Abstract

The main goal of this paper is to prove analytically the existence of strange attractors in a family of vector fields consisting of two Brusselators linearly coupled by diffusion. We will show that such a family contains a generic unfolding of a 4-dimensional nilpotent singularity of codimension 4. On the other hand, we will prove that in any generic unfolding $X_\mu$ of an $n$-dimensional nilpotent singularity of codimension $n$ there are bifurcation curves of $(n-1)$-dimensional nilpotent singularities of codimension $n-1$ which are in turn generically unfolded by $X_\mu$. Arguments conclude recalling that any generic unfolding of the 3-dimensional nilpotent singularity of codimension 3 exhibits strange attractors.

Keywords: Coupled systems; Nilpotent singularities; Strange attractor; Shil'nikov homoclinic orbit

1. Introduction

Consider the equations

$$u'_i = F(u_i) + \sum_{j=1}^{m} a_{ij} D(u_j - u_i), \quad i = 1, \ldots, m,$$

(1.1)

where $u_i \in \mathbb{R}^k$ for each $i = 1, \ldots, m$, $D$ is a $k \times k$ matrix and coefficients $a_{ij}$ can be either 0 or 1 but satisfying that $a_{ii} = 0$ and $a_{ij} = a_{ji}$. Moreover, $F$ is a $C^\infty$ vector field in $\mathbb{R}^k$, maybe...
depending on some fixed parameters. In the literature a set of equations as above is called a network of linearly coupled systems. We are mainly focused on diffusion problems and hence $D$ is considered as a diagonal matrix with positive elements. A factor $\kappa$, representing the coupling strength, is commonly introduced multiplying the matrix $D$. We do not follow this approach.

Coupled systems arise in biology, chemistry, physics, . . . , leading to several research topics: synchronization of oscillators, synchronization of chaotic systems, symmetries and dynamics, network topology and dynamics, dynamics of coupled oscillators, dynamics of coupled chaotic systems. . . . Both [8,10] are good references to learn about the state of the field.

The Turing’s paper [20] about morphogenesis is usually referred in the literature as the point where the interest in the study of coupled systems begins. One of his models corresponds to a “ring” of coupled systems. By a ring we mean system (1.1) with $k$ where the interest in the study of coupled systems begins. One of his models corresponds to $D$, representing the coupling strength, is commonly introduced multiplying the matrix $D$. We do not follow this approach.

Motivated by the Turing’s ideas, Smale wonders in [18] if oscillations can be generated by coupling identical systems which tend to the equilibrium and gives an example with $m = 2$ and $k = 4$ where the answer is positive. Later, in [11] and [1], examples are obtained with $k = 3$ and $k = 2$, respectively. In contrast with [11,18], where systems are constructed ad hoc, in [1] the Brusselator is used. It is a model of chemical reactions whose equations are given by $u' = F(u)$ with $u = (x, y)$ and $F(u) = (A - (B + 1)x + x^2y, Bx - x^2y)$, where $A$ and $B$ are positive parameters. When two Brusselators are coupled by diffusion we get (1.1) with $m = k = 2$,

\[
\begin{align*}
  x'_1 &= A - (B + 1)x_1 + x_1^2y_1 + \lambda_1(x_2 - x_1), \\
  y'_1 &= Bx_1 - x_1^2y_1 + \lambda_2(y_2 - y_1), \\
  x'_2 &= A - (B + 1)x_2 + x_2^2y_2 + \lambda_1(x_1 - x_2), \\
  y'_2 &= Bx_2 - x_2^2y_2 + \lambda_2(y_1 - y_2).
\end{align*}
\]  

The dynamics of the Brusselator as an isolated system is very well known. All the forward orbits starting at $P^+ = \{(x, y) \in \mathbb{R}^2 | x \geq 0, \ y \geq 0\}$ are bounded and remain inside $P^+$. In that way the isolated systems fulfill a very natural requirement, as examples [11,18] also do. A Hopf bifurcation arises for $B = A^2 + 1$. If $B < A^2 + 1$ there is a unique equilibrium point which is a global attractor in $P^+$ whereas if $B > A^2 + 1$ the equilibrium point becomes unstable at the same time that the periodic orbit arising from the bifurcation becomes the new global attractor. According to [1], “After the coupling spontaneous oscillations occur as secondary Hopf bifurcation branches (from nontrivial stationary solutions).” These stationary solutions arise from a pitchfork bifurcation at the unique equilibrium point which stays at the invariant plane $\{x_1 = x_2, \ y_1 = y_2\}$ for all parameter values.

Once we know that oscillations can arise from the coupling of identical systems with equilibrium points playing the role of global attractors, it makes sense to wonder whether some other interesting behaviour, particularly chaotic dynamics, can be generated in the same way. It is a rather involved problem, even more if tackled in such a so simple framework as that given by conditions $m = k = 2$. Moreover, as far as we know, there is no numerical evidence supporting the chance for a positive answer. Nevertheless such evidences exist when the role of global attractor is played by a periodic orbit. System (1.2) is already an example (see [2,19]). In this paper we consider that model and give an analytical proof of the existence of strange attractors.
The ideas that we develop are the following. The first one is that in order to explain the dynamics in a given system one should search for the organizing centers. The second idea is that the role of organizing center is played in many cases by singularities. Certainly, in [2,3] it was already suggested that singularities could explain chaotic dynamics in the context of coupled systems. The first step in this direction is to know if there are singularities of low codimension which generically unfold chaotic dynamics. In papers [12,13] the first proofs were given. Namely, in [13] it was proved that Shil’nikov configurations are present in any generic unfolding of the nilpotent singularity of codimension three in $\mathbb{R}^3$. Shil’nikov configurations are the simplest ones that imply, under generic assumptions, the existence of strange attractors (see [9,16,17]). By a 3-dimensional nilpotent singularity of codimension 3 we mean any one that can be reduced to the following normal form

$$y \frac{\partial}{\partial x} + z \frac{\partial}{\partial y} + (ax^2 + bxy + cxz + dy^2 + O(\|x, y, z\|^3)) \frac{\partial}{\partial z},$$

with $a \neq 0$.

In this paper we will prove that there are parameter values $(\hat{A}, \hat{B}, \hat{\lambda}_1, \hat{\lambda}_2)$ where the system (1.2) has a 4-dimensional nilpotent singularity of codimension 4 which is generically unfolded inside the family. After proving that any generic unfolding of an $n$-dimensional nilpotent singularity of codimension $n$ contains generic unfoldings of $(n-1)$-dimensional nilpotent singularities of codimension $n-1$, we can conclude our main result:

**Theorem 1.1.** Consider system (1.2) consisting of two Brusselators linearly coupled by diffusion. There exists a point $(\hat{A}, \hat{B}, \hat{\lambda}_1, \hat{\lambda}_2)$ in the parameter space such that there are values $(A, B, \lambda_1, \lambda_2)$ arbitrarily close for which, restricted to a normally attracting 3-dimensional invariant manifold, the system has Shil’nikov homoclinic orbits and hence strange attractors.

**Remark 1.2.** A 3-dimensional nilpotent singularity of codimension 3 involves much more dynamical richness than that mentioned in this paper (see [6,7]). It should be noticed that in fact many aspects and parts of its bifurcation diagram remain still unexplained.

**Remark 1.3.** When two systems with periodic orbits are coupled, as in the case of two Brusselators, there is, even before the coupling, an invariant torus. The evolution of this torus could be a mechanism to explain chaotic dynamics. Our approach in this paper has nothing to do with it since we will look for singularities. In fact, we want to emphasize that periodic orbits play no role in this paper.

Similar results to Theorem 1.1 should be obtained in other systems as, for instance, that studied in [4]. Generalizations to canonical models of coupled systems will also be of great interest. That should be compared with [3] where a canonical model is considered for the coupling of two Hopf bifurcations.

In Section 2 we introduce the notion of $n$-dimensional nilpotent singularity of codimension $n$ and prove that in any generic unfolding of such singularity there are two bifurcation curves of $(n - 1)$-dimensional nilpotent singularities of codimension $(n - 1)$, each one of them again generically unfolded inside the complete family. In Section 3 we prove that system (1.2) is a generic unfolding of the nilpotent singularity of codimension 4 in $\mathbb{R}^4$ and hence, as already argued, our main result.
2. $n$-Dimensional nilpotent singularity of codimension $n$

In this section we will first obtain a normal form for singularities in $\mathbb{R}^n$ whose 1-jet is linearly conjugate to

$$\sum_{k=1}^{n-1} x_{k+1} \frac{\partial}{\partial x_k}.$$ \hspace{1cm} (2.1)

Using such a normal form, a generic condition will be stated to characterize what is called $n$-dimensional nilpotent singularity of codimension $n$. Moreover, conditions will be given for an unfolding of one of such singularities to be considered as generic. The techniques that we use to get the normal form are those in [5] generalized to arbitrary dimension. It will be noticed that further reductions are possible but only the expression that we need to get our results is used. Later on, using a reduction to a center manifold, we will prove that in any generic unfolding $X_\mu$, with $\mu \in \mathbb{R}^n$, of an $n$-dimensional nilpotent singularity of codimension $n$, there exist two bifurcation curves of $(n-1)$-dimensional nilpotent singularities of codimension $n-1$ which are generically unfolded by $X_\mu$.

2.1. Reduction to a normal form

Let $X$ be a $C^\infty$ vector field in $\mathbb{R}^n$ with $X(0) = 0$ and 1-jet at 0 linearly conjugated to (2.1). Without loss of generality we assume that the linear part at 0 is already written in canonical form. Consequently, the equations we have to manage are

\[
\begin{cases}
  x'_k = x_{k+1} + f_k(x) & \text{for } k = 1, \ldots, n-1, \\
  x'_n = f_n(x),
\end{cases}
\] \hspace{1cm} (2.2)

with $x = (x_1, \ldots, x_n)$ and $f_k(x) = O(\|x\|^2)$ for $k = 1, \ldots, n$. Taking new coordinates $\hat{x}_k = d^{k-1}x_1/dt^{k-1} = x_k + O(\|x\|^2)$ for $k = 1, \ldots, n$, (2.2) transforms into

\[
\begin{cases}
  x'_k = x_{k+1} & \text{for } k = 1, \ldots, n-1, \\
  x'_n = f(x),
\end{cases}
\]

where $f(x) = O(\|x\|^2)$ and we preserve the notation for coordinates.

Singularities in $\mathbb{R}^n$ whose linear part is conjugated to (2.1) form a set of codimension $n$ in the space of germs of singularities in $\mathbb{R}^n$. We will refer to them as $n$-dimensional nilpotent singularities of codimension $n$ when the following generic condition is satisfied:

$$\frac{\partial^2 f}{\partial x_1^2}(0) \neq 0.$$ \hspace{1cm} (2.3)

Let $X_\mu$ be a $C^\infty$ family of vector fields in $\mathbb{R}^n$, with $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n$, such that 0 is an $n$-dimensional nilpotent singularity of codimension $n$ when $\mu = 0$. As in the case of the singu-
larity, it easily follows that a $\mu$-dependent change of coordinates permits to write the equations as

$$\begin{cases}
x_k' = x_{k+1} & \text{for } k = 1, \ldots, n - 1, \\
x_n' = f(\mu, x),
\end{cases}$$

(2.4)

with $f(0, 0) = (\partial f/\partial x_1)(0, 0) = 0$ and $(\partial^2 f/\partial x_1^2)(0, 0) = c \neq 0$. Note that we can assume that $c = 1$, otherwise one would scale by a factor $c$.

Let us write $f(\mu, x) = g(\mu, x_1) + h(\mu, x)$, with $g(\mu, x_1) = f(\mu, x_1, 0, \ldots, 0)$. Note that $h(\mu, x) = O(\| (\mu, x) \|^2)$ and also $h(\mu, x) = O(\| (x_2, \ldots, x_n) \|)$. Since $(\partial^2 g/\partial x_1^2)(0, 0) = 1$, it follows from the Malgrange Preparation Theorem that there exist $C^\infty$ functions $a_0(\mu)$, $a_1(\mu)$ and $B(\mu, x_1)$ with $a_0(0) = a_1(0) = 0$ and $B(0, 0) = 1$ such that

$$B(\mu, x_1)g(\mu, x_1) = a_0(\mu) + a_1(\mu)x_1 + x_1^2.$$

Define $A(\mu, x_1) = \sqrt[B(\mu, x_1)]{\cdot}$. As $A$ is a positive function in a neighbourhood of $(0, 0)$, we can multiply each component of (2.4) by $A(\mu, x_1)$ to obtain a topologically equivalent system

$$\begin{cases}
x_k' = A(\mu, x_1)x_{k+1} & \text{for } k = 1, \ldots, n - 1, \\
x_n' = A(\mu, x_1)f(\mu, x).
\end{cases}$$

To recover normalized equations for $k = 1, \ldots, n - 1$ we must take again coordinates $\hat{x}_k = d^{k-1}x_1/dt^{k-1}$, with $k = 1, \ldots, n$. A simple calculation shows that $dx_1/dt = (A(\mu, x_1))x_2$ and, for $k = 3, \ldots, n$,

$$d^{k-1}x_1/dt^{k-1} = (A(\mu, x_1))^{k-1}x_k + \varphi_{k-1}(\mu, x_1, \ldots, x_{k-1}),$$

with $\varphi_{k-1}(\mu, x_1, \ldots, x_{k-1}) = O((x_2, \ldots, x_{k-1})^2)$ and therefore $(\hat{x}_1, \ldots, \hat{x}_n)$ can be used as new local coordinates. On the other hand,

$$\hat{x}_n' = \frac{d}{dt}[(A(\mu, x_1))^{n-1}x_n + \varphi_{n-1}(\mu, x_1, \ldots, x_{n-1})]$$

$$= B(\mu, x_1)(g(\mu, x_1) + h(\mu, x)) + O(\| (x_2, \ldots, x_n) \|^2)$$

$$= a_0(\mu) + a_1(\mu)\hat{x}_1 + \hat{x}_1^2 + B(\mu, \hat{x}_1)h(\mu, \hat{x}) + O(\| (\hat{x}_2, \ldots, \hat{x}_n) \|^2),$$

where $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n)$. Reduction concludes after we use the translation $\tilde{x}_1 = \hat{x}_1 + a_1(\mu)/2$. Thus, we show that (2.4) can be brought into

$$\begin{cases}
\tilde{x}_k' = \tilde{x}_{k+1} & \text{for } k = 1, \ldots, n - 1, \\
\tilde{x}_n' = a_1(\mu) + \sum_{j=2}^n \tilde{a}_j(\mu)\tilde{x}_j + \tilde{x}_1^2 + \tilde{h}(\mu, \tilde{x}),
\end{cases}$$

(2.5)

where $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n)$, functions $\tilde{a}_j(\mu)$ represent exact coefficient in a development with respect to $\tilde{x}$ and $\tilde{h}$ is again $O(\| (\mu, \tilde{x}) \|^2)$ and $O(\| (\tilde{x}_2, \ldots, \tilde{x}_n) \|)$. 


Let $\bar{a}(\mu) = (\bar{a}_1(\mu), \ldots, \bar{a}_n(\mu))$. The unfolding (2.4) is said to be generic if $\bar{a}$ is a local diffeomorphism at the origin or, in other words, if the following generic condition is satisfied:

$$\det[D_\mu \bar{a}(0)] \neq 0.$$  

(2.6)

In such a case $v_1 = \bar{a}_1(\mu)$, ..., $v_n = \bar{a}_n(\mu)$ can be used as new parameters. So, in short, it is stated that

Lemma 2.1. Any generic unfolding of an $n$-dimensional nilpotent singularity of codimension $n$ can be written as

$$\begin{align*}
x_k' &= x_{k+1} \quad \text{for } k = 1, \ldots, n - 1, \\
x_n' &= \mu_1 + \sum_{j=2}^{n} \mu_j x_j + x_1^2 + h(\mu, x),
\end{align*}$$

(2.7)

where $\mu_1, \ldots, \mu_n$ and the coefficient in front of $x_1^2$ represent exact coefficients in a development with respect to $x$, $h$ is $O(\|(\mu, x)\|^2)$ and $O(\|(x_2, \ldots, x_n)\|)$.

Remark 2.2. Important to notice is that $B(\mu, x_1)$ plays no role in the generic condition (2.6). In fact it is easy to prove that (2.6) is equivalent to

$$\begin{vmatrix}
\frac{\partial^2 f}{\partial \mu_1 \partial x_2} - \frac{\partial f}{\partial \mu_1} & \frac{\partial^2 f}{\partial \mu_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial \mu_1 \partial x_n} \\
\frac{\partial^2 f}{\partial \mu_1 \partial x_2} & \frac{\partial^2 f}{\partial \mu_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial \mu_1 \partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 f}{\partial \mu_1 \partial x_n} & \frac{\partial^2 f}{\partial \mu_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial \mu_1 \partial x_n} \\
\frac{\partial^2 f}{\partial \mu_n \partial x_2} & \frac{\partial^2 f}{\partial \mu_1 \partial x_n} & \cdots & \frac{\partial^2 f}{\partial \mu_n \partial x_n} \\
\cdots & \cdots & \cdots & \cdots
\end{vmatrix} \neq 0,$$

where all the derivatives are evaluated at $(0, 0)$.

Remark 2.3. In Section 2.3, using Lemma 2.1, we will prove that any generic unfolding of an $n$-dimensional nilpotent singularity of codimension $n$ contains generic unfoldings of $(n - 1)$-dimensional singularities. Note that when $\mu_1 = 0$ family (2.7) has a unique singularity at the origin. Unless additional degeneracies occur, $N_1 = \{\mu_1 = 0\}$ is a linear manifold of saddle-node bifurcation at the origin. On the other hand, for $k = 2, \ldots, n - 1$, each linear manifold $N_k = \{\mu_1 = \cdots = \mu_k = 0\}$ contains $k$-dimensional nilpotent singularities of codimension $k$. Therefore the hierarchy of nilpotent singularities is clear. We should observe that additional degeneracies indeed occur. For instance $N_1$ will contain $p$-Hopf-saddle-node singularities for $p \in \mathbb{N}$ and $p \leq \frac{n-1}{2}$, i.e., singularities with $p$ pairs of purely imaginary eigenvalues and one zero eigenvalue. The same we can say for each $N_k$ with $k = 2, \ldots, n - 1$. Of course there are also manifolds corresponding to (maybe multiple) Hopf bifurcations at the equilibrium points arising from the saddle-node bifurcation. For instance, when $n = 3$ (see [5,6]) a Hopf-saddle-node bifurcation line is given by $HNS = \{\mu_3 = 0, \mu_2 < 0\}$. $N_1 \setminus \{N_2 \cup HNS\}$ corresponds to saddle-node bifurcation and $N_2$ to Bogdanov–Takens bifurcation. There are also two Hopf bifurcation surfaces for the equilibrium points arising at the saddle-node bifurcation. They connect $HNS$ to each one of the branches of $N_2$. When $n = 4$ besides the manifolds of nilpotent bifurcations and...
Hopf bifurcations one can find Hopf–Bogdanov–Takens bifurcation lines and Hopf-saddle-node bifurcation surfaces. A more detailed discussion is out of the scope of this paper. To understand the involved difficulties note that for instance the unfolding of the 3-dimensional nilpotent singularity of codimension 3 is nowadays only partially understood (see [5–7,13]).

2.2. Verification of the genericity

To simplify both calculations and wording in the next section, it is useful to have simple formulas to check the generic assumptions. We start again with a family $X_{\mu}$ such that $X_0(0) = 0$ and the linear part of $X_0$ at 0 is given in the canonical form (2.1). Note that our generic conditions (2.3) and (2.6) involve, at most, second order terms and therefore we only need to pay attention to the truncation up to such order.

Consider the following expansions of the equations

\[
\begin{aligned}
&x_k' = x_{k+1} + p_1^{(k)}(\mu) + p_2^{(k)}(\mu, x) + O\left(\|\mu, x\|^3\right) \quad \text{for } k = 1, \ldots, n - 1, \\
&x_n' = p_1^{(n)}(\mu) + p_2^{(n)}(\mu, x) + O\left(\|\mu, x\|^3\right),
\end{aligned}
\]

where, for $k = 1, \ldots, n$, $p_1^{(k)}(\mu) = \sum_{i=1}^{n} \alpha_i^{(k)} \mu_i$ and

\[
p_2^{(k)}(\mu, x) = \sum_{i=1}^{n} \sum_{j=1}^{n} \Gamma_{ij}^{(k)} \mu_i \mu_j + \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij}^{(k)} \mu_i x_j + \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij}^{(k)} x_i x_j.
\]

Using new coordinates $\tilde{x}_1 = x_1$ and $\tilde{x}_k = x_k + p_1^{(k-1)}(\mu)$ for $k = 2, \ldots, n$, we recover an expression as (2.8) but with $p_1^{(k)}(\mu) \equiv 0$ for $k = 1, \ldots, n - 1$. With that transformation, coefficients $\alpha_i^{(n)}$ in $p_1^{(n)}$ and $A_{ij}^{(k)}$ in $p_2^{(k)}$ remain unchanged whereas coefficients $A_{ij}^{(k)}$ are replaced according with the following rules:

\[
\begin{aligned}
&\Lambda_{i1}^{(k)} \leftarrow \Lambda_{i1}^{(k)} - \sum_{s=2}^{n} A_{1s}^{(k)} \alpha_i^{(s-1)} \quad \text{for } i = 1, \ldots, n, \\
&\Lambda_{ij}^{(k)} \leftarrow \Lambda_{ij}^{(k)} - \sum_{s=j}^{n} A_{js}^{(k)} \alpha_i^{(s-1)} - \sum_{s=2}^{j} \alpha_i^{(s-1)} A_{sj}^{(k)} \quad \text{for } i = 1, \ldots, n \text{ and } j = 2, \ldots, n.
\end{aligned}
\]

Coefficients $\Gamma_{ij}^{(k)}$ will be irrelevant in order to check the generic conditions. To remove the quadratic terms in the first $n - 1$ equations we use coordinates $\tilde{x}_1 = x_1$ and $\tilde{x}_k = x_k + p_2^{(k-1)}(\mu, x)$ for $k = 2, \ldots, n$ to obtain, preserving the notation

\[
\begin{aligned}
&x_k' = x_{k+1} + O\left(\|\mu, x\|^3\right) \quad \text{for } k = 1, \ldots, n - 1, \\
&x_n' = p_1^{(n)}(\mu) + p_2^{(n)}(\mu, x) + O\left(\|\mu, x\|^3\right),
\end{aligned}
\]

where polynomials $p_2^{(k)}(\mu, x)$ in both the change of coordinates and the above expression are given by formula (2.9) but substituting coefficients $A_{ij}^{(k)}$ and $A_{ij}^{(k)}$, for $k = 2, \ldots, n$, according to the following rules: For $i = 1, \ldots, n$,
\[
\begin{align*}
A_{i1}^{(k)} & \leftarrow A_{i1}^{(k)} + \alpha_i^{(n)} A_{in}^{(k-1)}, \\
A_{ij}^{(k)} & \leftarrow A_{ij}^{(k)} + A_{i,j-1}^{(k)} + \alpha_i^{(n)} A_{jn}^{(k-1)} \quad \text{for } j = 2, \ldots, n-1, \\
A_{in}^{(k)} & \leftarrow A_{in}^{(k)} + A_{i,n-1}^{(k)} + 2\alpha_i^{(n)} A_{nn}^{(k-1)},
\end{align*}
\]
\begin{align*}
A_{11}^{(k)} & \leftarrow A_{11}^{(k)}, \\
A_{12}^{(k)} & \leftarrow A_{12}^{(k)} + 2A_{11}^{(k-1)}, \\
A_{1j}^{(k)} & \leftarrow A_{1j}^{(k)} + A_{1,j-1}^{(k-1)} \quad \text{for } j = 3, \ldots, n, \\
A_{ii}^{(k)} & \leftarrow A_{ii}^{(k)} + A_{i-1,i}^{(k-1)} \quad \text{for } i = 2, \ldots, n, \\
A_{i,i+1}^{(k)} & \leftarrow A_{i,i+1}^{(k)} + A_{i-1,i+1}^{(k-1)} + 2A_{i,i}^{(k-1)} \quad \text{if } n \geq 3 \text{ for } i = 2, \ldots, n-1, \\
A_{ij}^{(k)} & \leftarrow A_{ij}^{(k)} + A_{i-1,j}^{(k-1)} + A_{i,j-1}^{(k-1)} \quad \text{if } n \geq 4 \text{ for } i = 2, \ldots, n-2, \quad j = i+2, \ldots, n.
\end{align*}
\]

Note that coefficient \(A_{11}^{(n)}\) remains unchanged in all the previous transformations. Hence, condition (2.3) is equivalent to check that \(A_{11}^{(n)}\) is nonzero in (2.8), immediately after reducing the linear part of the singularity to canonical form. Moreover, without loss of generality we assume that such coefficient was already normalized from the beginning and hence that \(A_{11}^{(n)} = 1\). To conclude we have to consider the translation \(\tilde{x}_1 = x_1 + \frac{1}{2} \sum_{i=1}^n A_{11}^{(n)} \mu_i\) to obtain a family as in (2.11) where coefficients \(\alpha_i^{(n)}\) for \(i = 1, \ldots, n\) are still the original ones, \(A_{11}^{(n)} = 0\) for \(i = 1, \ldots, n\) and coefficients \(\Lambda_{ij}\) are the result of transformations (2.10), (2.13) and
\[
\Lambda_{ij}^{(n)} \leftarrow A_{ij}^{(n)} - \frac{1}{2} A_{ij}^{(n)} A_{11}^{(n)}
\]
for \(i = 1, \ldots, n\) and \(j = 2, \ldots, n\).

It is now easy to understand that functions \(\tilde{a}_j(\mu)\) in (2.5) can be written as \(\tilde{a}_1(\mu) = \sum_{i=1}^n \alpha_i^{(n)} \mu_i + O(\|\mu\|^2)\) and \(\tilde{a}_j(\mu) = \sum_{i=1}^n \Lambda_{ij}^{(n)} \mu_i + O(\|\mu\|^2)\) for \(j = 2, \ldots, n\). Hence the generic condition (2.6) can be expressed as
\[
\begin{vmatrix}
\alpha_1^{(n)} & \cdots & \alpha_n^{(n)} \\
\Lambda_{12}^{(n)} & \cdots & \Lambda_{n2}^{(n)} \\
\vdots & \ddots & \vdots \\
\Lambda_{1n}^{(n)} & \cdots & \Lambda_{nn}^{(n)}
\end{vmatrix} \neq 0.
\]

2.3. Reducing to a center manifold

As stated in Lemma 2.1, any generic unfolding of the \(n\)-dimensional nilpotent singularity of codimension \(n\) can be written as in (2.7). It immediately follows that when \(\mu_1 = 0\) the system has a singularity at the origin whose 1-jet, when \(\mu_1 = \cdots = \mu_{n-1} = 0\) and \(\mu_n \neq 0\), is linearly conjugate to
\[
\sum_{k=1}^{n-2} x_{k+1} \frac{\partial}{\partial x_k} + \mu_n x_n \frac{\partial}{\partial x_n}.
\]
We will prove that each time that $\mu_n \neq 0$ is fixed, family (2.7) is a generic unfolding of the $(n - 1)$-dimensional nilpotent singularity of codimension $n - 1$. With that goal in mind we first carry out a reduction to the center manifold. Afterwards we will reduce the system to a normal form to check the generic assumptions. To emphasize that $\mu_n$ is no longer a parameter we will write $c = \mu_n$.

Introducing new coordinates $y_k = x_k - x_n/c^{n-k}$ for $k = 1, \ldots, n - 1$ and $y_n = x_n$ we transform the linear part of (2.7) into the canonical form (2.16). More precisely, we get

$$
\begin{align*}
\begin{cases}
y_k' = y_{k+1} - F(\mu, y)/c^{n-k} & \text{for } k = 1, \ldots, n - 2, \\
y_{n-1}' = -F(\mu, y)/c, \\
y_n' = cy_n + F(\mu, y),
\end{cases}
\end{align*}
$$

(2.17)

where

$$
F(\mu, y) = \mu_1 + \sum_{i=2}^{n-1} \mu_i y_i + \left( \sum_{i=2}^{n-1} \mu_i / c^{n-i} \right) y_n + y_1^2 + \sum_{i=2}^{n} y_i \sum_{j=1}^{i} \hat{A}_{ij} y_j + O(\| (\mu, y) \|^3).
$$

In view of expression (2.17), it should be noticed that only a first order expansion of the center manifold is needed to get a second order reduction. In the sequel we will write $\mu = (\mu_1, \ldots, \mu_{n-1})$ and $y = (y_1, \ldots, y_{n-1})$. It easily follows that the required expansion is $y_n = h(\mu, y) = -\mu_1/c + O(\| (\mu, y) \|^2)$ and hence the reduction to the center manifold is given by

$$
\begin{align*}
\begin{cases}
y_k' = y_{k+1} + \alpha^{(k)} \mu_1 + p_2^{(k)}(\mu, y) + O(\| (\mu, y) \|^3) & \text{for } k = 1, \ldots, n - 2, \\
y_{n-1}' = \alpha^{(n-1)} \mu_1 + p_2^{(n-1)}(\mu, y) + O(\| (\mu, y) \|^3),
\end{cases}
\end{align*}
$$

(2.18)

where, for $k = 1, \ldots, n - 1$,

$$
p_2^{(k)}(\mu, y) = \sum_{i=1}^{n-1} \Gamma_{k i}^{(k)} \mu_1 \mu_i + \sum_{i=2}^{n-1} \Lambda_{k i}^{(k)} \mu_i y_i + \sum_{i=1}^{n-1} \Lambda_{1 i}^{(k)} \mu_1 y_i + A_{11}^{(k)} y_1^2 + \sum_{i=2}^{n} y_i \sum_{j=1}^{i} \hat{A}_{ij}^{(k)} y_j
$$

(2.19)

and $\alpha^{(n-1)} = A_{11}^{(n-1)} = A_{ii}^{(n-1)} = -1/c$ for $i = 2, \ldots, n - 1$. Note that the first generic assumption (2.3) is fulfilled. Scaling by a factor $-1/c$ we get $A_{11}^{(n-1)} = 1$ and $\alpha^{(n-1)} = 1/c^2$ whereas it remains $A_{ii}^{(n-1)} = -1/c$. Once more again, the specific value of the remaining coefficients will be irrelevant.

To remove the first order terms in the first $n - 2$ equations we must take coordinates $\tilde{y}_1 = y_1$ and $\tilde{y}_k = y_k + \alpha^{(k-1)} \mu_1$ for $k = 2, \ldots, n - 1$ to obtain, preserving notations, an expression like in (2.18) but with $\alpha^{(k)} = 0$ for $k = 1, \ldots, n - 2$ and $p_2^{(k)}(\mu, x)$ still expanded as in (2.19) with
The system is invariant under the symmetry

$$\tilde{\alpha}(n_{i}) = \alpha(n_{i})$$

expression as in (2.20) but with checking the generic conditions. To remove the second order terms in the first equation we introduce new coordinates

$$\tilde{y}_1 = y_1$$

and all $$A_{ij}$$ unchanged. The remaining coefficients are irrelevant when

$$\Lambda_{ii}$$

for $$i = 2, \ldots, n - 1$$ and all $$A_{ij}$$ unchanged. The remaining coefficients are irrelevant when checking the generic conditions. To remove the second order terms in the first equation we introduce new coordinates $$\tilde{y}_1 = y_1$$ and $$\tilde{y}_k = y_k + \tilde{p}_{2}^{(k)}(\mu, y)$$ for $$k = 2, \ldots, n - 1$$, where

$$\tilde{p}_{2}^{(k)}(\mu, y) = \sum_{i=1}^{n-1} \tilde{F}_{ii}^{(k-1)} \mu_{i} \mu_{i} + \sum_{j=2}^{k} \sum_{i=j}^{n-1} \tilde{A}^{(k-1)}_{i-j+2,i} \mu_{i-j+2} y_{i} + \sum_{i=1}^{n-1} \tilde{A}^{(k-1)}_{ii} \mu_{i} y_{i} + \tilde{A}^{(k-1)}_{11} y_{1}^2 + \sum_{i=2}^{n-1} y_{i} \sum_{j=1}^{n-1} \tilde{A}^{(k-1)}_{ij} y_{j}.$$  

When $$k = 2$$ the coefficients in the above expression are the correspondent ones in $$p_{2}^{(1)}(\mu, y)$$. For $$k > 2$$ they are determined by the subsequence derivatives. We get finally

$$\left\{ \begin{array}{l} \tilde{y}_k = \tilde{y}_{k+1} + O(\| (\mu, \tilde{y}) \|^3) \quad \text{for } k = 1, \ldots, n - 2, \\ \tilde{y}_{n-1} = \alpha^{(n-1)} \mu_{1} + p_{2}^{(n-1)}(\mu, \tilde{y}) + O(\| (\mu, \tilde{y}) \|^3), \end{array} \right. \quad (2.20)$$

where

$$\tilde{p}_{2}^{(n-1)}(\mu, \tilde{y}) = \sum_{i=1}^{n-1} \tilde{F}_{ii}^{(n-1)} \mu_{i} \mu_{i} + \sum_{i=1}^{n-1} \tilde{A}^{(n-1)}_{ii} \mu_{i} \tilde{y}_{i} + \sum_{i=2}^{n-1} \tilde{A}^{(n-1)}_{i} \mu_{i} \tilde{y}_{i} + \sum_{j=2}^{n-2} \sum_{i=j}^{n-1} \tilde{A}^{(n-1)}_{i-j+2,i} \mu_{i-j+2} \tilde{y}_{i+1} + \tilde{y}_{1}^2 + \sum_{i=2}^{n-1} \tilde{y}_{i} \sum_{j=1}^{n-1} \tilde{A}^{(n-1)}_{ij} \tilde{y}_{j}.$$  

To conclude we translate by $$\tilde{y}_1 = \tilde{y}_1 - \tilde{A}^{(n-1)}_{11} \mu_{1}$$ and consider again $$\tilde{y}_1 = \tilde{y}_1$$ to obtain an expression as in (2.20) but with $$\tilde{A}^{(n-1)}_{11} = 0$$ and new coefficients $$\tilde{A}^{(n-1)}_{ii}$$ for $$i = 2, \ldots, n$$ and $$\tilde{A}^{(n-1)}_{ij}$$ for $$i = 1, \ldots, n - 1$$. It easily follows that functions $$\tilde{\alpha}_i(\mu)$$ in (2.5) can be expanded as

$$\left\{ \begin{array}{l} \tilde{a}_1(\mu) = \alpha^{(n-1)} \mu_{1} + O(\| \mu \|^2), \\ \tilde{a}_k(\mu) = \sum_{i=1}^{k-1} \tilde{A}^{(n-1)}_{ik} \mu_{i} + \tilde{A}^{(n-1)}_{kk} \mu_{k} + O(\| \mu \|^2) \quad \text{for } k = 2, \ldots, n - 1. \end{array} \right. \quad (2.21)$$

To check the second generic hypothesis we observe that the determinant in (2.15) is equal to

$$\alpha^{(n-1)} \prod_{k=2}^{n-1} \tilde{A}^{(n-1)}_{kk} = (-1/c)^n \neq 0.$$  

### 3. Coupled Brusselators

Consider Eqs. (1.2) corresponding to the coupling of two Brusselators by diffusion. Recall that $$A, B, \lambda_1$$ and $$\lambda_2$$ are positive parameters and that we are only interested in the dynamics when all variables are positive. Our goal is to find a 4-dimensional nilpotent singularity of codimension 4. The system is invariant under the symmetry $$(x_1, y_1, x_2, y_2) \rightarrow (x_2, y_2, x_1, y_1)$$ and hence the plane $$\{x_1 = x_2, y_1 = y_2\}$$ is invariant. On that plane there is always a unique equilibrium point at
(A, B/A, A, B/A). It will never be a 4-dimensional nilpotent singularity of codimension 4 since this is not compatible with the existence of a 2-dimensional invariant manifold. Therefore we have to look at bifurcations occurring outside that plane.

We will obtain the bifurcation equation for saddle-node bifurcations; that means one zero eigenvalue. Additional degenerations will be successively imposed until a zero eigenvalue of multiplicity four be achieved. The last step will be to prove that the singularity is generic and generically unfolded by (1.2).

To simplify calculations let us consider new coordinates

\[
\xi_1 = (x_2 - x_1)/2, \quad \xi_2 = (y_2 - y_1)/2, \quad \eta_1 = (x_2 + x_1)/2 \quad \text{and} \quad \eta_2 = (y_2 + y_1)/2.
\]

System (1.2) transforms into

\[
\begin{align*}
\xi_1' & = -(B + 1)\xi_1 + (\eta_1^2 + \xi_1^2)\xi_2 + 2\eta_1\eta_2\xi_1 - 2\lambda_1\xi_1, \\
\xi_2' & = B\xi_1 - (\eta_1^2 + \xi_1^2)\xi_2 - 2\eta_1\eta_2\xi_1 - 2\lambda_2\xi_2, \\
\eta_1' & = A - (B + 1)\eta_1 + (\eta_1^2 + \xi_1^2)\eta_2 + 2\xi_1\xi_2\eta_1, \\
\eta_2' & = B\eta_1 - (\eta_1^2 + \xi_1^2)\eta_2 - 2\xi_1\xi_2\eta_1, 
\end{align*}
\]

which has an equilibrium point at \(T = (0, 0, A, B/A)\) for all parameter values and leaves invariant the plane \{\(\xi_1 = 0, \xi_2 = 0\)\}.

The equations for the equilibrium points are

\[
\begin{align*}
-(B + 1)\xi_1 + (\eta_1^2 + \xi_1^2)\xi_2 + 2\eta_1\eta_2\xi_1 - 2\lambda_1\xi_1 & = 0, \\
B\xi_1 - (\eta_1^2 + \xi_1^2)\xi_2 - 2\eta_1\eta_2\xi_1 - 2\lambda_2\xi_2 & = 0, \\
A - (B + 1)\eta_1 + (\eta_1^2 + \xi_1^2)\eta_2 + 2\xi_1\xi_2\eta_1 & = 0, \\
B\eta_1 - (\eta_1^2 + \xi_1^2)\eta_2 - 2\xi_1\xi_2\eta_1 & = 0.
\end{align*}
\]

Adding the first two equations we get \((1 + 2\lambda_1)\xi_1 + 2\lambda_2\xi_2 = 0\) and hence:

\[
\xi_2 = -(1 + 2\lambda_1)\xi_1/2\lambda_2.
\]

On the other hand, adding the last two equations of (3.2) we get \(A - \eta_1 = 0\) and therefore,

\[
\eta_1 = A.
\]

Using (3.3) and (3.4) in the third equation of (3.2) we get

\[
\eta_2 = (AB - 2A\xi_1\xi_2)/(A^2 + \xi_1^2).
\]

Finally, taking (3.3)–(3.5) to the second equality of (3.2) we get the equation for the equilibrium points

\[
\xi_1 \left[(A^2 + \xi_1^2)^2 + (A^2 + \xi_1^2)p + q\right] = 0,
\]
with
\[ p = \left( 2\lambda_2(B + 2\lambda_1 + 1) - 4A^2(1 + 2\lambda_1) \right) /(1 + 2\lambda_1) \]

and
\[ q = \left( 4A^2(1 + 2\lambda_1) - B\lambda_2 \right) /[1 + 2\lambda_1] . \]

It easily follows that the family (3.1) undergoes a pitchfork bifurcation at \( T \) when \( A^4 + pA^2 + q = 0 \) and a saddle-node bifurcation at
\[ \xi_1 = \pm \sqrt{-p/2 - A^2} \] (3.7)

and \( \xi_2, \eta_1 \) and \( \eta_2 \) given by (3.3), (3.4) and (3.5), respectively, when
\[ p^2 - 4q = 0 \text{ and } 2A^2 + p < 0 . \]

The above conditions are equivalent to
\[ \lambda_2 = \left[ 2A(1 + 2\lambda_1)/(1 + B + 2\lambda_1) \right]^2 , \] (3.8)
\[ B - 6\lambda_1 - 3 > 0 . \] (3.9)

If we substitute (3.8) into (3.1) and calculate the characteristic polynomial of the Jacobian at the saddle-node bifurcation point we obtain
\[ r^4 + c_3r^3 + c_2r^2 + c_1r \]

with
\[ c_1 = 4(A + 2\lambda_1)^2[-B^3 + B^2(-1 + 2\lambda_1) + (1 + 2\lambda_1)^2(1 + 6\lambda_1) + B(1 + 8A^2 + 12\lambda_1 + 20\lambda_1^2)]/(1 + B + 2\lambda_1)^3 , \]
\[ c_2 = \left[ 32A^4B(1 + 2\lambda_1)^2 + (1 + B + 2\lambda_1)^4(1 + 3\lambda_1) - 8A^2(B^3\lambda_1 + (1 + 2\lambda_1)^3(2 + 5\lambda_1)
+ B(1 + 2\lambda_1)^2(4 + 9\lambda_1) + B^2(2 + 9\lambda_1 + 10\lambda_1^2)) \right] /(1 + B + 2\lambda_1)^3 , \]
\[ c_3 = \left[ -(1 + B + 2\lambda_1)^2(1 + B + 4\lambda_1) + 4A^2(B^2 + (1 + 2\lambda_1)^2) \right] /(1 + B + 2\lambda_1)^2 . \]

A nilpotent singularity appears when there is a solution of \( c_1 = c_2 = c_3 = 0 \) satisfying certain open conditions. From \( c_1 = 0 \) it follows
\[ A = \frac{\sqrt{-1 + B - 6\lambda_1(1 + B + 2\lambda_1)}}{2\sqrt{2}\sqrt{B}} . \] (3.10)

Substituting \( A \) in the equations \( c_2 = 0 \) and \( c_3 = 0 \) we see that they are equivalent to
\[ B^3(1 + 2\lambda_1 + 4\lambda_1^2) - B^2(3 + 22\lambda_1 + 48\lambda_1^2 + 40\lambda_1^3) + (1 + 2\lambda_1)^2(5 + 48\lambda_1 + 120\lambda_1^2 + 72\lambda_1^3)
+ B(1 + 16\lambda_1 + 60\lambda_1^2 + 88\lambda_1^3 + 48\lambda_1^4) = 0 , \] (3.11)
Fig. 1. Curves defined by (3.11) and (3.12); solid and dashed, respectively.

\[ B^3 - 3B(1 + 2\lambda_1)(1 + 6\lambda_1)(1 - 4\lambda_1 + 4\lambda_1^2) = 0, \]  

(3.12)

respectively. The correspondent curves on the plane \((\lambda_1, B)\) are shown in Fig. 1.

**Remark 3.1.** The curve given by (3.11) corresponds to 3-dimensional nilpotent singularities after the vector fields are reduced to the center manifold. Results in [13] about the existence of strange attractors apply to that singularities when they are of codimension 3 and generically unfolded. Such properties follow after we prove that, at the intersection point, our family is a generic unfolding of a 4-dimensional nilpotent singularity of codimension 4.

We finally get the bifurcation point \((\hat{A}, \hat{B}, \hat{\lambda}_1, \hat{\lambda}_2)\) given in Theorem 1.1. Parameters \(\hat{B}\) and \(\hat{\lambda}_1\) are obtained by solving, using the Newton Method, the system given by (3.11) and (3.12). The other parameters \(\hat{A}\) and \(\hat{\lambda}_2\) are given by formulas (3.8) and (3.10). The values are:

\[ \hat{A} \approx 2.6021429, \quad \hat{B} \approx 11.2982917, \quad \hat{\lambda}_1 \approx 1.2506766 \quad \text{and} \quad \hat{\lambda}_2 \approx 1.5159733. \]

Using (3.3)–(3.5) and (3.7) we obtain the coordinates of the singularity

\[ \hat{\xi}_1 \approx 0.6028083, \quad \hat{\xi}_2 \approx -0.6961352, \quad \hat{\eta}_1 \approx 2.6021429 \quad \text{and} \quad \hat{\eta}_2 \approx 4.4268781, \]

or, in the original coordinates,

\[ x_1 \approx 1.9993346, \quad x_2 \approx 3.2049512, \quad y_1 \approx 5.1230133 \quad \text{and} \quad y_2 \approx 3.7307429. \]
Note that all parameters and variables are, as required, positive. Recall that our families are invariant under certain symmetries and hence the above are not the only equilibria, but their symmetric counterparts are nilpotent equilibria too.

We finally want to check that the above singularity is generic and generically unfolded by the family (3.1). First step is to move our bifurcation point to the origin introducing new parameters

\[ \mu_1 = A - \hat{A}, \quad \mu_2 = B - \hat{B}, \quad \mu_3 = \lambda_1 - \hat{\lambda}_1 \quad \text{and} \quad \mu_4 = \lambda_2 - \hat{\lambda}_2, \]

and new variables

\[ v_1 = \xi_1 - \hat{\xi}_1, \quad v_2 = \xi_2 - \hat{\xi}_2, \quad v_3 = \eta_1 - \hat{\eta}_1 \quad \text{and} \quad v_4 = \eta_2 - \hat{\eta}_2. \]

Let \( L \) be the linear part at the origin of the vector field obtained when parameters vanish. To reduce \( L \) to a canonical form we use a change of coordinates \( v = C w \) where \( v = (v_1, v_2, v_3, v_4) \), \( w = (w_1, w_2, w_3, w_4) \) and \( C \) is a matrix with columns given by \( L^3 \delta, L^2 \delta, L \delta \) and \( \delta \), respectively, for any choice of a vector \( \delta \) not in the kernel of \( L^3 \). For our calculations we take \( \delta = (0, 0, 0, 1) \). After this change we can check that the first generic assumption (2.3) is indeed satisfied since the coefficient of \( w_1^2 \) in the last component of the new family is \( \gamma \approx 175.163 \). Such coefficient is normalized to 1 using \( w \rightarrow w/\gamma \). Hence we have a family as in (2.8) with \( A_{11}^{(4)} = 1 \) in formula (2.9). To conclude we must apply (2.10), (2.12)–(2.14). We finally obtain that the approximate value of the determinant in (2.15) is \(-365.263\) and therefore that the second generic condition (2.6) is also fulfilled.

**Remark 3.2.** To get the previous numerical values we have used very well-known software for scientific computation. To compute the coordinates of the intersection point in Fig. 1 we wrote a Fortran code to implement the Newton Method. The result was also checked using the Matlab [15] tools. Formulas to translate the bifurcation point to the origin are easy to get by hand but they were also obtained using Mathematica [14]. Calculations to reduce the linear part to a canonical form and application of formulas in Section 2 were done in two different ways, using Matlab and also using our own Fortran code. Note that the only “numerical” approach is the use of the Newton Method. All the other calculations are given by algebraic formulas. Complete calculations are not included to not increase the length of the paper unnecessarily. They are, of course, available on request.

**References**


