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Spectral radius and Hamiltonicity of graphs

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ABSTRACT

Let G be a graph of order n and $\mu(G)$ be the largest eigenvalue of its adjacency matrix. Let $\overline{G}$ be the complement of G.

Write $K_{n-1}+v$ for the complete graph on $n-1$ vertices together with an isolated vertex, and $K_{n-1}+e$ for the complete graph on $n-1$ vertices with a pendent edge.

We show that:

If $\mu(G) \geqslant n-2$, then G contains a Hamiltonian path unless $G = K_{n-1}+v$; if strict inequality holds, then G contains a Hamiltonian cycle unless $G = K_{n-1}+e$.

If $\mu(\overline{G}) \leqslant \sqrt{n-1}$, then G contains a Hamiltonian path unless $G = K_{n-1}+v$.

If $\mu(\overline{G}) \leqslant \sqrt{n-2}$, then G contains a Hamiltonian cycle unless $G = K_{n-1}+e$.

1. Introduction

The spectral radius of a graph is the largest eigenvalue of its adjacency matrix. In this note we give tight conditions on the spectral radius for the existence of Hamiltonian paths and cycles. Other spectral conditions for Hamiltonian cycles have been given in [2,3,5,6], but they all are far from our results.

In [7] Ore showed that if the inequality

$$d(u)+d(v) \geqslant n-1 \tag{1}$$

holds for every pair of nonadjacent vertices u and v, then G contains a Hamiltonian path. If the inequality (1) is strict, then G contains a Hamiltonian cycle.

Write $K_{n-1}+v$ for the complete graph on $n-1$ vertices together with an isolated vertex, and $K_{n-1}+e$ for the complete graph on $n-1$ vertices with a pendent edge.

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Note that $K_{n-1}+v$ has no Hamiltonian path as it is disconnected, and $K_{n-1}+e$ has no Hamiltonian cycle as it has a vertex of degree 1. As it turns out, these are the only graphs of order n with such properties and having maximum number of edges.

More precisely, Ore's approach can be used to prove the following extremal result.

Fact 1. Let G be a graph with n vertices and m edges. If

$$m \geqslant \binom{n-1}{2} \tag{2}$$

then G contains a Hamiltonian path unless $G = K_{n-1}+v$. If the inequality (2) is strict, then G contains a Hamiltonian cycle unless $G = K_{n-1}+e$.

We can now easily deduce a straightforward spectral version of this assertion.

Theorem 2. *Let G be a graph of order n and spectral radius $\mu(G)$. If*

$$\mu(G) \geqslant n-2, \tag{3}$$

then G contains a Hamiltonian path unless $G = K_{n-1}+v$. If the inequality (3) is strict, then G contains a Hamiltonian cycle unless $G = K_{n-1}+e$.

Proof. Stanley's inequality [8]

$$\mu(G) \leqslant -\frac{1}{2} + \sqrt{2m+\frac{1}{4}},$$

together with (3), implies that

$$2m \geqslant \left(n-\frac{3}{2}\right)^2 - \frac{1}{4} = n^2-3n+2.$$

Hence, we obtain

$$m \geqslant \binom{n-1}{2}$$

with strict inequality if (3) is strict. Now Theorem 2 follows from Fact 1.

The proof of Theorem 2 is so short because Stanley's inequality becomes equality precisely for complete graphs together with isolated vertices.

Another, subtler condition for Hamiltonicity can be obtained using the spectral radius of the complement of a graph.

Theorem 3. *Let G be a graph of order n and $\mu(\overline{G})$ be the spectral radius of its complement. If*

$$\mu(\overline{G}) \leqslant \sqrt{n-1}, \tag{4}$$

then G contains a Hamiltonian path unless $G = K_{n-1}+v$.

If

$$\mu(\overline{G}) \leqslant \sqrt{n-2}, \tag{5}$$

then G contains a Hamiltonian cycle unless $G = K_{n-1}+e$.

Our proof of Theorem 3 is based on the concept of k-closure of a graph, used implicitly by Ore in [7], and formally introduced by Bondy and Chvatal in [1]. We write $E(G)$ for the edge set of a graph G and $e(G)$ for $|E(G)|$; $d_G(u)$ stands for the degree of the vertex u in G.

Fix an integer $k \geqslant 0$. Given a graph G, perform the following operation: if there are two nonadjacent vertices u and v with $d_G(u)+d_G(v) \geqslant k$, add the edge uv to $E(G)$. A *k-closure* of G is a graph obtained from G by successively applying this operation as long as possible. Somewhat surprisingly, it turns out that the k-closure of G is unique, that is to say, it does not depend on the order in which edges are added; see [1] for details.

Write $\mathcal{G}_k(G)$ for the k-closure of G and note its main property:

$d_{\mathcal{G}_k(G)}(u) + d_{\mathcal{G}_k(G)}(v) \leqslant k - 1$ *for every pair of nonadjacent vertices* u *and* v *of* $\mathcal{G}_k(G)$.

The usefulness of the closure concept is demonstrated by the following two facts, due essentially to Ore [7]:

Fact 4. A graph G has a Hamiltonian path if and only if $\mathcal{G}_{n-1}(G)$ has one.

Fact 5. A graph G has a Hamiltonian cycle if and only if $\mathcal{G}_n(G)$ has one.

Armed with these facts we can carry out the proof of Theorem 3 .

Proof of Theorem 3. For short, let $H = \mathcal{G}_{n-1}(G)$. Assume that (4) holds but G has no Hamiltonian path. Then, by Fact 4, H has no Hamiltonian path either. Now the main property of $\mathcal{G}_{n-1}(G)$ gives $d_H(u) + d_H(v) \leqslant n - 2$ for every pair of nonadjacent vertices u and v of H; thus,

$$d_{\overline{H}}(u) + d_{\overline{H}}(v) = n - 1 - d_H(u) + n - 1 - d_H(v) \geqslant n$$

for every edge $uv \in E(\overline{H})$. Summing these inequalities for all edges $uv \in E(\overline{H})$, we obtain

$$\sum_{uv \in E(\overline{H})} d_{\overline{H}}(u) + d_{\overline{H}}(v) \geqslant ne(\overline{H})$$

and since each term $d_{\overline{H}}(u)$ appears in the left-hand sum precisely $d_{\overline{H}}(u)$ times, we see that

$$\sum_{v \in V(\overline{H})} d^2_{\overline{H}}(v) = \sum_{uv \in E(\overline{H})} d_{\overline{H}}(u) + d_{\overline{H}}(v) \geqslant ne(\overline{H}).$$

Using the inequality of Hofmeister [4], we obtain

$$n\mu^2(\overline{H}) \geqslant \sum_{u \in V(\overline{H})} d^2_{\overline{H}}(u) \geqslant ne(\overline{H}).$$

Since $\overline{H} \subset \overline{G}$, we have

$$\mu(\overline{H}) \leqslant \mu(\overline{G}) \leqslant \sqrt{n-1}$$

and so,

$$n(n-1) \geqslant n\mu^2(\overline{G}) \geqslant n\mu^2(\overline{H}) \geqslant ne(\overline{H}).$$

This easily gives $e(\overline{H}) \leqslant n - 1$ and

$$e(H) = \binom{n}{2} - e(\overline{H}) \geqslant \binom{n-1}{2}.$$

Since H has no Hamiltonian path, Fact 1 implies that $H = K_{n-1} + v$. If $G = H$, the proof is completed, so assume that G is a proper subgraph of $K_{n-1} + v$. Then $\overline{G}$ is a star $K_{1,n-1}$ of order n together with some additional edges; therefore $\overline{G}$ is connected. Hence, by the Perron–Frobenius theorem,

$$\mu(\overline{G}) > \mu(K_{1,n-1}) = \sqrt{n-1}$$

contradicting (4) and completing the proof for Hamiltonian paths.

Assume now that (5) holds but G has no Hamiltonian cycle. Using Fact 5 and arguing as above, we see that

$$e(H) > \binom{n-1}{2},$$

and since H has no Hamiltonian cycle, Fact 1 implies that $H = K_{n-1} + e$. If $G = H$, the proof is completed, so assume that G is a proper subgraph of $K_{n-1} + e$. Then $\overline{G}$ is a star $K_{1,n-2}$ together with some additional edges; therefore, $\overline{G}$ contains a connected proper supergraph of $K_{1,n-2}$. Hence, by the Perron-Frobenius theorem,

$$\mu(\overline{G}) > \mu(K_{1,n-2}) = \sqrt{n-2},$$

contradicting (5) and completing the proof.

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