Delta move and polynomial invariants of links

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Abstract

We investigate how a self-delta move, which is a delta move on the same component, influences the HOMFLY polynomial of a link. Then we reveal some relationships among finite type invariants, which are coming from the derivatives of the Jones polynomials and the first HOMFLY coefficient polynomials, of the four links involving in a self-delta move.

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1. Introduction

The HOMFLY polynomial, which is a topological invariant for an oriented link in $S^3$, is defined by a recursive formula involving a skein triple $(K_+, K_-, K_0)$ as shown in Fig. 1. The crossing change, that is, the move $K_+ \leftrightarrow K_-$, is an unknotting operation. The delta move as illustrated in Fig. 2 is another unknotting operation; see [15,18]. If further the three strands in each of the links in Fig. 2 belong to the same component, the move is called a self delta move; cf. [19–22,27,28]. In this paper, we consider the polynomial invariants of the four links involving a self-delta move.

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Let \( L, M, L_0, M_0 \) be four oriented links, which are identical except inside the depicted regions as illustrated in Fig. 3. We call \( (L, M, L_0, M_0) \) a \textit{delta skein quadruple}. If further the three strands in each of the links \( L, M, L_0 \) belong to the same component and those in \( M_0 \) belong to different components, then such a delta skein quadruple is called a \textit{self delta skein quadruple}; see Fig. 4, where the dotted arcs show how the strands connect. Thus in this case, \( M \) is obtained from \( L \) by a self-delta move, and vice versa, and if \( L \) and \( M \) have \( n \) components, then \( L_0 \) has \( n \) components and \( M_0 \) has \( n + 2 \) components. Note that the oriented delta moves given in Fig. 5(b) and (c) are realized by using the delta move given in Fig. 5(a); see [18, Fig. 1.1].

We give a relation among the HOMFLY polynomials of the delta skein quadruple (Theorem 3.1), which implies relations for the Jones and Conway polynomials of the delta-skein quadruple (Corollary 3.2) and a relation for the first HOMFLY coefficient polynomials of the self delta skein quadruple (Theorem 4.1). Since the coefficients of
the Conway polynomials, the derivatives of the Jones and the first HOMFLY coefficient polynomials are finite type invariants (or Vassiliev invariants) [1,2,29,30], we may have some relationships among these invariants of the (self) delta skein quadruple (Theorems 4.4–4.7). We also obtain a relation for the constant terms of the Q polynomials concerning a delta move of a knot (Corollary 4.9).

In Section 2, we give definitions of polynomial invariants of links and their properties, and explain finite type invariants derived from polynomial invariants. In Sections 3 and 4, we give relations among polynomial invariants and finite type invariants of the delta skein quadruple. In Section 5, we apply our results to twist knots.

2. Preliminaries

2.1. Polynomial invariants

The HOMFLY polynomial $P(L; t, z) \in \mathbb{Z}[t^\pm 1, z^\pm 1]$ is an invariant of the isotopy type of an oriented link $L$, which is defined, as in [8], by the following formulas:

\[
P(U; t, z) = 1,
\]

\[
t^{-1}P(K_+; t, z) - tP(K_-; t, z) = zP(K_0; t, z),
\]

where $U$ is a trivial knot and $(K_+, K_-, K_0)$ is a skein triple; see [5,26].

The Conway polynomial $\nabla_L(z) \in \mathbb{Z}[z]$ and the Jones polynomial $V(L; t) \in \mathbb{Z}[t^{\pm 1/2}]$ of an oriented link $L$ are given by the following formulas; see [4,8]:

\[
\nabla_L(z) = P(L; 1, z),
\]

\[
V(L; t) = P(L; t, t^{1/2} - t^{-1/2}).
\]

Let $L = K_1 \cup K_2 \cup \cdots \cup K_r$ be an oriented $r$-component link and $\text{Lk}(L)$ be the total linking number of $L$, $\text{Lk}(L) = \sum_{i<j} \text{lk}(K_i, K_j)$ with $\text{lk}(K_i, K_j)$ the linking number of $K_i$ and $K_j$. By [14, Proposition 22], the HOMFLY polynomial of $L$ is of the form

\[
P(L; t, z) = \sum_{n \geq 0} P_{2n-r+1}(L; t)z^{2n-r+1},
\]
where each \( P_{2n-r+1}(L; t) \in \mathbb{Z}[t^{\pm 1}] \) is called the coefficient polynomial; the powers of \( t \) which appear in it are all even or odd, depending on whether \( 2n - r + 1 \) is even or odd. In particular, the first coefficient polynomial has the following relation:

\[
P_{1-r}(L; t) = t^{2\text{Lk}(L)}(t^{-1} - t)^{r-1} \prod_{i=1}^{r} P_0(K_i; t) .
\]

(2.6)

Furthermore, the Conway polynomial of \( L \) is of the form

\[
\nabla_L(z) = \sum_{n \geq 0} a_{2n+r-1}(L) z^{2n+r-1} ,
\]

(2.7)

where \( a_{2n+r-1}(L) \in \mathbb{Z} \); cf. [12]. Note that \( P_{2n-r+1}(L; t) = 0 \) for \( 0 \leq n \leq r - 2 \).

2.2. Finite type invariants derived from polynomial invariants

Some finite type invariants of knots and links are derived from the polynomial invariants. In fact, the \( m \)th coefficient of the Conway polynomial of a link \( L \), \( a_m(L) \), is an order \( m \) invariant; the \( m \)th derivative of the Jones polynomial of a link \( L \) at \( t = 1, V^{(m)}(L; 1) \), is an order \( m \) invariant; the \( r \)th derivative at \( t = 1 \) of the \( q \)th coefficient polynomial of a link \( L \), \( P^{(q)}(L; 1) \), is an order \( q + r \) invariant; see [1, 2, 10, 29].

For \( V^{(m)}(L; 1), m = 0, 1, 2 \), we have the following equations; see [8, 16, 17]. For \( m = 3 \), such an equation has been given in [16].

**Proposition 2.1.** Let \( L = K_1 \cup K_2 \cup \cdots \cup K_n \) be an \( n \)-component link. Then

\[
V(L; 1) = (-2)^{n-1} ,
\]

(2.8)

\[
V^{(1)}(L; 1) = -3(-2)^{n-2} \text{Lk}(L) ,
\]

(2.9)

\[
V^{(2)}(L; 1) = (n - 1)(-2)^{n-3} + 3(-2)^{n-2} \text{Lk}(L) + 3(-2)^{n-1} \text{Lk}^2(L)
\]

\[
- 9(-2)^{n-2} \sum_{i<j, s<t, (i,j) \prec (s,t)} \text{lk}(K_i, K_j) \text{lk}(K_s, K_t) ,
\]

(2.10)

where \( \text{Lk}(L) = \sum_{i<j} \text{lk}(K_i, K_j) \), the total linking number of \( L \), \( \text{Lk}^2(L) = \sum_{i<j} \text{lk}(K_i, K_j)^2 \), and \( (i,j) \prec (s,t) \) means either \( i < s \) or \( i = s, j < t \).

Let \( \mathcal{V}_n \) be the space of the finite type invariants for knots of order less than or equal to \( n \). Then we have the following; see, for example, [10].

**Proposition 2.2.**

(i) \( \mathcal{V}_0 = \mathcal{V}_1 \), which consists of a constant map.

(ii) \( \mathcal{V}_2 / \mathcal{V}_1 \) is spanned by \( a_2(\cdot) \).
(iii) $V_3/V_2$ is spanned by $P_0^{(3)}(\cdot; 1)$.
(iv) $V_4/V_3$ is spanned by $\{a_2(\cdot)^2, a_4(\cdot), P_0^{(4)}(\cdot; 1)\}$, and $V_4$ is determined by the HOMFLY polynomial.

Proposition 2.1 gives relations among finite type invariants of links. Propositions 2.1 and 2.2 yield some relations among finite type invariants of knots; see [9,10].

Proposition 2.3. For a knot $K$, the following hold:

\begin{align*}
V^{(2)}(K; 1) &= -6a_2(K), \quad (2.11) \\
P_0^{(2)}(K; 1) &= -8a_2(K), \quad (2.12) \\
V^{(3)}(K; 1) &= \frac{3}{4}P_0^{(3)}(K; 1), \quad (2.13) \\
V^{(4)}(K; 1) &= -6a_2(K) - 72a_4(K) + \frac{3}{4}P_0^{(4)}(K; 1). \quad (2.14)
\end{align*}

Notice also that $P_0(K; 1) = 1$ and $P_0^{(1)}(K; 1) = 0$ for a knot $K$; see [14, Propositions 22 and 23]. Eq. (2.14) has been given in [10, (5.11)] in a wrong form. The correct one has been given in [9, Appendix].

3. Polynomials of a delta skein quadruple

In contrast with (2.2), we have the following.

Theorem 3.1. Let $(L, M, L_0, M_0)$ be a delta skein quadruple. Then

\begin{equation}
P(L; t, z) - P(M; t, z) = t^2z^2\left(P(L_0; t, z) - P(M_0; t, z)\right). \tag{3.1}
\end{equation}

Proof. Consider the skein tree as illustrated in Fig. 6. First, we have the two skein triples $(L, L_2, L_1)$ and $(M, M_2, M_1)$, and thus from (2.2), we obtain

\begin{align*}
t^{-1}P(L; t, z) - tP(L_2; t, z) &= zP(L_1; t, z), \quad (3.2) \\
t^{-1}P(M; t, z) - tP(M_2; t, z) &= zP(M_1; t, z). \quad (3.3)
\end{align*}

Since $L_2$ and $M_2$ are isotopic by the Reidemeister move, (3.2) and (3.3) imply

\begin{equation}
P(L; t, z) - P(M; t, z) = tz\left(P(L_1; t, z) - P(M_1; t, z)\right). \tag{3.4}
\end{equation}

Next, from the two skein triples $(L_1, L_3L_0)$ and $(M_1, L_3, M_0)$, we obtain

\begin{align*}
t^{-1}P(L_1; t, z) - tP(L_3; t, z) &= zP(L_0; t, z), \quad (3.5) \\
t^{-1}P(M_1; t, z) - tP(L_3; t, z) &= zP(M_0; t, z), \quad (3.6)
\end{align*}

which imply

\begin{equation}
P(L_1; t, z) - P(M_1; t, z) = tz\left(P(L_0; t, z) - P(M_0; t, z)\right). \tag{3.7}
\end{equation}
Substituting (3.7) into (3.4), we obtain (3.1), completing the proof.

By using (2.3) and (2.4), (3.1) implies the following formulas, where (3.9) has been given in [23, Lemma 3.3].

**Corollary 3.2.** Let \((L, M, L_0, M_0)\) be a delta skein quadruple. Then

\[
\nabla_L(z) - \nabla_M(z) = z^2(\nabla_{L_0}(z) - \nabla_{M_0}(z)), \tag{3.8}
\]

\[
V_L(t) - V_M(t) = t(t - 1)^2(V_{L_0}(t) - V_{M_0}(t)). \tag{3.9}
\]

Differentiating the both sides of (3.9) in Corollary 3.2, we have the following, which we will use in the next section.

**Corollary 3.3.** Let \((L, M, L_0, M_0)\) be a delta skein quadruple. Then

\[
V^{(m)}(L; 1) - V^{(m)}(M; 1) = m(m - 1)(V^{(m-2)}(L_0; 1) - V^{(m-2)}(M_0; 1)) + m(m - 1)(m - 2)(V^{(m-3)}(L_0; 1) - V^{(m-3)}(M_0; 1)). \tag{3.10}
\]
4. Polynomials of a self delta quadruple

In this section, we apply the formulas (3.1) and (3.8)–(3.10) obtained in the previous section to a self delta skein quadruple. We use the following notation through this section: Let \((L, M, L_0, M_0)\) denote a self delta skein quadruple such that \(L = J \cup J_1 \cup \cdots \cup J_r, M = K \cup J_1 \cup \cdots \cup J_r, L_0 = J_0 \cup J_1 \cup \cdots \cup J_r\) are \((r + 1)\)-component links and \(M_0 = K_1 \cup K_2 \cup K_3 \cup J_1 \cup \cdots \cup J_r\) is an \((r + 3)\)-component link, \(r \geq 0\), where \((J, K, J_0, K_0)\) is a self delta skein quadruple with \(K_0 = K_1 \cup K_2 \cup K_3\) a 3-component link. Also, we use the notation “\(\text{Lk}\)” and “\(\text{Lk}^{+}\)” given in Proposition 2.1.

4.1. HOMFLY polynomials of a self delta quadruple

Considering the coefficient polynomials of (3.1) in Theorem 3.1, we have
\[
P_m(L; t) - P_m(M; t) = t^2 \left( P_{m-2}(L_0; t) - P_{m-2}(M_0; t) \right)
\] (4.1)
for each \(m\). Although this holds for any delta skein quadruple, we consider for a self delta skein quadruple \((L, M, L_0, M_0)\). As a special case of this formula, we have the following.

**Theorem 4.1.**

\[
P_0(J; t) - P_0(K; t) = -t^{2\text{Lk}(K_0)} (t^2 - 1)^2 P_0(K_1; t) P_0(K_2; t) P_0(K_3; t).
\] (4.2)

**Proof.** If \(m = r = 0\), then (4.1) becomes \(P_0(J; t) - P_0(K; t) = t^2 (P_{-2}(J_0; t) - P_{-2}(K_0; t))\). Since \(J_0\) is a knot, \(P_{-2}(J_0; t) = 0\). From (2.6), we have \(P_{-2}(K_0; t) = t^{2\text{Lk}(K_0)} (t^{-1} - t)^2 P_0(K_1; t) P_0(K_2; t) P_0(K_3; t)\). Thus we obtain (4.2). \(\square\)

**Remark 4.2.** Since \(P_{-r-4}(L_0; t) = 0\), from (4.1) with \(m = -r - 2\) we have
\[
P_{-r}(L; t) - P_{-r}(M; t) = -t^2 P_{-r-2}(M_0; t).
\] (4.3)
Notice that \(L\) and \(M\) have the equal total linking numbers, which we denote by \(\mu\); \(\mu = \text{Lk}(L) = \text{Lk}(M)\). Then \(\text{Lk}(M_0) = \mu + \text{Lk}(K_0)\). From (2.5) and (2.6), we have
\[
P_{-r}(L; t) = t^{2\mu} (t^{-1} - t)^r P_0(J_0; t) \prod_{i=1}^r P_0(J_i; t),
\] (4.4)
\[
P_{-r}(M; t) = t^{2\mu} (t^{-1} - t)^r P_0(K_0; t) \prod_{i=1}^r P_0(J_i; t),
\] (4.5)
\[
P_{-r-2}(M_0; t) = t^{2\text{Lk}(K_0)+2\mu} (t^{-1} - t)^{r+2} P_0(K_1; t) P_0(K_2; t) P_0(K_3; t) \prod_{i=1}^r P_0(J_i; t).
\] (4.6)

Substituting these equations into (4.3), we obtain (4.2).

**Remark 4.3.** Using (2.2) and (2.6), we have
\[
t^{r-1} P_0(K_1; t) - t P_0(K; t) = t^{2\text{Lk}(K_0)} (t^{-1} - t) P_0(K_1; t) P_0(K_2; t);
\] (4.7)
see [13]. Compare this equation with (4.2).
4.2. Jones polynomials of a self delta quadruple

From (3.10), we have the following.

**Theorem 4.4.**

\[
V^{(2)}(L; 1) - V^{(2)}(M; 1) = 3(-2)^{r+1},
\]
\[
V^{(3)}(L; 1) - V^{(3)}(M; 1) = -9(-2)^r(3\text{Lk}(L_0) + 4\text{Lk}(K_0) + 2).
\]

**Proof.** Using (2.8) and (2.9) in (3.10) with \(m = 2\), we obtain (4.8).

Next, from (3.10), we have

\[
V^{(3)}(L; 1) - V^{(3)}(M; 1) = 6((-2)^r - (-2)^{r+1} \text{Lk}(L_0) - (-2)^{r+1} \text{Lk}(M_0)) + 6((-2)^r - (-2)^{r+2})
\]
\[
= 9(-2)^r(\text{Lk}(L_0) - 4\text{Lk}(M_0) - 2),
\]
(4.10)

where we use (2.9) and (2.8). Now,

\[
\text{Lk}(L_0) = \sum_{1 \leq j \leq r} \text{lk}(J_0, J_j) + \sum_{1 \leq i < j \leq r} \text{lk}(J_i, J_j),
\]
\[
\text{Lk}(M_0) = \sum_{1 \leq i \leq 3, 1 \leq j \leq r} \text{lk}(K_i, J_j) + \sum_{1 \leq i < j \leq r} \text{lk}(J_i, J_j) + \text{Lk}(K_0).
\]

Since \(\text{lk}(J_0, J_j) = \sum_{1 \leq i \leq 3} \text{lk}(K_i, J_j)\) for each \(j\), we have

\[
\text{Lk}(L) = \text{Lk}(M) = \text{Lk}(L_0) = \text{Lk}(M_0) - \text{Lk}(K_0).
\]
(4.11)

Using this, (4.10) yields (4.9). \(\square\)

**Theorem 4.5.**

\[
V^{(4)}(J; 1) - V^{(4)}(K; 1) = 72 \left(-a_2(J_0) + 4 \sum_{i=1}^3 a_2(K_1) - 2\text{Lk}^2(K_0) - 3a_2(K_0) - \text{Lk}(K_0)\right) - 24.
\]
(4.12)

Notice that

\[
a_2(K_0) = \text{lk}(K_1, K_2) \text{lk}(K_1, K_3) + \text{lk}(K_1, K_2) \text{lk}(K_2, K_3)
\]
\[+ \text{lk}(K_1, K_3) \text{lk}(K_2, K_3),
\]
(4.13)

which is due to Hoste [7]; cf. [11, Section 8]. Thus we have

\[
\text{Lk}(K_0)^2 = \text{Lk}^2(K_0) + 2a_2(K).
\]
(4.14)

**Proof.** From (3.10), we have
\[ V^{(4)}(J; 1) - V^{(4)}(K; 1) = 12(V^{(2)}(J_0; 1) - V^{(2)}(K_0; 1)) + 24(V^{(1)}(J_0; 1) - V^{(1)}(K_0; 1)). \]

By using (2.10) and (2.9), this becomes
\[ 12 \left( -6a_2(J_0) - 2 - 3(-2) \text{Lk}(K_0) - 3(-2)^3 \sum_{i=1}^{3} a_2(K_i) \right. \]
\[ \left. - 3(-2)^2 \text{Lk}^2(K_0) + 9(-2)a_2(K_0) \right) + 24(-3)(-2) \text{Lk}(K_0), \]
which implies (4.12). \(\Box\)

### 4.3. Conway polynomials of a self delta skein quadruple

Putting \(t = 1\) in (4.1), we have
\[ a_m(L) - a_m(M) = a_{m-2}(L_0) - a_{m-2}(M_0). \tag{4.15} \]
For a self skein quadruple, this yields the following, where (4.16) has been given by Nakanishi [19] as a generalization of Okada’s formula (4.17) below.

**Theorem 4.6.**
\[ a_{r+2}(L) - a_{r+2}(M) = a_r(L_0). \tag{4.16} \]

### 4.4. Finite type knot invariants of a self delta skein quadruple

We summarize the relations of small order finite type invariants given as a basis in Proposition 2.2 for a self delta skein quadruple \((J, K, J_0, K_0)\). In the following, (4.17) is due to Okada [25] and (4.21) (equivalently, (4.18)) is due to Nikkuni [23,24].

**Theorem 4.7.**
\[ a_2(J) - a_2(K) = 1. \tag{4.17} \]
\[ P^{(3)}_0(J; 1) - P^{(3)}_0(K; 1) = -24(2 \text{Lk}(K_0) + 1). \tag{4.18} \]
\[ a_4(J) - a_4(K) = a_2(J_0) - a_2(K_0). \tag{4.19} \]
\[ P^{(4)}_0(J; 1) - P^{(4)}_0(K; 1) = 24 \left( 16 \sum_{i=1}^{3} a_2(K_i) - 8 \text{Lk}(K_0)^2 - 4 \text{Lk}(K_0) - 1 \right). \tag{4.20} \]

**Proof.** As special cases of (4.8), we obtain (4.17) and (4.19); (4.17) is also obtained from (4.8) with \(r = 0\) by using (2.11).

Next, from (4.9) with \(r = 0\), we have
\[ V^{(3)}(J; 1) - V^{(3)}(K; 1) = -18(2 \text{Lk}(K_0) + 1), \tag{4.21} \]
which yields (4.18) by using (2.13).
Lastly, using (2.14), we have
\[ V^{(4)}(J; 1) - V^{(4)}(K; 1) = -6(a_2(J) - a_2(K)) - 72(a_4(J) - a_4(K)) + \frac{3}{4}(P^{(4)}(J; 1) - P^{(4)}(K; 1)) \]
\[ = -6 - 72(a_2(J_0) - a_2(K_0)) + \frac{3}{4}(P^{(4)}(J; 1) - P^{(4)}(K; 1)), \tag{4.22} \]
where we use (4.17) and (4.19). By combining this with (4.12) in Theorem 4.5, we obtain
\[ P^{(4)}(J; 1) - P^{(4)}(K; 1) = 8 + 96(a_2(J_0) - a_2(K_0)) + 96(-a_2(J_0) - \text{Lk}(K_0) + 4 \sum_{i=1}^{3} a_2(K_i) - 2\text{Lk}^2(K_0) - 3a_2(K_0)) - 32 \]
\[ = 96(4 \sum_{i=1}^{3} a_2(K_i) - \text{Lk}(K_0) - 2\text{Lk}^2(K_0)^2) - 24, \tag{4.23} \]
where we use (4.14). This completes the proof. □

**Remark 4.8.** Differentiating the both sides of (4.2) in Theorem 4.1, we obtain
\[ P^{(2)}(J; 1) - P^{(2)}(K; 1) = -8, \tag{4.24} \]
(4.18), and (4.20). Again, we obtain (4.17) from (4.24) by using (2.12).

### 4.5. Q polynomials of a self delta quadruple

The \( Q \) polynomial \( Q(M; x) \in \mathbb{Z}[x^{\pm 1}] \) is an invariant of an unoriented link \( M \) defined by the following formulas:
\[ Q(U; x) = 1, \tag{4.25} \]
\[ Q(M_+; x) + Q(M_-; x) = x(Q(M_0; x) + Q(M_{\infty}; x)). \tag{4.26} \]
where \( U \) is the unknot and \( (M_+, M_-, M_0, M_{\infty}) \) are four links which are identical except inside the depicted regions as illustrated in Fig. 7; see [3,6].

It is known that
\[ Q(K; 0) = P_0(K; \sqrt{-1}) \equiv 1 \pmod{4} \tag{4.27} \]
for a knot \( K \); see [3, Property 7], [13, Theorem 4.12(i)]. Then by (4.2) in Theorem 4.1 we have the following corollary.
Corollary 4.9. Let $J$ and $K$ be two knots such that $K$ is obtained from $J$ by a single delta move. Then
\[ Q(J; 0) \neq Q(K; 0). \quad (4.28) \]
In particular, a non-trivial knot $K$ with $Q(K; 0) = 1$ does not become unknotted by a single delta move.

5. Example

As mentioned in Section 1, the oriented delta move $L \leftrightarrow M$, where $L$ and $M$ are as in Fig. 3, is an unknotting operation. That is, for any knot $K$, there is a sequence of knots:
\[ K = K^0 \xrightarrow{\Delta_1} K^1 \xrightarrow{\Delta_2} \ldots \xrightarrow{\Delta_{m-1}} K^{m-1} \xrightarrow{\Delta_m} K^m = U, \]
where $K^{i-1} \xrightarrow{\Delta_i} K^i$ means that $K^i$ is obtained from $K^{i-1}$ by a single delta move $\Delta_i$, $i = 1, 2, \ldots, m$, and $U$ denotes a trivial knot. For a skein delta quadruple $(L, M, L^0, M^0)$, we define the sign of the oriented delta move $\Delta$, $\varepsilon(\Delta)$, as follows:
\[ \varepsilon(\Delta) = \begin{cases} 
1 & \text{if } L \xrightarrow{\Delta} M, \\
-1 & \text{if } M \xrightarrow{\Delta} L.
\end{cases} \quad (5.1) \]

For each $i$, we obtain a knot $J^i_0$ and a 3-component link $K^i_0$ by the move $\Delta_i$, that is, we obtain a skein delta quadruple $(K^{i-1}, K^i, J^i_0, K^i_0)$ or $(K^i, K^{i-1}, J^i_0, K^i_0)$ according as if $\varepsilon(\Delta_i) = 1$ or $\varepsilon(\Delta_i) = -1$. Then we have the following; cf. [12, Chapter III].

Theorem 5.1.
\[ a_2(K) = \sum_{i=1}^{m} \varepsilon(\Delta_i), \quad (5.2) \]
\[ P_0^{(3)}(K; 1) = -48 \sum_{i=1}^{m} \varepsilon(\Delta_i) \text{Lk}(K^i_0) - 24a_2(K). \quad (5.3) \]

Proof. From (4.17) and (4.18), we have
\[ a_2(K^{i-1}) - a_2(K^i) = \varepsilon(\Delta_i), \quad (5.4) \]
\[ P_0^{(3)}(K^{i-1}; 1) - P_0^{(3)}(K^i, 1) = -24 \varepsilon(\Delta_i)(2 \text{Lk}(K^i_0) + 1). \quad (5.5) \]
Fig. 8.

which yield (5.2) and

\[ P_0^{(3)}(K; 1) = -24 \sum_{i=1}^{m} \varepsilon(\Delta_i)(2 \text{Lk}(K_i^0) + 1), \]  

(5.6)

respectively. Together (5.2) and (5.6) yield (5.3). □

We apply Theorems 5.1 and 4.1 to twist knots; see [25, Example 2.1]. Let \( T_m, m \geq 0, \) be the twist knot as illustrated in Fig. 8. If \( m \) is odd, then we have a delta skein quadruple \((T_{2i-1}, T_{2i-3}, J_{2i-1}, K_{2i-3})\), where \( J_{2i-1} \) is a trivial knot and \( K_{2i-3} \) is a split union of torus link of type \((2, 2i - 2)\) with linking number \( i - 1 \) and a trivial knot; see Fig. 9. Then we have the following sequence of twist knots:

\[ T_{2n-1} \Delta_n \rightarrow T_{2n-3} \Delta_{n-1} \rightarrow \cdots \rightarrow T_1 \Delta_1 \rightarrow U. \]

Since \( \varepsilon(\Delta_i) = 1 \) and \( \text{Lk}(K_{2i-3}) = i - 1 \), we have the following by (5.2) and (5.3):

\[ a_2(T_{2n-1}) = n, \]

(5.7)

\[ P_0(T_{2n-1}; 1) = -48 \sum_{i=1}^{n} (i - 1) - 24n = -24n^2. \]

(5.8)

Moreover, since each component of \( K_{2i-3} \) is a trivial knot, by (4.2) in Theorem 4.1, we have

\[ P_0(T_{2i-1}; t) - P_0(T_{2i-3}; t) = -t^{2(i-1)}(t^2 - 1)^2, \]

(5.9)

and thus

\[ P_0(T_{2n-1}; t) = 1 + \sum_{i=1}^{n} (-t^{2(i-1)})(t^2 - 1)^2 = t^2 + t^{2n} - t^{2n+2}. \]

(5.10)

Next, if \( m \) is even, then we have a delta skein quadruple \((T_{2i-2}, T_{2i}, J_{2i}, K_{2i-2})\), where \( J_{2i-1} \) is a twist knot \( T_{2i-2} \) and \( K_{2i-2} \) is a connected sum of torus link of type \((2, 2i - 2)\) with linking number \( i - 1 \) and the Hopf link with linking number \(-1\); see Fig. 10. Then we have the following sequence of twist knots:

\[ T_{2n} \Delta_n \rightarrow T_{2n-2} \Delta_{n-1} \rightarrow \cdots \rightarrow T_2 \Delta_1 \rightarrow U, \]

Since \( \varepsilon(\Delta_i) = -1 \) and \( \text{Lk}(K_{2i-2}) = i - 2 \), we have the following by (5.2) and (5.3):
For all $n$, we have
\[ a_2(T_{2n}) = -n. \]  
\[ \beta_0^{(3)}(T_{2n}; 1) = 48 \sum_{i=1}^{n} (i - 2) + 24n = 24n(n - 2). \]  

Moreover, since each component of $K_{2i-2}$ is a trivial knot, by (4.2) in Theorem 4.1, we have
\[ \beta_0(T_{2i-2}; t) - \beta_0(T_{2i}; t) = -t^{2(i-2)}(t^2 - 1)^2, \]  
and thus
\[ \beta_0(T_{2n}; t) = 1 + \sum_{i=1}^{n} t^{2(i-2)}(t^2 - 1)^2 = t^{-2} - t^{2n-2} + t^{2n}. \]
References