, \mathscr{B}_E) \to (\mathscr{E}, \mathscr{F}_\mathscr{E})$ is measurable. Thus any probability measure on $(\mathscr{E}_E, \mathscr{B}_E)$ can trivially be extended to $(\mathscr{E}, \mathscr{F}_\mathscr{E})$.

Given a countable set E and a family $(\varepsilon_e)_{e \in E}$ of real numbers in $[0, 1]$ the Bernoulli measure on $(\mathscr{P}(E), \mathscr{B}_E)$ is defined as the unique probability measure for which the events $(\{E' \subset E : e \in E'\})_{e \in E}$ are independent with probabilities $(\varepsilon_e)_{e \in E}$.

3.3. Interaction and superstability

Our next step is to introduce the interaction between particles. As mentioned before we will consider a ψ-dominated potential corresponding to $J, K, \tilde{U}$ as defined in and below of (2.1). The energy of a finite configuration $Y \in \mathscr{Y}$ is defined as

$$H^U(Y) := \sum_{y_1 y_2 \in E(Y)} U(y_1, y_2)$$

and for two finite configurations $Y, Y' \in \mathscr{Y}$ let

$$W^U(Y, Y') := \sum_{y_1 \in Y} \sum_{y_2 \in Y'} U(y_1, y_2) \tag{3.1}$$

be the interaction energy of the configurations. Definition (3.1) can be extended to infinite configuration Y' whenever $W^U(Y, Y'_\Lambda)$ converges as $\Lambda \uparrow \mathbb{R}^2$ through the net $\mathscr{B}_\mathrm{b}^2$.

For a configuration $Y \in \mathscr{Y}$ let $\mathbb{Z}^2(Y) := \{r \in \mathbb{Z}^2 : N_{C_r}(Y) > 0\}$ be the minimal set of lattice points such that the corresponding squares cover Y. Then a potential is called superstable if there are real constants $A > 0$ and $B \geqslant 0$ such that for all finite configurations $Y \in \mathscr{Y}$

$$H^U(Y) \geqslant \sum_{r \in \mathbb{Z}^2(Y)} [A N_{C_r}(Y)^2 - B N_{C_r}(Y)].$$

A potential is called lower regular if there is a decreasing function $\Psi : \mathbb{N} \to \mathbb{R}_+$ such that

$$\sum_{r \in \mathbb{Z}^2} \Psi(\|r\|) < \infty$$

and

$$W^U(Y, Y') \geqslant -\sum_{r \in \mathbb{Z}^2(Y)} \sum_{s \in \mathbb{Z}^2(Y')} \Psi(\|r - s\|) \left[\tfrac{1}{2} N_{C_r}(Y)^2 + \tfrac{1}{2} N_{C_s}(Y')^2\right]$$

for all finite configurations $Y, Y' \in \mathcal{Y}$. Note that any ψ-dominated potential corresponding to $J, K, \tilde{U}$ such that also $K(x) \geqslant -\psi(\|x\|)$ for all $x \in \mathbb{R}^2$ is lower regular.

It is well known that for any superstable and lower regular potential U, any finite configuration $Y \in \mathcal{Y}$ and any tempered configuration $Y' \in \mathcal{Y}_t$ the interaction energy $W^U(Y, Y')$ exists in $]-\infty, \infty]$ (see [15] for example).

3.4. Gibbs measures

Given a superstable and lower regular potential U, the Hamiltonian of a configuration $Y \in \mathcal{Y}$ in $\Lambda \in \mathcal{B}_b^2$ with boundary condition $\bar{Y}_{\Lambda^c} \in \mathcal{Y}_t$ is defined by

$$H_\Lambda^U(Y_\Lambda \bar{Y}_{\Lambda^c}) := H^U(Y_\Lambda) + W^U(Y_\Lambda, \bar{Y}_{\Lambda^c}) = \sum_{y_1 y_2 \in E(Y_\Lambda \bar{Y}_{\Lambda^c}): y_1^o y_2^o \cap \Lambda \neq \emptyset} U(y_1, y_2).$$

The integral

$$Z_\Lambda^U(\bar{Y}) := \int v(\mathrm{d}Y)\, \mathrm{e}^{-H_\Lambda^U(Y_\Lambda \bar{Y}_{\Lambda^c})}$$

is called the partition function in $\Lambda \in \mathcal{B}_b^2$ for the boundary condition $\bar{Y}_{\Lambda^c} \in \mathcal{Y}_t$. Using superstability and lower regularity of U and temperedness of $\bar{Y}$ one can show that $Z_\Lambda^U(\bar{Y})$ is finite (see [15] for example), and considering the empty configuration Y one can show that $Z_\Lambda^U(\bar{Y})$ is positive. The Gibbs distribution $\gamma_\Lambda^U(.|\bar{Y})$ in $\Lambda \in \mathcal{B}_b$ with boundary condition $\bar{Y}_{\Lambda^c} \in \mathcal{Y}_t$, potential U and activity z is thus well defined by

$$\gamma_\Lambda^U(A|\bar{Y}) := Z_\Lambda^U(\bar{Y})^{-1} \int v(\mathrm{d}Y)\, \mathrm{e}^{-H_\Lambda^U(Y_\Lambda \bar{Y}_{\Lambda^c})} 1_A(Y_\Lambda \bar{Y}_{\Lambda^c}) \quad \text{for } A \in \mathcal{F}_{\mathcal{Y}}.$$

γ_Λ^U is a probability kernel from $(\mathcal{Y}, \mathcal{F}_{\mathcal{Y}})$ to $(\mathcal{Y}, \mathcal{F}_{\mathcal{Y}})$. Let $\gamma_\Lambda := \gamma_\Lambda^U$ if it is clear which potential is considered. Let

$$\begin{aligned} \mathcal{G}(U) := \{\mu \in \mathcal{P}(\mathcal{Y}, \mathcal{F}_{\mathcal{Y}}) : \mu(\mathcal{Y}_t) = 1 \\ \mu(A|\mathcal{F}_{\mathcal{Y}, \mathbb{R}^2 \setminus \Lambda}) = \gamma_\Lambda^U(A|.)\ \mu\text{-a.s.}\ \forall A \in \mathcal{F}_{\mathcal{Y}}, \Lambda \in \mathcal{B}_b^2\} \end{aligned}$$

be the set of all tempered Gibbs measures for the potential U and the activity z. It is easy to see that for any probability measure $\mu \in \mathcal{P}(\mathcal{Y}, \mathcal{F}_{\mathcal{Y}})$ such that $\mu(\mathcal{Y}_t) = 1$ the equivalence

$$\mu \in \mathcal{G}(U) \iff (\mu\gamma_\Lambda^U = \mu\ \forall \Lambda \in \mathcal{B}_b^2)$$

holds. So for every $\mu \in \mathcal{G}(U)$, $f : \mathcal{Y} \to \mathbb{R}_+$ measurable and $\Lambda \in \mathcal{B}_b^2$ we have

$$\int \mu(\mathrm{d}Y) f(Y) = \int \mu(\mathrm{d}\bar{Y}) \int \gamma_\Lambda^U(\mathrm{d}Y|\bar{Y}) f(Y_\Lambda \bar{Y}_{\Lambda^c}). \tag{3.2}$$

For a superstable and lower regular potential U and a tempered Gibbs measure $\mu \in \mathcal{G}(U)$, the correlation function $\rho^{U,\mu}$ of μ is defined by

$$\rho^{U,\mu}(Y) = \mathrm{e}^{-H^U(Y)} \int \mu(\mathrm{d}\bar{Y}) \mathrm{e}^{-W^U(Y,\bar{Y})}$$

for any finite configuration Y. It is a remarkable consequence of Ruelle's superstability estimates that there is a constant $\xi \in \mathbb{R}$ such that

$$\rho^{U,\mu}(Y) \leqslant \xi^{|Y|} \tag{3.3}$$

for any finite configuration $Y \in \mathscr{Y}$. (For a proof see [15].) We will call a $\xi \in \mathbb{R}$ satisfying (3.3) a Ruelle bound. Actually we will need this bound on the correlation function in the following way.

Lemma 1. *Let U be a superstable and lower regular potential, $\mu \in \mathscr{G}(U)$ a tempered Gibbs measure and $\xi \in \mathbb{R}$ a Ruelle bound. Then we have*

$$\int \mu(\mathrm{d}Y) \sum_{x_1,\ldots,x_m \in Y^o}^{\neq} f(x_1,\ldots,x_m) \leqslant (z\xi)^m \int \mathrm{d}x_1 \cdots \mathrm{d}x_m f(x_1,\ldots,x_m) \tag{3.4}$$

for every integer $m \geqslant 0$ and every measurable function $f : (\mathbb{R}^2)^m \to \mathbb{R}_+$.

We use $\sum^{\neq}$ as a shorthand notation for a multiple sum such that the summation indices are assumed to be pairwise distinct.

3.5. Transformations of spins

Now let the spin space S be a compact topological space, and G be a compact, connected Lie-group operating on S,

$$\mathrm{op} : G \times S \to S, \quad (\tau,\sigma) \mapsto \mathrm{op}(\tau,\sigma) =: \tau(\sigma),$$

such that the operation is measurable.

For every $\tau \in G$ we also consider $\tilde{\tau} : \mathscr{Y} \to \mathscr{Y}$, $\tilde{\tau}(Y) = \{(x,\tau(\sigma)) : (x,\sigma) \in Y\}$, and $\bar{\tau} : D \to D$, $\bar{\tau}(x_1,\sigma_1;x_2,\sigma_2) = (x_1,\tau(\sigma_1);x_2,\tau(\sigma_2))$. Usually these mappings will again be denoted by τ. Furthermore, for a configuration $Y \in \mathscr{Y}$ and $\tau : Y^o \to G$ we write $\tau Y := \tau(Y) := \{(x,\tau(x)(\sigma)) : (x,\sigma) \in Y\}$.

$\tau \in G$ is called a symmetry of a given pair potential U if $U \circ \tau = U$. If this holds for every $\tau \in G$, then U is said to be G-invariant. The reference measure λ_S is called G-invariant if $\lambda_S \circ \tau^{-1} = \lambda_S$ for all $\tau \in G$, and a Gibbs measure $\mu \in \mathscr{G}(U)$ is called G-invariant if $\mu \circ \tau^{-1} = \mu$ for all $\tau \in G$.

4. The case of S^1-action

We will first consider the mark space $(S,\mathscr{B}(S),\lambda_S) := (S^1,\mathscr{B}(S^1),\lambda_{S^1})$, where S^1 is the unit circle, $\mathscr{B}(S^1)$ is the Borel-σ-algebra on S^1 and λ_{S^1} is the Lebesgue-measure on S^1, and transformations $\tau \in G := \{\tau_\sigma : \sigma \in S^1\}$, where τ_σ is defined to be the rotation with angle σ. For $\sigma,\sigma' \in S^1 = \mathbb{R}/(2\pi\mathbb{Z})$ we write $\tau_\sigma(\sigma') =: \sigma' + \sigma$. In order to simplify notation we identify a rotation with its angle, i.e. we identify $S^1 = \mathbb{R}/(2\pi\mathbb{Z})$ with $[0,2\pi[$, and so we consider functions on S^1 as 2π-periodic functions on $\mathbb{R}$ whenever possible.

If all rotations $\tau \in G$ are symmetries of the ψ-dominated potential U corresponding to $J,K,\tilde{U}$, then U can also be written in the form

$$U(x_1,\sigma_1;x_2,\sigma_2) = J(x_1-x_2)V(\sigma_1-\sigma_2) + K(x_1-x_2),$$

where $V : S \to \mathbb{R}$ is defined by $V(\sigma) := \tilde{U}(\sigma, 0)$. On the other hand a potential of the above form is G-invariant. It is called the ψ-dominated potential corresponding to J, K, V. As an additional preliminary simplification we assume that $J \geqslant 0$. So we consider the following special case of Theorem 1.

Theorem 2. *Let U be a superstable, lower regular ψ-dominated potential corresponding to J, K, V such that $J \geqslant 0$ and V is continuous.*

Then every tempered Gibbs measure corresponding to U is G-invariant.

In the following subsections we will give a proof of this theorem.

4.1. Constants and decomposition of V

Let U be a potential with the properties stated in Theorem 2, $\mu \in \mathscr{G}(U)$ a tempered Gibbs measure and $\xi \in \mathbb{R}$ a Ruelle bound satisfying (3.3) and $1 < 2z\xi$, where again z is the intensity of the underlying Poisson point process. As a consequence of the ψ-domination of J and the integrability condition on ψ there is a real constant c_J such that

$$1 + \psi(0) + \int J(x)(1 + \|x\|^2)\, \mathrm{d}x \leqslant c_J$$

and there are real constants $c(R)$ for $R \geqslant 0$ such that $\lim_{R\to\infty} c(R) = 0$ and for all $R \geqslant 0$

$$\int 1_{\{|x| \geqslant R\}} J(x)\, \mathrm{d}x \leqslant c(R). \tag{4.1}$$

We want to show the G-invariance of μ by an argument similar to the one given in [5, Chapter 9.1, Proposition 9.1]. So we fix a transformation $\tau \in]0, \pi[$, a test cylinder event $B \in \mathscr{F}_{\mathscr{Y}, \Lambda_{n'}}$ ($n' \in \mathbb{N}$) and a real $\delta > 0$. Furthermore let $1 > \varepsilon > 0$ such that

$$c_J \varepsilon < 2 c_J z \xi \varepsilon < 1. \tag{4.2}$$

As the above parameters are fixed for the whole proof we will ignore the dependence of any variable on any of the above parameters.

As V is a continuous function on S^1, V can be approximated by trigonometric polynomials due to the Weierstraß theorem. So we have the decomposition $V = \tilde{V} - \tilde{v}$, such that $\tilde{V}$ is smooth (i.e. twice continuously differentiable), and $|\tilde{v}| < \varepsilon/2$. Defining $v := \tilde{v} + \varepsilon/2$ and $\bar{V} := \tilde{V} + \varepsilon/2$ we get the decomposition

$$V = \bar{V} - v \quad \text{with smooth } \bar{V} \text{ and } 0 < v < \varepsilon.$$

By symmetrizing $\bar{V}$ and v we can assume $\bar{V}$ and v to be symmetric. Let $\bar{U}$ be the ψ-dominated potential corresponding to $J, K, \bar{V}$.

4.2. Decomposition of μ and the bond process

For $n \in \mathbb{N}$ and $X \in \mathscr{X}$ we consider the bond set

$$E(X, n) := \{x_1 x_2 \in E(X) : J(x_1 - x_2) \neq 0, x_1 x_2 \cap \Lambda_n \neq \emptyset\}.$$

In order to be able to extend the decomposition of the potential function V to a decomposition of the Hamiltonian we need:

Lemma 2. *For each $n \in \mathbb{N}$ there is a set $\mathcal{X}_n \in \mathcal{F}_{\mathcal{X}}$ such that $\mu(\mathcal{X}_n) = 1$ and*

$$\sum_{x_1x_2 \in E(X,n)} J(x_1 - x_2) < \infty \quad \forall X \in \mathcal{X}_n.$$

Now let $n \in \mathbb{N}$ and $Y \in \mathcal{X}_n$ be fixed. Because of Lemma 2 we have

$$H^U_{\Lambda_n}(Y) = H^{\bar{U}}_{\Lambda_n}(Y) - \sum_{x_1x_2 \in E(Y^o,n)} J(x_1 - x_2) v(\sigma_{x_1}(Y) - \sigma_{x_2}(Y))$$

and therefore

$$\mathrm{e}^{-H^U_{\Lambda_n}(Y)} = \sum_{A \subset E(Y^o,n)}{}' \mathcal{V}_n(A, Y), \tag{4.3}$$

where we have used the shorthand notation

$$\mathcal{V}_n(A, Y) := \mathrm{e}^{-H^{\bar{U}}_{\Lambda_n}(Y)} \prod_{x_1x_2 \in A} [\mathrm{e}^{J(x_1-x_2)\, v(\sigma_{x_1}(Y)-\sigma_{x_2}(Y))} - 1]$$

for $n \in \mathbb{N}$, $Y \in \mathcal{Y}$ and finite $A \subset E(Y^o, n)$. The summation symbol $\sum'$ in (4.3) indicates that the sum extends over finite subsets only. For $n \in \mathbb{N}$, $X \in \mathcal{X}_{\Lambda_n}$, $\bar{Y} \in \mathcal{Y}_{\Lambda_n^c}$ such that $X\bar{Y}^o \in \mathcal{X}_n$, finite $A \subset E_n := E(X\bar{Y}^o, n)$, $\mathcal{E}' \in \mathcal{B}_{E_n}$ and $D \in \mathcal{F}_{\mathcal{Y}}$ we define

$$\mathcal{W}_n(X, \bar{Y}) := \int \mathrm{d}\sigma_X \mathrm{e}^{-H^U_{\Lambda_n}((X,\sigma)\bar{Y})},$$

$$\mathcal{W}_n(A, X, \bar{Y}) := \int \mathrm{d}\sigma_X\, \mathcal{V}_n(A, (X,\sigma)\bar{Y}),$$

$$\pi_n(\mathcal{E}'|X, \bar{Y}) := \sum_{A \in \mathcal{E}'}{}' \frac{\mathcal{W}_n(A, X, \bar{Y})}{\mathcal{W}_n(X, \bar{Y})},$$

$$\alpha_n(D|A, X, \bar{Y}) := \frac{1}{\mathcal{W}_n(A, X, \bar{Y})} \int \mathrm{d}\sigma_X \mathcal{V}_n(A, (X,\sigma)\bar{Y}) 1_D((X,\sigma)\bar{Y}).$$

As J and v are nonnegative the above factors and integrands are nonnegative, too, and so all products and integrals are well defined. If $\mathcal{W}_n(X, \bar{Y}) = 0$ or $X\bar{Y}^o \notin \mathcal{X}_n$ we define $\pi_n(.|X, \bar{Y})$ to be the probability measure on $(\mathcal{E}_{E_n}, \mathcal{B}_{E_n})$ with whole weight on the empty set. If $\mathcal{W}_n(A, X, \bar{Y}) = 0$ or $X\bar{Y}^o \notin \mathcal{X}_n$ or $A \in \mathcal{E}$ is not a finite subset of E_n let $\alpha_n(.|A, X, \bar{Y})$ be an arbitrary fixed probability measure on $(\mathcal{Y}, \mathcal{F}_{\mathcal{Y}})$. For $n \in \mathbb{N}$, $X \in \mathcal{X}_{\Lambda_n}$ and $\bar{Y} \in \mathcal{Y}_{\Lambda_n^c}$ such that $\mathcal{W}_n(X, \bar{Y}) > 0$ and $X\bar{Y}^o \in \mathcal{X}_n$ we have by (4.3)

$$\pi_n(\mathcal{E}_{E_n}|X, \bar{Y}) = \frac{1}{\mathcal{W}_n(X, \bar{Y})} \sum_{A \subset E_n}{}' \int \mathrm{d}\sigma_X \mathcal{V}_n(A, (X,\sigma)\bar{Y}) = 1.$$

Therefore $\pi_n(.|X, \bar{Y})$ is a probability measure on $(\mathcal{E}_{E_n}, \mathcal{B}(\mathcal{E}_{E_n}))$ and can be considered as a probability measure on $(\mathcal{E}, \mathcal{F}_{\mathcal{E}})$ as remarked earlier. All above functions are measurable in their arguments with respect to the given σ-algebras, which is an easy application of the measurability parts of Fubini's theorem and Campbell's theorem (see [11, Proposition 5.1.2]. for example). Hence both π_n and α_n are probability kernels. By the above definitions and by (4.3) for every $D \in \mathscr{F}_{\mathscr{Y}}$ and $\bar{Y} \in \mathscr{Y}$ one has the decomposition

$$\begin{aligned} &\gamma_{\Lambda_n}(D \cap \mathscr{X}_n | \bar{Y}) \\ &= \frac{1}{Z^U_{\Lambda_n}(\bar{Y})} \int v^o(\mathrm{d}X) \int_{S^{X_{\Lambda_n}}} \mathrm{d}\sigma_{X_{\Lambda_n}} \mathrm{e}^{-H^U_{\Lambda_n}((X_{\Lambda_n},\sigma)\bar{Y}_{\Lambda_n^c})} 1_{(D\cap\mathscr{X}_n)}((X_{\Lambda_n},\sigma)\bar{Y}_{\Lambda_n^c}) \\ &= \int \gamma^o_{\Lambda_n}(\mathrm{d}X|\bar{Y}) 1_{\mathscr{X}_n}(X_{\Lambda_n}\bar{Y}^o_{\Lambda_n^c}) \int \pi_n(\mathrm{d}A|X_{\Lambda_n}, \bar{Y}_{\Lambda_n^c}) \alpha_n(D|A, X_{\Lambda_n}, \bar{Y}_{\Lambda_n^c}), \end{aligned} \tag{4.4}$$

where $\gamma^o_{\Lambda_n}(.|\bar{Y}) := \gamma_{\Lambda_n}(.|\bar{Y}) \circ o^{-1}$. Now we want to examine the percolation process given by π_n. So let $n \in \mathbb{N}$, $Y \in \mathscr{Y}$ and $E_n := E(Y^o, n)$. $\pi_n(.|Y^o_{\Lambda_n}, Y_{\Lambda_n^c})$ has its whole weight on the countable set of finite subsets $A \subset E_n$, but this measure shows a strong dependence of different bonds. Fortunately, this measure is stochastically dominated ($\preceq$) by a Bernoulli measure, where the order on the underlying space $\mathscr{E}_{E_n}$ is given by the inclusion. This stochastic domination will be an important tool for evaluating bond probabilities. For a definition of stochastic domination see [7], for example.

More precisely, for given $X \in \mathscr{X}$ let $\pi(.|X)$ be the Bernoulli measure on $(\mathscr{E}_{E(X)}, \mathscr{B}_{E(X)})$ with bond probabilities $\varepsilon_{x_1x_2} := J(x_1 - x_2)\varepsilon$ for $x_1x_2 \in E(X)$. Note that $0 \leqslant \varepsilon_{x_1x_2} \leqslant 1$ for all bonds $x_1x_2 \in E(X)$, which is a consequence of the condition on ε in (4.2), and even $0 < \varepsilon_{x_1x_2}$ for all $x_1x_2 \in E(X, n)$. Again $\pi(.|X)$ can be considered as a probability measure on $(\mathscr{E}, \mathscr{F}_{\mathscr{E}})$, and indeed is a probability kernel. We now have:

Lemma 3. *For all $n \in \mathbb{N}$ and $Y \in \mathscr{Y}$,*

$$\pi_n(.|Y^o_{\Lambda_n}, Y_{\Lambda_n^c}) \preceq \pi(.|Y^o). \tag{4.5}$$

4.3. Deforming the spin transformations

For a configuration of positions $X \in \mathscr{X}$ and a bond set $A \subset E(X)$ let $\overset{A}{\longleftrightarrow} := \overset{A,X}{\longleftrightarrow}$ be the equivalence relation on X such that for all $x_1, x_2 \in X$ we have $x_1 \overset{A,X}{\longleftrightarrow} x_2$ iff either $x_1 = x_2$ or there is a finite path in X joining x_1 and x_2 and using bonds in A only. For $x_1 \neq x_2 \in X$, the inequality

$$\pi(x_1 \overset{\cdot}{\longleftrightarrow} x_2|X) \leqslant \sum_{m\geqslant 1} \sideset{}{^{\neq}}\sum_{\substack{x'_0,\ldots,x'_m \in X:\\ x'_0 = x_1, x'_m = x_2}} \varepsilon^m \prod_{i=1}^m J(x'_i - x'_{i-1}) \tag{4.6}$$

is an easy consequence of the above definition. For a configuration $X \in \mathscr{X}$, a bond set $A \subset E(X)$ and a point $x \in X$ let

$$C_{A,X}(x) := \{x' \in X : x \overset{A}{\longleftrightarrow} x'\}$$

be the percolation cluster of x in (X, A). Furthermore, we want to consider the range of clusters, so for $x \in X$ and $A \in \mathscr{B}^2_\mathrm{b}$ let

$$r_{A,X}(x) := \sup\{\|x'\| : x' \in C_{A,X}(x)\}$$

and

$$r_{A,X}(\Lambda) := \begin{cases} \max\{r_{A,X}(x') : x' \in \Lambda \cap X\} & \text{for } \Lambda \cap X \neq \emptyset, \\ 0 & \text{for } \Lambda \cap X = \emptyset. \end{cases}$$

Obviously $\|x\| \leqslant r_{A,X}(x) \leqslant \infty$ and $r_{A,X}(\Lambda) \leqslant \infty$. Now we have an estimate for the range of the cluster of the given set $\Lambda_{n'}$, where n' is the natural number fixed in Section 4.1.

Lemma 4. *There exists an integer $R > n'$ and a set $\mathscr{X}_R \in \mathscr{F}_{\mathscr{X}}$ such that $\mu(\mathscr{X}_R) \geqslant 1 - 2\delta$ and, for every $Y \in \mathscr{X}_R$ and $n \geqslant n'$,*

$$\pi_n(\{A : r_{A,Y^o}(\Lambda_{n'}) \geqslant R\} | Y^o_{\Lambda_n}, Y_{\Lambda_n^c}) \leqslant \delta. \tag{4.7}$$

From now on let an integer $R \geqslant 2$ with the above property be fixed. In order to construct the spin deformation we define the functions $q : \mathbb{R} \to \mathbb{R}$, $Q : \mathbb{R} \to \mathbb{R}$, $r : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ and $\tau_n : \mathbb{R}^2 \to S^1$ for $n > R$ by

$$q(s) := 1_{\{s \leqslant 2\}} + \frac{1}{s \log(s)} 1_{\{s > 2\}}, \quad Q(k) := \int_0^k q(s)\,\mathrm{d}s,$$

$$r(s,k) := 1_{\{s \leqslant 0\}} + \int_s^k \frac{q(s')}{Q(k)}\,\mathrm{d}s'\, 1_{\{0 < s < k\}}, \quad \tau_n(x) := \tau \cdot r(\|x\| - R, n - R).$$

Lemma 5. *For all $n > R$ and $x, x' \in \mathbb{R}^2$ such that $\|x'\| \geqslant \|x\|$ we have*

$$0 \leqslant \tau_n(x) - \tau_n(x') \leqslant \tau \|x - x'\| \frac{q(\|x\| - R)}{Q(n - R)}, \tag{4.8}$$

$\lim_{n\to\infty} Q(n) = \infty$,

$$\tau_n(x) = \tau \text{ for } \|x\| \leqslant R \quad \text{and} \quad \tau_n(x) = 0 \text{ for } \|x\| \geqslant n. \tag{4.9}$$

However, what we really need here is a spin deformation which is constant on points joined by a bond of a given set A. So, for $n \in \mathbb{N}$, $X \in \mathscr{X}$ and $A \subset E(X,n)$ we define $\tau_n^{X,A} : X \to S^1$ by

$$\tau_n^{X,A}(x) := \min\{\tau_n(x') : x' \in X \text{ and } x \stackrel{A}{\longleftrightarrow} x'\}.$$

This spin deformation can be seen to be measurable in x, X and A with respect to the given σ-algebras using Campbell's theorem. Because of (4.9) we have $\tau_n(x') = 0$ for $\|x'\| \geqslant n$, so the minimum is attained at some point $t_A(x) \in X$ ($t_A(x) := x$ for $\|x\| \geqslant n$). By construction we have

$$\begin{aligned} &\|t_A(x)\| \geqslant \|x\|, \quad t_A(x) \stackrel{A}{\longleftrightarrow} x, \quad \tau_n^{X,A}(x) = \tau_n(t_A(x)) \quad \forall x \in X \\ &\text{and} \quad \tau_n^{X,A}(x) = \tau_n^{X,A}(x') \quad \forall x, x' \in X \text{ such that } x \stackrel{A}{\longleftrightarrow} x'. \end{aligned} \tag{4.10}$$

4.4. Proof of Theorem 2

In order to simplify notation, for $n \in \mathbb{N}$, $X \in \mathscr{X}$ and $E_n := E(X,n)$ let $f_{n,X} : \mathscr{E}_{E_n} \to \mathbb{R}$ be defined by

$$f_{n,X}(A) := \sum_{xx' \in E_n} J(x - x')(\tau_n^{X,A}(x) - \tau_n^{X,A}(x'))^2. \tag{4.11}$$

Lemma 6. *There exists an integer $n > R$ and a set of configurations $\mathscr{X}_{R,n} \in \mathscr{F}_{\mathscr{X}}$ such that $\mu(\mathscr{X}_{R,n}) \geqslant 1-\delta$ and, for every $Y \in \mathscr{X}_{R,n}$,*

$$\pi_n\left(f_{n,Y^o} \geqslant \frac{2}{\|\bar{V}''\|}\middle| Y^o_{\Lambda_n}, Y_{\Lambda_n^c}\right) \leqslant \delta. \tag{4.12}$$

Let such an n be fixed for the rest of the proof, let $\mathscr{X}_\delta := \mathscr{X}_{R,n} \cap \mathscr{X}_R \cap \mathscr{X}_n$ be the set of good configurations of positions, and for $X \in \mathscr{X}$ let

$$\mathscr{A}_{n,X} := \left\{A \subset E(X,n) : r_{A,X}(\Lambda_{n'}) < R,\ f_{n,X}(A) < \frac{2}{\|\bar{V}''\|}\right\}$$

be the set of good bond sets.

Lemma 7. *For every $Y \in \mathscr{X}_\delta$ and $A \in \mathscr{A}_{n,Y^o}$ we have*

$$\mu(\mathscr{X}_\delta) \geqslant 1 - 3\delta \quad \textit{and} \quad \pi_n(\mathscr{A}_{n,Y^o} \mid Y^o_{\Lambda_n}, Y_{\Lambda_n^c}) \geqslant 1 - 2\delta, \tag{4.13}$$

$$\tau_n^{Y^o,A}(x) = \tau \quad \forall x \in Y^o_{\Lambda_{n'}} \quad \textit{and} \quad \tau_n^{Y^o,A}(x) = 0 \quad \forall x \in Y^o_{\Lambda_n^c}, \tag{4.14}$$

$$\frac{\mathrm{e}}{2}\mathrm{e}^{-H^{\bar{U}}_{\Lambda_n}((\tau_n^{Y^o,A})^{-1}Y)} + \frac{\mathrm{e}}{2}\mathrm{e}^{-H^{\bar{U}}_{\Lambda_n}(\tau_n^{Y^o,A}Y)} \geqslant \mathrm{e}^{-H^{\bar{U}}_{\Lambda_n}(Y)}. \tag{4.15}$$

All these facts together imply

Lemma 8. *For the integer n and the set $\mathscr{X}_\delta$ we have*

$$\frac{\mathrm{e}}{2}\gamma_{\Lambda_n}(\tau^{-1}B \cap \mathscr{X}_\delta | \bar{Y}) + \frac{\mathrm{e}}{2}\gamma_{\Lambda_n}(\tau B \cap \mathscr{X}_\delta | \bar{Y}) \geqslant \gamma_{\Lambda_n}(B \cap \mathscr{X}_\delta | \bar{Y}) - 2\delta. \tag{4.16}$$

Now integrating (4.16)—using property (3.2) of μ and (4.13)—yields

$$\frac{\mathrm{e}}{2}\mu(\tau^{-1}B) + \frac{\mathrm{e}}{2}\mu(\tau B) \geqslant \mu(B) - 5\delta$$

for arbitrary $\mu \in \mathscr{G}(U)$, $\tau \in G$, $n' \in \mathbb{N}$, $B \in \mathscr{F}_{\mathscr{Y},\Lambda_{n'}}$ and $\delta > 0$. Letting $\delta \to 0$ the assertion of the theorem follows by using results from the general theory of Gibbs measures (see [5], Chapter 9.1, Proposition 9.1) for example.

5. Proofs of the lemmas

5.1. Property of the correlation function: Lemma 1

Let U be a superstable and lower regular potential, $\mu \in \mathscr{G}(U)$ a tempered Gibbs measure, $\xi \in \mathbb{R}$ a correlation bound, $m \geqslant 0$ an integer and $f : (\mathbb{R}^2)^m \to \mathbb{R}_+$ a measurable function. The Poisson point process v satisfies for every $N \in \mathbb{N}$ and every measurable $g : \mathscr{Y}_{\Lambda_N} \to \mathbb{R}_+$

$$\int \nu(\mathrm{d}Y) \sum_{x_1,\dots,x_m \in Y^o_{\Lambda_N}}^{\neq} f(x_1,\dots,x_m) g(Y)$$
$$= z^m \int_{\Lambda_N^m} \mathrm{d}x_1 \cdots \mathrm{d}x_m \int_{E^m} \mathrm{d}\sigma_1 \cdots \mathrm{d}\sigma_m f(x_1,\dots,x_m) \int \nu(\mathrm{d}Y') g((X,\sigma)_m Y'),$$

where $(X,\sigma)_m := \{(x_i,\sigma_i) : 1 \leqslant i \leqslant m\}$. Using this equality, the characterisation of Gibbs measures (3.2), the definition of the conditional Gibbs distribution and the definition of the correlation function we get

$$\int \mu(\mathrm{d}Y) \sum_{x_1,\dots,x_m \in Y^o_{\Lambda_N}}^{\neq} f(x_1,\dots,x_m)$$
$$= \int \mu(\mathrm{d}\bar{Y}) \frac{1}{Z^U_{\Lambda_N}(\bar{Y})} \int \nu(\mathrm{d}Y) \sum_{x_1,\dots,x_m \in Y^o_{\Lambda_N}}^{\neq} f(x_1,\dots,x_m) \mathrm{e}^{-H^U_{\Lambda_N}(Y_{\Lambda_N} \bar{Y}_{\Lambda^c_N})}$$
$$= \int_{\Lambda_N^m} \mathrm{d}x_1 \cdots \mathrm{d}x_m \int \mathrm{d}\sigma_1 \cdots \mathrm{d}\sigma_m f(x_1,\dots,x_m) z^m \rho^{U,\mu}((X,\sigma)_m)$$
$$\leqslant (z\xi)^m \int_{\Lambda_N^m} \mathrm{d}x_1 \cdots \mathrm{d}x_m f(x_1,\dots,x_m),$$

where we have used bound (3.3) on the correlation function in the last step. Letting $N \to \infty$ the assertion (3.4) follows from the monotone limit theorem.

5.2. *Convergence of energy sums: Lemma 2*

Let $n \in \mathbb{N}$. For every $X \in \mathscr{X}$ we have

$$\sum_{x_1 x_2 \subset E(X,n)} J(x_1 - x_2) \leqslant \sum_{x_1,\, x_2 \in X}^{\neq} 1_{\{x_1 \in \Lambda_n\}} J(x_1 - x_2),$$

so

$$\int \mu(\mathrm{d}Y) \sum_{x_1 x_2 \subset E(Y^o,n)} J(x_1 - x_2) \leqslant (z\xi)^2 \int \mathrm{d}x_1 \mathrm{d}x_2 1_{\{x_1 \in \Lambda_n\}} J(x_1 - x_2)$$

by Lemma 1, and the right-hand side of the last inequality is at most $c_J(2nz\xi)^2 < \infty$. So the assertion is true for

$$\mathscr{X}_n := \left\{ X \in \mathscr{X}\colon \sum_{x_1 x_2 \subset E(X,n)} J(x_1 - x_2) < \infty \right\}.$$

5.3. *Stochastic domination: Lemma 3*

A general sufficient condition for stochastic domination in a situation like the one considered is given by Holley (see e.g. [8]). The result is the following:

Lemma 9. *Let $Z=\{e_1,e_2,\cdots\}$ be a countable set, $(\varepsilon_e)_{e\in Z}$ a family of reals in* $]0,1]$, $\mathscr{B}_Z$ *the Bernoulli-σ-algebra on $\mathscr{P}(Z)$, and let $\mathscr{A}$ and $\mathscr{A}_\varepsilon$ be random variables with values in $(\mathscr{P}(Z),\mathscr{B}_Z)$ such that $\mathscr{A}_\varepsilon$ is a Bernoulli process with bond probabilities ε_e, and for every $e\in Z$ we have $P(e\in\mathscr{A}|\mathscr{A}\backslash e)\leqslant\varepsilon_e$ a.s. . Then $\mathscr{L}(\mathscr{A})\preceq\mathscr{L}(\mathscr{A}_\varepsilon)$.*

Proof. Let all assumptions of the lemma hold. First we consider the finite sets $Z^{(n)}:=\{e_1,\dots,e_n\}$ and let $\mathscr{A}^{(n)},\mathscr{A}_\varepsilon^{(n)}$ be the restrictions of $\mathscr{A},\mathscr{A}_\varepsilon$ to $Z^{(n)}$, i.e. $\mathscr{A}^{(n)}=\mathscr{A}\cap Z^{(n)}$ and $\mathscr{A}_\varepsilon^{(n)}=\mathscr{A}_\varepsilon\cap Z^{(n)}$. For any $n\in\mathbb{N}$ and $e\in Z^{(n)}$ we have $P(e\in\mathscr{A}^{(n)}|\mathscr{A}^{(n)}\backslash e)\leqslant\varepsilon_e$ a.s., which is a straightforward consequence of $P(e\in\mathscr{A}|\mathscr{A}\backslash e)\leqslant\varepsilon_e$ a.s. and the properties of conditional probabilities. Now the criterion of Holley (as presented in [7, Theorem 4.8], for example) gives $\mathscr{L}(\mathscr{A}^{(n)})\preceq\mathscr{L}(\mathscr{A}_\varepsilon^{(n)})$. If $\mathscr{L}(\mathscr{A}^{(n)})$ and $\mathscr{L}(\mathscr{A}_\varepsilon^{(n)})$ are considered as measures on $(\mathscr{P}(Z),\mathscr{B}_Z)$ we observe that

$$\mathscr{L}(\mathscr{A}^{(n)})\to\mathscr{L}(\mathscr{A})\quad\text{and}\quad\mathscr{L}(\mathscr{A}_\varepsilon^{(n)})\to\mathscr{L}(\mathscr{A}_\varepsilon)\quad\text{weakly as } n\to\infty.$$

As stochastic domination is preserved under weak limits (see [7, Corollary 4.7], for example) we get $\mathscr{L}(\mathscr{A})\preceq\mathscr{L}(\mathscr{A}_\varepsilon)$. □

Now, turning to the proof of Lemma 3 let $n\in\mathbb{N}$, $Y\in\mathscr{Y}$ and $E_n:=E(Y^o,n)$. In order to show that $\pi_n(.|Y^o_{\Lambda_n},Y_{\Lambda_n^c})\preceq\pi(.|Y^o)$ we may consider both measures as measures on $(\mathscr{E}_{E_n},\mathscr{B}_{E_n})$. We also may assume that $Y^o\in\mathscr{X}_n$ and $\mathscr{W}_n(Y_{\Lambda_n},Y^o_{\Lambda_n^c})>0$. By Lemma 9 it is sufficient to show that, for every bond $x_1x_2\in E_n$ and every finite bond set $D\subset E_n\backslash\{x_1x_2\}$,

$$\pi_n(\{x_1x_2\}\cup D\,|\,Y^o_{\Lambda_n},Y_{\Lambda_n^c})\leqslant\varepsilon_{x_1x_2}\pi_n(\{D,\{x_1,x_2\}\cup D\}\,|\,Y^o_{\Lambda_n},Y_{\Lambda_n^c}).$$

(Here we have used that the whole weight of $\pi_n(.|Y^o_{\Lambda_n},Y_{\Lambda_n^c})$ is on the countable set of finite bond sets.) So let $x_1x_2\in E_n$ and $D\subset E_n\backslash\{x_1x_2\}$ be finite. By the definition of π_n the last inequality is equivalent to

$$\int \mathrm{d}\sigma_{Y^o_{\Lambda_n}}\mathscr{V}_n(D,(Y^o_{\Lambda_n},\sigma)Y_{\Lambda_n^c})[\varepsilon_{x_1x_2} + (\varepsilon_{x_1x_2}-1)(\mathrm{e}^{J(x_1-x_2)v(\sigma_{x_1}((Y^o_{\Lambda_n},\sigma)\bar{Y}_{\Lambda_n^c})-\sigma_{x_2}((Y^o_{\Lambda_n},\sigma)\bar{Y}_{\Lambda_n^c}))}-1)]\geqslant 0.$$

But since $0<\varepsilon_{x_1x_2}\leqslant 1$ and $0<v<\varepsilon$, the term in the brackets is at least

$$\varepsilon_{x_1x_2}+(\varepsilon_{x_1x_2}-1)(\mathrm{e}^{\varepsilon_{x_1x_2}}-1)\geqslant 0,$$

which completes the proof of the Lemma 3.

5.4. Cluster bounds: Lemma 4

Let $n\geqslant n'$ be a fixed integer. For a given configuration $X\in\mathscr{X}$ and a bond set $A\in E(X,n)$ we consider the cardinality of the cluster of points from $\Lambda:=\Lambda_{n'}$, which is defined by

$$C_\Lambda(A):=\left|\bigcup_{x\in X_\Lambda} C_{A,X}(x)\right|.$$

For all $X \in \mathcal{X}$ we have the estimate

$$\begin{aligned}\int \pi(\mathrm{d}A|X)C_\Lambda(A) &\leqslant \int \pi(\mathrm{d}A|X) \sum_{x\in X_\Lambda} \sum_{x'\in X} 1_{\{x \overset{A}{\longleftrightarrow} x'\}} \\ &= \sum_{x\in X_\Lambda} \sum_{x'\in X} \pi(x \overset{\cdot}{\longleftrightarrow} x'|X) \\ &\leqslant \sum_{m\geqslant 0} \varepsilon^m \sum^{\neq}_{x_0,\dots,x_m\in X} 1_{x_0\in\Lambda} \prod\nolimits_{i=1}^m J(x_i - x_{i-1}) =: f(X),\end{aligned}$$

where we have used (4.6). By Lemma 1 we have

$$\begin{aligned}\int \mu(\mathrm{d}Y) f(Y^o) &\leqslant \sum_{m\geqslant 0} \varepsilon^m (z\xi)^{m+1} \int \mathrm{d}x_0 \cdots \mathrm{d}x_m 1_{x_0\in\Lambda} \prod_{i=1}^m J(x_i - x_{i-1}) \\ &\leqslant z\xi(2n')^2 \sum_{m\geqslant 0} (z\xi\varepsilon c_J)^m =: c < \infty\end{aligned}$$

due to (4.2). Letting

$$\mathcal{X}'_R := \left\{ X \in \mathcal{X} \colon f(X) \leqslant \frac{c}{\delta} \right\},$$

we get $\mu(\mathcal{X}'_R) \geqslant 1-\delta$ from Chebyshev's inequality, and for any $X \in \mathcal{X}'_R$ we have again by Chebyshev's inequality that

$$\pi\left(C_\Lambda > \frac{2c}{\delta^3} \middle| X \right) \leqslant \frac{\delta^3}{2c} \int \pi(\mathrm{d}A|X) C_\Lambda(A) \leqslant \frac{\delta^2}{2}.$$

Now let $n \geqslant n'$, $R > n'$ and $X \in \mathcal{X}'_R$. Then, by the above estimate,

$$\begin{aligned}&\pi(r_{\cdot,X}(\Lambda) \geqslant R \mid X) \\ &\quad \leqslant \pi\left(C_\Lambda > \frac{2c}{\delta^3} \middle| X \right) + \pi\left(C_\Lambda \leqslant \frac{2c}{\delta^3},\ r_{\cdot,X}(\Lambda) \geqslant R \middle| X \right) \\ &\quad \leqslant \frac{\delta^2}{2} + \pi\Bigg(\Bigg\{ A : \exists 1 \leqslant m \leqslant \frac{2c}{\delta^3}\ \exists \text{distinct } x_0, \dots, x_m \in X : \\ &\qquad \exists 1 \leqslant j \leqslant m : x_0 \in \Lambda, \|x_j - x_{j-1}\| \geqslant \frac{(R-n')\delta^3}{2c}, x_{i-1}x_i \in A\ \forall i \Bigg\} \Bigg| X \Bigg) \\ &\quad \leqslant \frac{\delta^2}{2} + \sum_{m\geqslant 1} \sum_{j=1}^m \sum^{\neq}_{x_0,\dots,x_m\in X} 1_{\left\{ x_0\in\Lambda, \|x_j - x_{j-1}\| \geqslant \frac{(R-n')\delta^3}{2c} \right\}} \varepsilon^m \prod_{i=1}^m J(x_i - x_{i-1}) \\ &\quad =: \frac{\delta^2}{2} + f_R(X)\end{aligned}$$

and Lemma 1 yields

$$\int \mu(\mathrm{d}Y) f_R(Y^o) \leqslant \sum_{m\geqslant 1} \varepsilon^m \sum_{j=1}^{m} (z\xi)^{m+1} \int \mathrm{d}x_0 \cdots \mathrm{d}x_m \times \left[1_{\left\{x_0\in\Lambda, \|x_j - x_{j-1}\| \geqslant \frac{(R-n')\delta^3}{2c}\right\}} \prod_{i=1}^{m} J(x_i - x_{i-1}) \right] \leqslant z\xi \sum_{m\geqslant 1} (z\xi\varepsilon)^m m (2n')^2 c_J^{m-1} c\left(\frac{(R-n')\delta^3}{2c}\right).$$

In the last step, the integrals have been estimated backwards from x_m to x_0, where integration over x_j gives the constant $c((R-n')\delta^3/2c)$ defined in (4.1). As $\lim_{R\to\infty} c([(R-n')\delta^3]/2c) = 0$ and the sum over m is finite by condition (4.2), we can fix an $R > n'$ such that

$$\int \mu(\mathrm{d}Y)\left(\frac{\delta^2}{2} + f_R(Y^o)\right) \leqslant \delta^2. \tag{5.1}$$

Now let

$$\mathscr{X}''_R := \left\{ X \in \mathscr{X} \colon \frac{\delta^2}{2} + f_R(X) \leqslant \delta \right\}$$

and $\mathscr{X}_R := \mathscr{X}''_R \cap \mathscr{X}'_R$, then by Chebyshev's inequality and (5.1) we have $\mu(\mathscr{X}''_R) \geqslant 1-\delta$, and hence $\mu(\mathscr{X}_R) \geqslant 1 - 2\delta$. For every $Y \in \mathscr{X}_R$ the event $\{A : r_{A,Y^o}(\Lambda) \geqslant R\}$ is increasing, so by stochastic domination (4.5) we have

$$\pi_n(\{A : r_{A,Y^o}(\Lambda) \geqslant R\} \mid Y^o_{\Lambda_n}, Y_{\Lambda_n^c}) \leqslant \pi(\{A : r_{A,Y^o}(\Lambda) \geqslant R\} \mid Y^o) \leqslant \frac{\delta^2}{2} + f_R(Y^o) \leqslant \delta.$$

5.5. Properties of τ_n and Q: Lemma 5

(4.9) is evident from the definition of τ_n, and $\lim_{n\to\infty} Q(n) = \infty$ is a consequence of $\log\log n \leqslant Q(n)$ for $n \geqslant 2$. For (4.8) let $x, x' \in \mathbb{R}^2$ such that $\|x'\| \geqslant \|x\|$. The left inequality is trivial and for the right inequality we may assume that $\|x'\| > R$ and $\|x\| < n$ because of (4.9). Hence

$$r(\|x\| - R, n-R) - r(\|x'\| - R, n-R) = \int_{\max\{R,\|x\|\}}^{\min\{\|x'\|,n\}} \frac{q(s'-R)}{Q(n-R)}\,\mathrm{d}s' \leqslant (\|x'\| - \|x\|)\frac{q(\|x\|-R)}{Q(n-R)} \leqslant \|x' - x\| \,\frac{q(\|x\|-R)}{Q(n-R)},$$

where we have used the monotonicity of q and the triangle inequality. Now (4.8) follows immediately.

5.6. Probability of bad bond sets: Lemma 6

First of all we state two easy facts. First,

$$\|x_m - x_0\|^2 \leqslant m \prod_{i=1}^{m} (\|x_i - x_{i-1}\|^2 + 1) \quad \forall m \geqslant 1, x_0, \ldots, x_m \in \mathbb{R}^2, \tag{5.2}$$

by the triangle inequality and the arithmetic–quadratic mean inequality. Secondly,

$$\int_{\Lambda_n} \mathrm{d}x q(\|x\| - R)^2 \leqslant 8(R+3)^2 + 8RQ(n-R) \quad \forall n \geqslant R, \tag{5.3}$$

which is obtained by the substitution $t := \|x\|$:

$$\int_{\Lambda_n} \mathrm{d}x q(\|x\| - R)^2 \leqslant \int_0^{R+3} \mathrm{d}t\, 8t + \int_3^{n-R} \mathrm{d}t\, 8(t+R)q(t)^2$$
$$\leqslant 8(R+3)^2 + 8R \int_0^{n-R} q(t)\,\mathrm{d}t = 8(R+3)^2 + 8RQ(n-R),$$

where we have used in the first step that $q(t) \leqslant 1\ \forall t \in \mathbb{R}$, and in the second step that $t + R \leqslant tR$ for $t, R \geqslant 2$, and $tq(t) \leqslant 1\ \forall t \geqslant 3$.

Now for the proof of Lemma 6 let $n > R$ and $Y \in \mathscr{Y}$ be arbitrary. Using the arithmetic–quadratic mean inequality to estimate $(\tau_n^{X,A}(x) - \tau_n^{X,A}(x'))^2$ we get

$$f_{n,Y^o}(A) \leqslant 6 \sum_{x,\, x' \in Y^o} 1_{\{x \neq x'\}} J(x - x')(\tau_n(t_A(x)) - \tau_n(x))^2$$
$$+ 3 \sum_{x,\, x' \in Y^o} 1_{\{\|x\| \leqslant \|x'\|\}} J(x - x')(\tau_n(x) - \tau_n(x'))^2.$$

Substituting $z := t_A(x)$ and introducing $1_{\{z = t_A(x)\}}$ in the first sum we need only consider $z \in Y^o$ such that $\|x\| \leqslant \|z\|$ and $x \neq z$. By distinguishing the cases $z \neq x, x'$ and $z = x'$ and by using $\{A : t_A(x) = z\} \subset \{A\colon\ x \overset{A}{\longleftrightarrow} z\}$ we can estimate the expectation value of f_{n,Y^o} by

$$\int \pi_n(\mathrm{d}A |\, Y^o_{\Lambda_n}, Y_{\Lambda_n^c}) f_{n,Y^o}(A)$$
$$\leqslant 6 \sideset{}{^{\neq}}\sum_{x,\, x',\, z \in Y^o} 1_{\{\|x\| \leqslant \|z\|\}} J(x - x')(\tau_n(z) - \tau_n(x))^2 \pi_n(x \overset{\cdot}{\longleftrightarrow} z |\, Y^o_{\Lambda_n}, Y_{\Lambda_n^c})$$
$$+ 9 \sideset{}{^{\neq}}\sum_{x, z \in Y^o} 1_{\{\|x\| \leqslant \|z\|\}} J(x - z)(\tau_n(x) - \tau_n(z))^2.$$

Next we use the stochastic domination (4.5) for the increasing events $x \overset{\cdot}{\longleftrightarrow} z$ to estimate $\pi_n(x \longleftrightarrow z |\, Y^o_{\Lambda_n}, Y_{\Lambda_n^c})$, and we use (4.8) from Lemma 5, noting that

$\tau_n(x) = 0 = \tau_n(z)$ for $n < \|x\| \leqslant \|z\|$. So we get

$$\int \pi_n(\mathrm{d}A | Y^o_{\Lambda_n}, Y_{\Lambda_n^c}) f_{n,Y^o}(A)$$
$$\leqslant 6 \sum_{x,x',z \in Y^o}^{\neq} 1_{\{x \in \Lambda_n\}} J(x-x')\tau^2 \|x-z\|^2 \frac{q(\|x\|-R)^2}{Q(n-R)^2} \pi(x \overset{\cdot}{\longleftrightarrow} z | Y^o)$$
$$+ 9 \sum_{x,z \in Y^o}^{\neq} 1_{\{x \in \Lambda_n\}} J(x-z)\tau^2 \|x-z\|^2 \frac{q(\|x\|-R)^2}{Q(n-R)^2}$$
$$=: \Sigma_1(Y^o, n) + \Sigma_2(Y^o, n).$$

In order to deal with $\Sigma_1(Y^o, n)$ we distinguish the paths $x_0, \ldots, x_m$ from x to z analogously to (4.6) and distinguish the cases $x_j = x'$ and $x_j \neq x'$ $\forall j$. Hence

$$\Sigma_1(Y^o, n) \leqslant 6 \sum_{m \geqslant 1} \varepsilon^m \sum_{x', x_0, \ldots, x_m \in Y^o}^{\neq} 1_{\{x_0 \in \Lambda_n\}} J(x_0 - x')$$
$$\tau^2 \|x_0 - x_m\|^2 \frac{q(\|x_0\|-R)^2}{Q(n-R)^2} \prod_{i=1}^m J(x_i - x_{i-1})$$
$$+ 6 \sum_{m \geqslant 1} \varepsilon^m \sum_{j=1}^{m-1} \sum_{x_0, \ldots, x_m \in Y^o}^{\neq} 1_{\{x_0 \in \Lambda_n\}} J(x_0 - x_j)$$
$$\tau^2 \|x_0 - x_m\|^2 \frac{q(\|x_0\|-R)^2}{Q(n-R)^2} \prod_{i=1}^m J(x_i - x_{i-1}).$$

Applying Lemma 1 we thus find

$$\int \mu(\mathrm{d}Y) \Sigma_1(Y^o, n)$$
$$\leqslant 6 \sum_{m \geqslant 1} \varepsilon^m (z\xi)^{m+1} \int \mathrm{d}x_0 \cdots \mathrm{d}x_m \left[1_{\{x_0 \in \Lambda_n\}} \tau^2 \|x_0 - x_m\|^2 \frac{q(\|x_0\|-R)^2}{Q(n-R)^2} \right.$$
$$\left. \prod_{i=1}^m J(x_i - x_{i-1}) \left(z\xi \int \mathrm{d}x' J(x_0 - x') + \sum_{j=1}^{m-1} J(x_0 - x_j) \right) \right].$$

After applying (5.2) to $\|x_0 - x_m\|^2$ and estimating the parentheses (.) by $z\xi c_J m$ we evaluate the integrals backwards from x_m to x_1:

$$\int \mu(\mathrm{d}Y) \Sigma_1(Y^o, n)$$
$$\leqslant 6 \sum_{m \geqslant 1} m^2 \varepsilon^m c_J (z\xi)^{m+2} (2c_J)^m \tau^2 \int \mathrm{d}x 1_{\{x \in \Lambda_n\}} \frac{q(\|x\|-R)^2}{Q(n-R)^2}$$
$$\leqslant 6 c_J (\xi z \tau)^2 \left[\sum_{m \geqslant 1} m^2 (2\varepsilon c_J \xi z)^m \right] \frac{8(R+3)^2 + 8RQ(n-R)}{Q(n-R)^2},$$

where we have used (5.3) in the last step. The expectation value of Σ_2 can be treated similarly, and the estimates together give

$$\int \mu(\mathrm{d}Y)[\Sigma_1(Y^o,n)+\Sigma_2(Y^o,n)]$$
$$\leqslant \left[6\sum_{m\geqslant 1} m^2(2\varepsilon c_J\xi z)^m+9\right] c_J(\tau z\xi)^2\,\frac{8(R+3)^2+8RQ(n-R)}{Q(n-R)^2}.$$

The sum over m is finite by the choice of ε (4.2), and because of $\lim_{k\to\infty} Q(k)=\infty$ the fraction on the right-hand side can be made arbitrarily small choosing n large enough. So there is an integer $n>R$ such that

$$\int \mu(\mathrm{d}Y)[\Sigma_1(Y^o,n)+\Sigma_2(Y^o,n)]\leqslant \frac{2\delta^2}{\|V''\|}. \tag{5.4}$$

Let

$$\mathscr{X}_{R,n} := \left\{X\in\mathscr{X}:\ \Sigma_1(X,n)+\Sigma_2(X,n)\leqslant \frac{2\delta}{\|V''\|}\right\},$$

then we have found n and $\mathscr{X}_{R,n}$ as desired, as by (5.4) and Chebyshev's inequality we have $\mu(\mathscr{X}_{R,n})\geqslant 1-\delta$, and for every $Y\in\mathscr{X}_{R,n}$ we have by the definition of $\mathscr{X}_{R,n}$, Σ_1 and Σ_2 and again by Chebyshev's inequality

$$\pi_n\left(f_{n,Y^o}\geqslant \frac{2}{\|\bar V''\|}\,\middle|\, Y^o_{\Lambda_n}, Y_{\Lambda_n^c}\right)\leqslant\delta.$$

5.7. *Properties of good configurations and bond sets: Lemma 7*

Let $Y\in\mathscr{X}_\delta$, $A\in\mathscr{A}_{n,Y^o}$ and $E_n:=E(Y^o,n)$. Inequalities (4.13) then follow immediately from Lemmas 4 and 6. (4.14) follows from (4.9) because $r_{A,Y^o}(\Lambda_{n'})<R$. For (4.15) we consider $\bar V$ as a 2π-periodic function on $\mathbb{R}$. By the smoothness of $\bar V$ we can use a Taylor expansion to obtain for all $a,b\in\mathbb{R}$

$$\bar V(a+b)+\bar V(a-b)-2\bar V(a)\leqslant\|\bar V''\|b^2,$$

where $\|\bar V''\|<\infty$, as $\bar V''$ is continuous on a compact space. W.l.o.g. we may assume the right-hand side of (4.15) to be positive, hence $|H^{\bar U}_{\Lambda_n}(Y)|<\infty$. So we have, introducing $\eta_{x_1,x_2}:=\sigma_{x_1}(Y)-\sigma_{x_2}(Y)$ and $\vartheta_{x_1,x_2}:=\tau^{X,A}_n(x_1)-\tau^{X,A}_n(x_2)$,

$$\begin{aligned}
&H^{\bar U}_{\Lambda_n}((\tau^{X,A}_n)^{-1}Y)+H^{\bar U}_{\Lambda_n}(\tau^{X,A}_n Y)-2H^{\bar U}_{\Lambda_n}(Y)\\
&\quad=\sum_{x_1x_2\in E_n} J(x_1-x_2)\big[\bar V(\eta_{x_1,x_2}-\vartheta_{x_1,x_2})+\bar V(\eta_{x_1,x_2}+\vartheta_{x_1,x_2})-2\bar V(\eta_{x_1,x_2})\big]\\
&\quad\leqslant\sum_{x_1x_2\in E_n} J(x_1-x_2)\|\bar V''\|\vartheta^2_{x_1,x_2}=\|\bar V''\|f_{n,Y^o}(A).
\end{aligned}$$

By the convexity of the exponential function we conclude

$$\frac{e}{2}e^{-H^{\bar U}_{\Lambda_n}((\tau_n^{X,A})^{-1}Y)}+\frac{e}{2}e^{-H^{\bar U}_{\Lambda_n}(\tau_n^{X,A}Y)}$$
$$\geqslant e^{1-(1/2)H^{\bar U}_{\Lambda_n}((\tau_n^{X,A})^{-1}Y)-(1/2)H^{\bar U}_{\Lambda_n}(\tau_n^{X,A}Y)}$$
$$\geqslant e^{1-\frac{\|\bar V''\|}{2}f_{n,Y^o}(A)}\cdot e^{-H^{\bar U}_{\Lambda_n}(Y)}\geqslant e^{-H^{\bar U}_{\Lambda_n}(Y)}.$$

5.8. Inequality for the specifications: Lemma 8

By (4.4) it is sufficient to prove that for every $Y\in\mathscr{X}_\delta$ we have

$$\int\pi_n(\mathrm{d}A|\,Y^o_{\Lambda_n},Y_{\Lambda_n^c})\Big[\frac{e}{2}\alpha_n(\tau^{-1}B|A,Y^o_{\Lambda_n},Y_{\Lambda_n^c})$$
$$+\frac{e}{2}\alpha_n(\tau B|A,Y^o_{\Lambda_n},Y_{\Lambda_n^c})-\alpha_n(B|A,Y^o_{\Lambda_n},Y_{\Lambda_n^c})\Big]+2\delta\geqslant 0,$$

and because of (4.13) it suffices to show that, for every $Y\in\mathscr{X}_\delta$ and every finite $A\in\mathscr{A}_{n,Y^o}$ such that $\mathscr{W}_n(A,Y^o_{\Lambda_n},Y_{\Lambda_n^c})>0$, the term in square brackets is nonnegative. By definition of α_n, this will follow once we have shown that for every $X\in\mathscr{X}_{\Lambda_n}$, $Y\in\mathscr{Y}_{\Lambda_n^c}$ and every finite $A\in\mathscr{A}_{n,XY^o}$ we have

$$\int \mathrm{d}\sigma'_X e^{-H^{\bar U}_{\Lambda_n}(Y\sigma')}\Big(\frac{e}{2}1_{\tau^{-1}B}(Y\sigma')+\frac{e}{2}1_{\tau B}(Y\sigma')-1_B(Y\sigma')\Big)$$
$$\cdot\prod_{x_1x_2\in A}(e^{J(x_1-x_2)v(\sigma_{x_1}(Y\sigma')-\sigma_{x_2}(Y\sigma'))}-1)\geqslant 0, \tag{5.5}$$

where we have used the notation $Y\sigma:=(X,\sigma)Y$. So let X, Y and A as above. The integral on the left-hand side of (5.5) can be split into the three parts I_-,I_+ and I_0 corresponding to the terms $\tau^{-1}B$, τB and B, and for any $x\in X$ we make the substitutions $\sigma_x:=\sigma'_x+\tau_n^{X,A}(x)$ and $\sigma_x:=\sigma'_x-\tau_n^{X,A}(x)$ in I_- and I_+, respectively. Because of (4.14) the spin transformation $\tau_n^{X,A}$ has no effect outside of Λ_n, so that $Y\sigma'=(\tau_n^{X,A})^{-1}(Y\sigma)$ and $Y\sigma'=\tau_n^{X,A}(Y\sigma)$, respectively. Because of (4.14) we have $\tau^{-1}B=(\tau_n^A)^{-1}B$ and $\tau B=\tau_n^AB$, so that after the substitution the indicator functions simplify to $1_B(Y\sigma)$. Because of (4.10), $\tau_n^{X,A}$ is constant on particles joined by bonds in A, so in I_- we have

$$\begin{aligned}\sigma_{x_1}(Y\sigma)-\sigma_{x_2}(Y\sigma)&=\sigma_{x_1}(\tau_n^{X,A}(Y\sigma'))-\sigma_{x_2}(\tau_n^{X,A}(Y\sigma'))\\&=\sigma_{x_1}(Y\sigma')+\tau_n^{X,A}(x_1)-\sigma_{x_2}(Y\sigma')-\tau_n^{X,A}(x_2)\\&=\sigma_{x_1}(Y\sigma')-\sigma_{x_2}(Y\sigma'),\end{aligned}$$

for every $\sigma\in(S^1)^X$ and for every bond $x_1x_2\in A$, and the same holds for I_+. Therefore the left-hand side of (5.5) is equal to

$$\int \mathrm{d}\sigma_X\Bigg[1_B(Y\sigma)\prod_{x_1x_2\in A}(e^{J(x_1-x_2)v(\sigma_{x_1}(Y\sigma)-\sigma_{x_2}(Y\sigma))}-1)$$
$$\cdot\Big(\frac{e}{2}e^{-H^{\bar U}_{\Lambda_n}((\tau_n^{X,A})^{-1}(Y\sigma))}+\frac{e}{2}e^{-H^{\bar U}_{\Lambda_n}(\tau_n^{X,A}(Y\sigma))}-e^{-H^{\bar U}_{\Lambda_n}(Y\sigma)}\Big)\Bigg]$$

which is nonnegative by (4.15) from Lemma 7. This proves (5.5) and completes the proof of the Lemma 8.

6. Proof of Theorem 1

Let the assumptions of Theorem 1 hold. First we observe that for every $\tau \in G$ there is a torus T such that $\tau \in T$ and T is a subgroup of G. Every torus is a finite product of compact one-dimensional subgroups of G, so w.l.o.g. we may assume that τ is contained in such a subgroup, i.e. we may assume that G is a compact one-dimensional Lie-group, and hence that $G = S^1$ (for details see [3] for example).

For general S we have to modify the decomposition of V. What we need is a decomposition $V = \bar{V} - v$ as guaranteed by Lemma 10 presented below.

In order to deal with general J we have to construct two different decompositions of V: For $(x_1, \sigma_1), (x_2, \sigma_2) \in \mathbb{R}^2 \times S$ such that $J(x_1 - x_2) \geqslant 0$ we decompose as before: $V(\sigma_1, \sigma_2) = \bar{V}_+(\sigma_1, \sigma_2) - v_+(\sigma_1, \sigma_2)$, but if $J(x_1 - x_2) < 0$ we decompose $V(\sigma_1, \sigma_2) = \bar{V}_-(\sigma_1, \sigma_2) + v_-(\sigma_1, \sigma_2)$, where v_- and $\bar{V}_-$ have the same properties as v_+ and $\bar{V}_+$, respectively. This decomposition is also obtained analogously to the following lemma.

The rest of the proof simply carries over. $\square$

We still need

Lemma 10. *Let E be a compact topological space and let S^1 operate on E continuously. Let $V : E^2 \to \mathbb{R}$ be a continuous mapping. Then we have a decomposition $V = \bar{V} - v$ such that $0 < v < \varepsilon$, $\bar{V}$ is symmetric and S^1-invariant and such that $\bar{V}(a, \tau b)$ is twice continuously differentiable with respect to τ such that $\partial_\tau^2 \bar{V}(a, \tau b)$ is bounded uniformly in a and b.*

Proof. Here we consider $S^1 = \mathbb{R}/\mathbb{Z}$ and we identify functions on S^1 with periodic functions on $\mathbb{R}$. As a function of all three arguments $V(a, \tau b)$ is continuous on the compact space $E^2 \times S^1$, and therefore uniformly continuous. Hence there exists a $\delta > 0$ such that

$$\forall a, b \in E\ \forall \tau', \tau \in \mathbb{R}:\ |\tau' - \tau| < 2\delta \Rightarrow |V(a, \tau' b) - V(a, \tau b)| < \frac{\varepsilon}{2}. \tag{6.1}$$

For this δ we choose a twice continuously differentiable symmetric probability density $f_\delta : \mathbb{R} \to \mathbb{R}_+$ with support in $[-\delta, \delta]$, for example

$$f_\delta(t) := c \cdot 1_{]-\delta,\delta[}(t) \cdot \mathrm{e}^{-\delta^2/(\delta^2 - t^2)} \quad \text{with } c := \int_{-\delta}^{\delta} \mathrm{e}^{-\delta^2/(\delta^2 - t^2)}\, \mathrm{d}t.$$

Setting

$$\bar{V}(a, b) := \int \mathrm{d}t\, f_\delta(t) V(a, tb) + \frac{\varepsilon}{2} \quad \text{and} \quad v := \bar{V} - V$$

gives us the desired decomposition. $\bar{V}$ is measurable by Fubini's theorem, and symmetric, because V is symmetric and S^1-invariant and f_δ is symmetric. $\bar{V}$ is

S^1-invariant because V is. $0<v<\varepsilon$ is a straightforward consequence of (6.1) and the small support of f_δ. Finally,

$$\bar{V}(a,\tau b) = \int \mathrm{d}t f_\delta(t-\tau)V(a,tb) + \frac{\varepsilon}{2},$$

which is twice continuously differentiable with respect to τ such that $\partial_\tau^2 \bar{V}(a,\tau b) = \int \mathrm{d}t f_\delta''(t-\tau)V(a,tb)$ is bounded by $2\delta\|f_\delta''\|\,\|V\|$. $\square$

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