The Contraction Number of a Multigrid Method with Mesh Ratio Two for Solving Model Problems

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ABSTRACT

The algorithm proposed in [6] is analyzed with the aid of model problem analysis. The explicit expression for the convergence factor and the exact values of the contraction number are obtained.

1. INTRODUCTION

In [6] Verfürth presented a multigrid algorithm with mesh ratio 2 in which no intermediate grids were used explicitly and estimated the contraction number by Braess’s technique [2]. In [3] we studied the behavior of Braess’s algorithm with the aid of model problem analysis and showed that this algorithm has better contradiction numbers (spectral norm and energy norm) than those of the mcr-ch[v] algorithm presented in [5]. In this paper Verfürth’s algorithm is analyzed by the method of model problem analysis, and the exact convergence factor and the contraction number are obtained. The results are useful for understanding the behavior of Verfürth’s algorithm and for appraising the sharpness of the estimation of the contraction number in [6].

In this paper we follow closely the notation presented in [5].

2. ITERATION MATRIX

Let \( \varphi_{kl} (k, l = 1, \ldots, N - 1) \) denote the discrete eigenfunctions of the model operator \( \Delta_h \) (with the Dirichlet boundary conditions):

\[
\varphi_{kl}(x, y) = 2 \sin k \pi x \sin l \pi y, \quad (x, y) \in \Omega_h, \tag{1}
\]

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which form a basis of the grid function space $G(\Omega_{2h})$, and the restrictions of which to $\Omega_{2h}$,

$$\Phi_{kl}(x, y) = \varphi_{kl}(x, y)|_{\Omega_{2h}} \quad (k, l = 1, \ldots, N/2),$$

(2)

form a basis of the grid function space $G(\Omega_{2h})$. Set

$$E_{kl} = \text{span}(\varphi_{kl}, \varphi_{kl} - \varphi_{kl}, - \varphi_{kl}, - \varphi_{kl}), \quad k, l \leq N/2,$$

(3)

where $k = N - k$ and $l = N - l$. $E_{kl}$ is 4-dimensional when $k, l \neq N/2$, and $E_{N/2,l} (l \neq N/2)$ and $E_{k,N/2} (k \neq N/2)$ are 2-dimensional, while $E_{N/2,N/2}$ is 1-dimensional.

Applying a relaxation operator, restriction operator, interpolation operator, etc., we can rewrite Verfürth's algorithm in [6] and get the iteration matrix of the algorithm (with $v = 0$):

$$M_{2h}^0[0] = R_h^2 R_h^0 R_h^c (I_h - I_{2h}^h (A_{2h}^2)^{-1} I_{2h}^h A_h^h) R_h^c R_h^0 R_h^c$$

(4)

where

$$R_h^c u_h(p) = \begin{cases} \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} u_h(p), & p \in \Omega_H, \\ u_h(p), & p \in \Omega_h, \end{cases}$$

$$R_h^0 u_h(p) = \begin{cases} \begin{bmatrix} u_h(p), \\ \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} u_h(p), & p \in \Omega_H, \\ u_h(p), & p \in \Omega_h, \end{cases}$$

$$R_h u_h(p) = \begin{cases} \begin{bmatrix} u_h(p), \\ \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} u_h(p), & p \in \Omega_H \setminus \Omega_{2h}, \\ u_h(p), & p \in \Omega_h \cup \Omega_{2h}, \end{cases}$$

$$A_h \doteq \frac{1}{4} \begin{bmatrix} 1 & -1 & 1 \\ -1 & 4 & 1 \\ 1 & -1 & 1 \end{bmatrix}_h,$$

$$I_{2h}^h \doteq \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}_h.$$
Obviously, the iteration matrix with $\nu > 0$ is as follows:

$$
M_{kn}^{2h}[\nu] = \begin{cases} 
( R_h^k R_h^O )^{\nu/2 + 1} R_h^O \left[ \left[ I_h - I_{2h}^h ( A^{2h} ) - 1 I_{2h}^h A^h \right] R_h^O ( R_h^k R_h^O )^{\nu/2 + 1} , 
\nu \text{ even}, \\
R_h^O ( R_h^k R_h^O )^{(\nu + 1)/2} \left[ \left[ I_h - I_{2h}^h ( A^{2h} ) - 1 I_{2h}^h A^h \right] R_h^O ( R_h^k R_h^O )^{(\nu + 1)/2} R_h^O , 
\nu \text{ odd}.
\end{cases}
$$

(5)

Now we can first derive the $E_{kl}$-representation of each operator in (5) by using the method of undetermined coefficients. Then applying the special form of the iteration matrix $M_{kn}^{2h}[\nu]$ in (5), we can further reduce the dimension of the invariant spaces.

Let $c_k = \cos k \pi h, c_{kl} = (c_k + c_l)/2 = a, c_{kl} = b$, and

$$
\psi_1 = (1 + a) \varphi_{kl} \quad (1 - a) \varphi_{kl}
$$

and

$$
\psi_2 = (1 + b)( - \varphi_{kl}) - (1 - b)( - \varphi_{kl}).
$$

Then we have

$$
M_{kn}^{2h}[0] [\psi_1, \psi_2] = [\psi_1, \psi_2] \tilde{M}_{kl}[0],
$$

that is, the (at most) 2-dimensional subspace spanned by $\psi_1$ and $\psi_2$ is invariant under $M_{kn}^{2h}[0]$, and the corresponding $2 \times 2$ matrix $\tilde{M}_{kl}[0]$ is of the form

$$
\tilde{M}_{kl}[0] = \begin{bmatrix} m_{11}^{(0)} & m_{12}^{(0)} \\
m_{21}^{(0)} & m_{22}^{(0)} \end{bmatrix}
$$

Through careful computation we get

$$
m_{11}^{(0)} = \frac{a^2}{4} \left[ (1 + b^2) - a^2 (a^2 + b^2) - \frac{4a^2 b^2}{1 - a^2 - b^2} \right],
$$

$$
m_{12}^{(0)} = \frac{ab(1 - b^2)}{4} \left[ 1 - a^2 - b^2 - \frac{4a^2 b^2}{1 - a^2 - b^2} \right],
$$

$$
m_{21}^{(0)} = \frac{ab(1 - a^2)}{4} \left[ 1 - a^2 - b^2 - \frac{4a^2 b^2}{1 - a^2 - b^2} \right],
$$

$$
m_{22}^{(0)} = \frac{b^2}{4} \left[ (1 + a^2) - b^2 (a^2 + b^2) - \frac{4b^4 a^2}{1 - a^2 - b^2} \right].
$$

(7)
When \( k = l = N/2 \), we have the \( 1 \times 1 \) matrix

\[
\tilde{M}_{N/2, N/2}[0] = 0.
\]

When \( \nu > 0 \) is even, \( \psi_1 \) and \( \psi_2 \) also form a basis of the 2-dimensional invariant subspace of \( M_{h}^{2h}[\nu] \). When \( \nu > 0 \) is odd, take

\[
\psi'_1 = (1 + a)\varphi_{kl} + (1 - a)\varphi_{kl} \quad \text{and} \quad \psi'_2 = (1 + b)(-\varphi_{kl}) + (1 - b)(-\varphi_{kl})
\]

to form the basis of the 2-dimensional invariant subspace. Then it is easy to show that the matrix representation \( \tilde{M}_{kl}[\nu] \), whether \( \nu \) is even or odd, has a uniform form:

\[
\tilde{M}_{kl}[\nu] = \begin{bmatrix}
m_{11}^{(\nu)} & m_{12}^{(\nu)} \\
m_{21}^{(\nu)} & m_{22}^{(\nu)}
\end{bmatrix},
\]

where

\[
\begin{align*}
m_{11}^{(\nu)} &= a^{2\nu}m_{11}^{(0)}, & m_{12}^{(\nu)} &= (ab)^{\nu}m_{12}^{(0)}, \\
m_{21}^{(\nu)} &= (ab)^{\nu}m_{21}^{(0)}, & m_{22}^{(\nu)} &= b^{2\nu}m_{22}^{(0)}.
\end{align*}
\]

3. CONVERGENCE FACTOR AND CONTRACTION NUMBER

From the above discussion we have for the convergence factor \( \rho^*(\nu) \):

\[
\rho^*(\nu) = \rho^*(M_{h}^{2h}[\nu]) = \sup_{0 < h < h^*} \rho(M_{h}^{2h}[\nu]) = \sup_{0 < h < h^*} \max_{k,l} \rho(\tilde{M}_{kl}[\nu]),
\]

where \( h^* \) is a positive constant, usually \( h^* = \frac{1}{4} \). Since \( M_{N/2, N/2}[\nu] = 0 \), it is sufficient to discuss \( 2 \times 2 \) matrices of the form (8).

From (7) and (9) we know that we can assume \( b \geq 0 \). Hence the ranges of \( a \) and \( b \) are

\[
0 \leq a \leq 1, \quad 0 \leq b \leq \min(a, 1 - a).
\]

**Lemma 1.** The eigenvalues of matrix \( \tilde{M}_{kl}[\nu] \) are real and nonnegative.

**Proof.** Use polar coordinates, and let \( a = r \cos \theta \) and \( b = r \sin \theta \). From (10) we can determine the range \( GM \) of \((r, \theta)\) (cf. Figure 1). Through careful
computation we get

\[
m_{11}^{(0)} = \frac{r^2}{4} \cos^2 \theta \left[1 + r^2 \sin^2 \theta - r^4 \cos^2 \theta - \frac{r^6 \sin^2 2\theta \cos^2 \theta}{1 - r^2}\right],
\]

\[
m_{12}^{(0)} = \frac{r^2}{4} \sin \theta \cos \theta \left(1 - r^2 \sin^2 \theta\right) \left[1 - r^2 - \frac{r^4 \sin^2 2\theta}{1 - r^2}\right],
\]

\[
m_{21}^{(0)} = \frac{r^2}{4} \sin \theta \cos \theta \left(1 - r^2 \cos^2 \theta\right) \left[1 - r^2 - \frac{r^4 \sin^2 2\theta}{1 - r^2}\right],
\]

\[
m_{22}^{(0)} = \frac{r^2}{4} \sin^2 \theta \left[1 + r^2 \cos^2 \theta - r^4 \sin^2 \theta - \frac{r^6 \sin^2 2\theta \sin^2 \theta}{1 - r^2}\right],
\]

\[
T^{(0)} = m_{11}^{(0)} + m_{22}^{(0)} = \frac{r^2}{4} \left[1 - r^4 + \frac{r^2(1 - 3r^4)}{2(1 - r^2)} \sin^2 2\theta + \frac{r^6}{2(1 - r^2)} \sin^4 2\theta\right],
\]

\[
D^{(0)} = m_{11}^{(0)} m_{22}^{(0)} - m_{12}^{(0)} m_{21}^{(0)} = \frac{r^6 \sin^2 2\theta}{32} \left[2(1 - r^2) + \frac{r^2(1 - 2r^2)}{1 - r^2} \sin^2 2\theta\right].
\]

(11)
By using the condition \( a + b \leq 1 \) we deduce that

\[
\sin 2\theta \leq \frac{1 - r^2}{r^2}.
\]

(12)

Consequently, we have

\[
m^{(0)}_{11} \geq \frac{r^2}{4} \cos^2 \theta \left[ 1 + r^2 \sin^2 \theta - r^4 \cos^2 \theta - r^2 \cos^2 \theta (1 - r^2) \right] = \frac{r^2}{4} \cos^2 \theta \left[ 1 + r^2 (\sin^2 \theta - \cos^2 \theta) \right] \geq 0.
\]

Analogously, we have

\[
m^{(0)}_{22} \geq 0, \quad m^{(0)}_{12} \geq 0, \quad \text{and} \quad m^{(0)}_{21} \geq 0.
\]

Furthermore, we have

\[
T^{(0)} \geq 0 \quad \text{and} \quad D^{(0)} \geq 0.
\]

For \( \nu > 0 \), we have

\[
D^{(\nu)} = m^{(\nu)}_{11} m^{(\nu)}_{22} - m^{(\nu)}_{12} m^{(\nu)}_{21} = (ab)^2 D^{(0)} \geq 0
\]

and

\[
T^{(\nu)} = m^{(\nu)}_{11} + m^{(\nu)}_{22} = a^{2\nu} m^{(0)}_{11} + b^{2\nu} m^{(0)}_{22} \geq b^{2\nu} (m^{(0)}_{11} + m^{(0)}_{22}) = b^{2\nu} T^{(0)} \geq 0.
\]

Thus, the conclusion can be easily obtained.

\[\blacksquare\]

**Theorem 2.** If \( \nu \geq 1 \), then it holds that

\[
\rho^*(\nu) = \frac{1}{2} \cdot \frac{(\nu + 1)^{\nu + 1}}{(\nu + 3)^{\nu + 3}}
\]

(13)
Proof. First, let \( \nu = 1 \) and consider the trace \( T^{(1)} \) of the matrix \( M_{kl}[1] \). From (11) and (9) we get

\[
T^{(1)} = T^{(1)}(r, \theta)
\]

\[
= \frac{r^4}{4} \left[ 1 - \frac{1}{3} \sin^2 2\theta + \frac{r^2}{4} \sin^2 2\theta - r^4 \left( 1 + \frac{r^2 \sin^2 2\theta}{1 - r^2} \right) \left( 1 - \frac{3}{4} \sin^2 2\theta \right) \right]
\]

Let \( t = \sin^2 2\theta \); then we have

\[
T^{(1)} = \frac{r^4}{4} \left[ 1 - \frac{t}{2} + \frac{r^2}{4} t - r^4 \left( 1 + \frac{r^2}{1 - r^2 t} \right) \left( 1 - \frac{3}{4} t \right) \right].
\]

Consider the partial derivative \( \partial T^{(1)} / \partial t \), and in case \( 0 \leq r \leq 1 / \sqrt{2} \), use \( t \leq 1 \), while in case \( 1 / \sqrt{2} < r \leq 1 \), use \( t \leq [(1 - r^2) / r^2]^2 \) [cf. (12)]. Then we can deduce that for \( r \) fixed, \( T^{(1)}(r, \theta) \) achieves its maximum at \( \theta = 0 \):

\[
T^{(1)}(r, 0) = m_{11}(r, 0) = \frac{r^4(1 - r^4)}{4}.
\]

Since \( T^{(1)}(r, 0) \) achieves its maximum when \( r = \frac{\sqrt{1 - \frac{1}{2}}}{2} \), we have

\[
\max_{(r, \theta) \in G_M} T^{(1)}(r, \theta) = T^{(1)}\left( \frac{\sqrt{1 - \frac{1}{2}}}{2}, 0 \right) = \frac{1}{16}.
\]

But from (11) and (9) we have

\[
D^{(1)} = (ab)^2 D^{(0)} = (ab)^2 \frac{r^2 t}{32} \left[ 2(1 - r^2) + \frac{r^2(1 - 2r^2)}{1 - r^2 t} \right].
\]

Since \( D^{(1)} = 0 \) when \( (r, \theta) = \left( \frac{4}{\sqrt{3}}, 0 \right) \), we get

\[
\rho^*(1) = \max_{(r, \theta) \in G_M} T^{(1)}(r, \theta) = \frac{1}{16} = \frac{1}{2} \left( \frac{\sqrt{1 + 1}}{1 + 3} \right)^{1/3}.
\]

When \( \nu > 1 \), we have

\[
T^{(\nu)}(r, \theta) = a^{2\nu - 2} m_{11}^{(1)} + b^{2\nu - 2} m_{22}^{(1)} \leq \frac{r^{2\nu + 1}(1 - r^4)}{4}.
\]
The rightmost expression achieves its maximum when $r = r^*$:

$$r^* = \sqrt{\frac{\nu + 1}{\nu + 3}},$$

and we have

$$\max_{(r, \theta) \in C_M} T^{(r)}(r, \theta) = T^{(r)}(r^*, 0) = \frac{1}{2} \cdot \frac{(\sqrt{\nu + 1})^{\nu + 1}}{(\sqrt{\nu + 3})^{\nu + 3}}.$$  

Obviously, $D^{(r)}(r, \theta) = 0$ at $(r, \theta) = (r^*, 0)$, so we get

$$\rho^*(\nu) = \max_{(r, \theta) \in C_M} T^{(r)}(r, \theta) = \frac{1}{2} \cdot \frac{(\sqrt{\nu + 1})^{\nu + 1}}{(\sqrt{\nu + 3})^{\nu + 3}}.$$

Verfürth's algorithm is also convergent when $\nu = 0$. Applying (12) and Lemma 1, we have

**Theorem 3.** For $M^{2h}[0]$ it holds that

$$\left(\frac{1}{16} = \right) 0.0625 \leq \rho^*(0) \leq 0.125 \left( = \frac{1}{8}\right).$$

**Remark 4.** Applying the expressions for $T^{(0)}$ and $D^{(0)}$ in (11), we can verify that the maximum eigenvalue $\lambda(r, t)$ of the matrix $\bar{M}_{kt}[0]$,

$$\lambda(r, t) = \frac{1}{2} \left( T^{(0)} + \sqrt{(T^{(0)})^2 - 4D^{(0)}} \right),$$

achieves its maximum when $t = 0$ and $r = \frac{4}{\sqrt{3}}$. Therefore we get

$$\rho^*(0) = \lambda(\frac{4}{\sqrt{3}}, 0) = \frac{1}{6\sqrt{5}} = \frac{1}{2} \cdot \frac{(\sqrt{0 + 1})^{0 + 1}}{(\sqrt{0 + 3})^{0 + 3}},$$

i.e., the formula (13) holds for $\nu = 0$, too.
In order to compare our result with that in [6], we need to compute the supremum \( \sigma_E^*(v) \) of the energy norm of the iteration matrix \( M^{2h}[v] \):

\[
\sigma_E^*(v) = \sup_{0 < h \leq h^*} \left\| (A^h)^{1/2} M^{2h}[v] (A^h)^{-1/2} \right\|_2
\]

\[
= \sup_{h} \max_{k,l} \left\| (A_{kl}^h)^{1/2} (M^{2h}[v])_{kl} (A_{kl}^h)^{-1/2} \right\|_2.
\]

It is easy to deduce that

\[
\left[ (A_{kl}^h)^{1/2} (M^{2h}[v])_{kl} (A_{kl}^h)^{-1/2} \right]^T = (A_{kl}^h)^{1/2} (M^{2h}[v])_{kl} (A_{kl}^h)^{-1/2}.
\]

Thus we can get

**Theorem 5.** For Verfürth's algorithm,

\[
\sigma_E^*(v) = \rho^*(v) \quad (v \geq 0). \tag{14}
\]

In Table 1 we list our contraction numbers and those in [6] and in [4] for comparison.

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<th>( \rho^<em>(v) = \sigma_E^</em>(v) )</th>
<th>( \sigma_E^*(v) ) in [6]</th>
<th>( \sigma^*(v) ) in [4] (mg-ch(V, 1))</th>
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REFERENCES


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