An analogue of the zeta function and its applications

Cheon Seoung Ryoo\textsuperscript{a,}\textsuperscript{*}, Taekyun Kim\textsuperscript{b}

\textsuperscript{a}Department of Mathematics, Hannam University, Daejeon 306-791, Republic of Korea
\textsuperscript{b}Institute of Science Education, Kongju National University, Kongju 314-701, Republic of Korea

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Abstract

In this work we construct new analogues of Bernoulli numbers and polynomials. We define the $q$-extension of zeta function. Finally we give relation between the $q$-extension of zeta functions and the $(h, q)$-extension of Bernoulli polynomials.

Keywords: Bernoulli numbers and polynomials; $q$-Bernoulli numbers and polynomials; Zeta functions

1. Introduction

Using the $q$-Volkenborn integration and uniform differentiation on $\mathbb{Z}_p$, we construct $p$-adic $q$-zeta functions. These functions interpolate the $q$-Bernoulli numbers and $q$-Bernoulli polynomials. The values of $p$-adic $q$-zeta functions at negative integers are given; cf. [1–5]. The main purpose of this work is to consider the new analogues of Bernoulli numbers and polynomials. Finally we consider the $q$-zeta functions which interpolate the $(h, q)$-extension of Bernoulli numbers and polynomials. Throughout this work $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will be denoted by the ring of rational integers, the ring of $p$-adic integers, the field of $p$-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively; cf. [1–5]. Let $v_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = p^{-1}$. When one talks of $q$-extension, $q$ is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, and a $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}_p$, then we normally assume $|q - 1|_p < p^{-\frac{1}{p^1}}$, so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. If $q \in \mathbb{C}$, then we normally assume that $|q| < 1$. For $f \in UD(\mathbb{Z}_p, \mathbb{C}_p) = \{f : \mathbb{Z}_p \to \mathbb{C}_p \text{ is a uniformly differentiable function}\}$, the $p$-adic $q$-integral (or $q$-Volkenborn integration) is defined as

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{p^N q^1} \sum_{x=0}^{p^N-1} f(x) q^x,$$

$$1$$

\textsuperscript{*}Corresponding author.

E-mail addresses: ryooecs@hannam.ac.kr (C.S. Ryoo), tkim@kongju.ac.kr (T. Kim).

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where \([x]_q = \frac{1-q^x}{1-q}\); cf. [1–5]. Thus we note that

\[
I_1(f) = \lim_{q \to 1} I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_1(x) = \lim_{N \to \infty} \frac{1}{p^N} \sum_{0 \leq x < p^N} f(x), \text{ cf. [3]}.
\]  

(2)

By (2), we easily see that

\[
I_1(f_1) = I_1(f) + f'(0), \text{ cf. [4]},
\]

where \(f_1(x) = f(x + 1), f'(0) = \frac{d}{dx} f(x)|_{x=0}\).

In [5], the \(q\)-Bernoulli polynomials are defined by

\[
B_n^{(h)}(x, q) = \int_{\mathbb{Z}_p} [x + x_1]_q q^{x_1(h-1)}d\mu_q(x_1), \quad \text{for } h \in \mathbb{Z}.
\]  

(4)

2. An analogue of Bernoulli numbers and polynomials

In (3), if we take \(f(x) = q^{hx} e^{tx}\), then we have

\[
\int_{\mathbb{Z}_p} q^{hx} e^{tx}d\mu_1(x) = \frac{h \log q + t}{q^h e^t - 1}
\]

(5)

for \(|t| \leq p^{-\frac{1}{p-1}}, h \in \mathbb{Z}.

Let us define the \((h, q)\)-extension of Bernoulli numbers and polynomials as follows:

\[
F_q^{(h)}(t) = \frac{h \log q + t}{q^h e^t - 1} = \sum_{n=0}^{\infty} B_{n,q}^{(h)} \frac{t^n}{n!},
\]

\[
F_q^{(h)}(t, x) = \frac{h \log q + t}{q^h e^t - 1} e^{tx} = \sum_{n=0}^{\infty} B_{n,q}^{(h)}(x) \frac{t^n}{n!}.
\]  

(6)

Note that \(B_{n,q}^{(h)}(0) = B_{n,q}^{(h)}, \lim_{q \to 1} B_{n,q}^{(h)} = B_n\), where \(B_n\) are the \(n\)-th Bernoulli numbers. By (5) and (6), we obtain the following Witt formula.

**Theorem 1.** For \(h \in \mathbb{Z}, q \in \mathbb{C}_p\) with \(|1 - q|_p \leq p^{-\frac{1}{p-1}}\), we have

\[
\int_{\mathbb{Z}_p} q^{hx} x^n d\mu_1(x) = B_{n,q}^{(h)},
\]

\[
\int_{\mathbb{Z}_p} q^{hy}(x+y)^n d\mu_1(y) = B_{n,q}^{(h)}(x).
\]  

(7)

By the above theorem, we easily see that

\[
B_{n,q}^{(h)}(x) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} B_{k,q}^{(h)}.
\]  

(8)

Let \(d\) be any fixed positive integer with \((p, d) = 1\). Then we set

\[
X = X_d = \lim_{N \to \infty} (\mathbb{Z}/dp^N\mathbb{Z}), X_1 = \mathbb{Z}_p,
\]

\[
X^* = \bigcup_{0 < a < dp} a + dp\mathbb{Z}_p,
\]

\[
a + dp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^N}\},
\]
where \( a \in \mathbb{Z} \) with \( 0 \leq a < dp^N \). Note that
\[
\int_{\mathbb{Z}_p} f(x) d\mu_1(x) = \int_X f(x) d\mu_1(x),
\]
for \( f \in UD(\mathbb{Z}_p, \mathbb{C}_p) \); cf. [1–5].

By (12) and (9), we can give the generation function of
\[
F_{q, \chi}^{(h)}(t) = \sum_{n=0}^{\infty} \frac{(te^{it})^n}{q^{hd}e^{dt} - 1} = \sum_{n=0}^{\infty} B_{n, q, \chi}^{(h)} \frac{t^n}{n!}.
\]

Therefore we have the theorem below.

**Theorem 2.** For any positive integer \( m \), we obtain
\[
B_{k, q}^{(h)}(x) = m^{k-1} \sum_{i=0}^{m-1} q^{hi} B_{k, q}^{(h)} \left( \frac{x + i}{m} \right), \quad \text{for } k \geq 0.
\]

Let \( \chi \) be the Dirichlet character with conductor \( d \in \mathbb{Z}_+ \). Then we define the \((h, q)\)-extension of generalized Bernoulli numbers attached to \( \chi \). For \( n \geq 0 \), define
\[
B_{n, q, \chi}^{(h)} = \int_X \chi(x) q^{hx} x^n d\mu_1(x).
\]

By (9), we easily see that
\[
B_{n, q, \chi}^{(h)} = \lim_{l \to \infty} \frac{1}{dp^l} \sum_{i=0}^{dp^l-1} \chi(x) q^{hx} x^n = \frac{1}{d} \lim_{l \to \infty} \frac{1}{p^l} \sum_{i=0}^{p^l-1} \chi(i + dx) q^{h(i + dx)}(i + dx)^n
\]
\[
= \frac{1}{d} \sum_{i=0}^{d-1} \chi(i) q^{hi} \int_{\mathbb{Z}_p} q^{hdx}(i + dx)^n d\mu_1(x) = d^{n-1} \sum_{i=0}^{d-1} \chi(i) q^{hi} B_{n, q}^{(h)} \left( \frac{i}{d} \right).
\]

Therefore we obtain the lemma below.

**Lemma 3.** For \( d \in \mathbb{Z}_+ \), we have
\[
B_{k, q, \chi}^{(h)} = d^{k-1} \sum_{i=0}^{d-1} \chi(i) q^{hi} B_{k, q}^{(h)} \left( \frac{i}{d} \right), \quad \text{for } n \geq 0.
\]

By induction in Eq. (3), we easily see that
\[
I_1(f_b) = I_1(f) + \sum_{i=0}^{b-1} f'(i), \quad \text{where } f_b(x) = f(x + b), \quad b \in \mathbb{Z}_+.
\]

In Eq. (11), if we take \( f(x) = q^{hx}e^{tx}\chi(x) \), then we have
\[
I_1(e^{tx} q^{hx}\chi(x)) = \frac{\sum_{i=0}^{d-1} (te^{it}\chi(i) q^{hi} + e^{iti}(h \log q) q^{hi} \chi(i))}{q^{hd}e^{dt} - 1}.
\]

By (12) and (9), we can give the generation function of \( B_{n, q, \chi}^{(h)} \) as follows:
3. The analogue of the zeta function

In this section we assume that \( q \in \mathbb{C} \) with \( |q| < 1 \). Let \( \Gamma(s) \) be the gamma function. By (6), we can readily see that

\[
\frac{1}{\Gamma(s)} \int_0^\infty t^{s-2} e^{-t} F_q^{(h)}(-t) dt = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-2} e^{-t} \left\{ \frac{-t}{q^h e^{-t} - 1} + \frac{h \log q}{q^h e^{-t} - 1} \right\} dt
\]

\[
= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-t} \frac{1}{1 - q^h e^{-t}} dt - \frac{h \log q}{\Gamma(s)} \int_0^\infty t^{s-2} e^{-t} \frac{1}{1 - q^h e^{-t}} dt
\]

\[
= \sum_{n=0}^{\infty} q^{nh} \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-(n+1)t} dt - h \log q \sum_{n=0}^{\infty} q^{nh} \int_0^\infty t^{s-1} e^{-(n+1)t} dt
\]

\[
= \sum_{n=1}^{\infty} \frac{q^{(n-1)h}}{n^s} - \frac{\log q}{s - 1} \sum_{n=1}^{\infty} \frac{q^{(n-1)h}}{n^{s-1}}.
\]

Using (14), we define the new \( q \)-extensions of zeta functions as follows:

**Definition 4.** For \( s \in \mathbb{C}, x \in \mathbb{R}^+ \), we define

\[
\zeta_q^{(h)}(s) = \sum_{n=1}^{\infty} \frac{q^{(n-1)h}}{n^s} - \frac{h \log q}{s - 1} \sum_{n=1}^{\infty} \frac{q^{(n-1)h}}{n^{s-1}}.
\]

\[
\zeta_q^{(h)}(s, x) = \sum_{n=0}^{\infty} \frac{q^{nh}}{(n+x)^s} - \frac{h \log q}{s - 1} \sum_{n=0}^{\infty} \frac{q^{nh}}{(n+x)^{s-1}}.
\]

**Remark.** By (15) and (16), we easily see that \( \zeta_q^{(h)}(s) = \zeta_q^{(h)}(s, 1) \). Also, we note that \( \zeta_q^{(h)}(s) \) is an analytic continuation for \( R(s) > 1 \).

Using Mellin transforms in Eq. (6), we obtain

\[
\frac{1}{\Gamma(s)} \int_0^\infty t^{s-2} e^{-t} F_q^{(h)}(-t, x) dt = \zeta_q^{(h)}(s, x).
\]

By (17) and (6), we readily see that

\[
\zeta_q^{(h)}(s, x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} B_{n,q}^{(h)}(x) \frac{1}{\Gamma(s)} \int_0^\infty t^{s-2+n} dt.
\]

Therefore we obtain the following:

**Theorem 5.** For \( n \in \mathbb{N} \), we have

\[
\zeta_q^{(h)}(1 - n, x) = -\frac{B_{n,q}^{(h)}(x)}{n}.
\]

By Mellin transforms and (13), we note that

\[
\frac{1}{\Gamma(s)} \int_0^\infty F_q^{(h)}(-t) t^{s-2} dt = \sum_{n=1}^{\infty} \frac{q^{nh}\chi(n)}{n^s} - \frac{h \log q}{s - 1} \sum_{n=1}^{\infty} \frac{q^{nh}\chi(n)}{n^{s-1}}.
\]

Thus we can define the new \( q \)-extension of Dirichlet \( L \)-function as follows:

**Definition 6.** For \( s \in \mathbb{C} \), we define

\[
L_q^{(h)}(s, \chi) = \sum_{n=1}^{\infty} \frac{q^{nh}\chi(n)}{n^s} - \frac{h \log q}{s - 1} \sum_{n=1}^{\infty} \frac{q^{nh}\chi(n)}{n^{s-1}}.
\]
By (18) and (13), we have the following:

**Theorem 7.** For \( n \in \mathbb{N} \), we obtain

\[
L_q^{(h)}(1 - n, \chi) = -\frac{B_{n,q,\chi}^{(h)}}{n}.
\]

**References**