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The Terwilliger algebra of an almost-bipartite P- and Q-polynomial association scheme

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Abstract

Let Y denote a D-class symmetric association scheme with $D \geqslant 3$, and suppose Y is almost-bipartite P- and Q-polynomial. Let x denote a vertex of Y and let $T = T(x)$ denote the corresponding Terwilliger algebra. We prove that any irreducible T-module W is both thin and dual thin in the sense of Terwilliger. We produce two bases for W and describe the action of T on these bases. We prove that the isomorphism class of W as a T-module is determined by two parameters, the dual endpoint and diameter of W. We find a recurrence which gives the multiplicities with which the irreducible T-modules occur in the standard module. We compute this multiplicity for those irreducible T-modules which have diameter at least $D - 3$.

Keywords: Association scheme; Distance-regular graph; Almost-bipartite; Terwilliger algebra; Subconstituent algebra

1. Introduction

The Terwilliger algebra of a commutative association scheme was introduced in [23]. This algebra is a finite-dimensional, semisimple $\mathbb{C}$-algebra, and is noncommutative in general. The Terwilliger algebra has been used to study P- and Q-polynomial schemes [5,23], group schemes [1,3], strongly regular graphs [27], Doob schemes [21], and schemes over

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the Galois rings of characteristic four [17]. Other work involving this algebra can be found in [6–9,11,13–16,18,25,26,28].

The Terwilliger algebra is particularly well-suited for studying P- and Q-polynomial schemes; nevertheless, it is apparent from [23] that the intersection numbers of these schemes do not completely determine the structure of the algebra. In this article, we consider the Terwilliger algebra of an almost-bipartite P- and Q-polynomial scheme. We show that with the added almost-bipartite assumption, the intersection numbers of the scheme completely determine the structure of the algebra.

To describe our results, let $Y=(X,\{R_i\}_{0\leqslant i\leqslant D})$ denote a symmetric association scheme with $D\geqslant 3$. Suppose Y is almost-bipartite P- and Q-polynomial. Fix any $x\in X$, and let $T=T(x)$ denote the Terwilliger algebra of Y with respect to x. T acts faithfully on the vector space $\mathbb{C}^X$ by matrix multiplication; we refer to $\mathbb{C}^X$ as the *standard module*. Since T is semisimple, $\mathbb{C}^X$ decomposes into a direct sum of irreducible T-modules.

Let W denote an irreducible T-module contained in $\mathbb{C}^X$. We show that W is thin and dual thin in the sense of Terwilliger (Lemma 10.3). We produce two bases for W with respect to which the action of T is particularly simple (Theorem 8.1). To describe this action, we use two sets of scalars, the *intersection numbers* of W and the *dual intersection numbers* of W. We compute these scalars in terms of the eigenvalues of Y, the dual eigenvalues of Y, and two additional parameters, called the *dual endpoint* and *diameter* of W (Theorems 11.4, 12.4). We show that the dual endpoint and diameter of W determine its isomorphism class as a T-module (Theorem 14.1).

Combining our above results, we find a recurrence which gives the multiplicities with which the irreducible T-modules occur in $\mathbb{C}^X$ (Theorem 16.6). We compute this multiplicity for those irreducible T-modules which have diameter at least $D-3$. (Example 16.8).

In a future paper, we intend to use these results to study the subconstituents of an almost-bipartite P- and Q-polynomial scheme. We hope this will produce a classification of these schemes. Our work is closely related to that of Curtin concerning 2-thin distance-regular graphs [8].

2. Association schemes

Definition 2.1. By a *symmetric association scheme* (or *scheme* for short) we mean a pair $Y=(X,\{R_i\}_{0\leqslant i\leqslant D})$, where X is a nonempty finite set, D is a nonnegative integer, and $R_0,\ldots,R_D$ are nonempty subsets of $X\times X$ such that

(i) $\{R_i\}_{0\leqslant i\leqslant D}$ is a partition of $X\times X$;
(ii) $R_0=\{xx \mid x\in X\}$;
(iii) $R_i^{\mathrm{t}}=R_i$ for $0\leqslant i\leqslant D$, where $R_i^{\mathrm{t}}=\{yx \mid xy\in R_i\}$;
(iv) For all h,i,j $(0\leqslant h,i,j\leqslant D)$, and for all $x,y\in X$ such that $xy\in R_h$, the scalar

$$p^h_{ij}=|\{z\in X|xz\in R_i,\text{ and } zy\in R_j\}|$$

is independent of x,y.

The constants p^h_{ij} are called the *intersection numbers of* Y.

For the rest of this section, let $Y=(X,\{R_i\}_{0\leqslant i\leqslant D})$ denote a scheme. We begin with a few comments about the intersection numbers of Y. For all integers i $(0\leqslant i\leqslant D)$, set $k_i := p^0_{ii}$, and note that $k_i\neq 0$ since R_i is nonempty. We refer to k_i as the ith *valency* of Y. Observe that $p^0_{ij}=\delta_{ij}k_i$ $(0\leqslant i,j\leqslant D)$.

We now recall the Bose–Mesner algebra of Y. Let $\mathrm{Mat}_X(\mathbb{C})$ denote the $\mathbb{C}$-algebra of matrices with entries in $\mathbb{C}$, where the rows and columns are indexed by X. For each integer i $(0\leqslant i\leqslant D)$, let A_i denote the matrix in $\mathrm{Mat}_X(\mathbb{C})$ with xy-entry

$$(A_i)_{xy}=\begin{cases}1 & \text{if } xy\in R_i\\ 0 & \text{if } xy\notin R_i\end{cases}\qquad (x,y\in X). \tag{1}$$

We refer to A_i as the ith *associate matrix of* Y. By Definition 2.1, the associate matrices satisfy: (i) $A_0=I$, where I is the identity matrix in $\mathrm{Mat}_X(\mathbb{C})$; (ii) the conjugate-transpose $\overline{A}_i^{\mathrm{t}}=A_i$ $(0\leqslant i\leqslant D)$; (iii) $A_0+A_1+\cdots+A_D=J$, where J is the all 1's matrix in $\mathrm{Mat}_X(\mathbb{C})$; (iv) $A_iA_j=\sum_{h=0}^{D}p^h_{ij}A_h$ $(0\leqslant i,j\leqslant D)$.

It follows from (i)–(iv) that $A_0,\ldots,A_D$ form a basis for a subalgebra M of $\mathrm{Mat}_X(\mathbb{C})$. M is known as the *Bose–Mesner algebra of* Y. Observe that M is commutative, since the associate matrices are symmetric.

By Brouwer et al. [4, p. 45], the algebra M has a second basis $E_0,\ldots,E_D$ satisfying: (i) $E_0=|X|^{-1}J$; (ii) $\overline{E}_i^{\mathrm{t}}=E_i$ $(0\leqslant i\leqslant D)$; (iii) $E_iE_j=\delta_{ij}E_i$ $(0\leqslant i,j\leqslant D)$; (iv) $E_0+E_1+\cdots+E_D=I$. We refer to E_i as the ith *primitive idempotent of* Y for $0\leqslant i\leqslant D$. For convenience we define $E_i:=0$ for $i>D$ and $i<0$.

For all integers i $(0\leqslant i\leqslant D)$, set $m_i:=\mathrm{rank}(E_i)$, and note that $m_i\neq 0$. We refer to m_i as the ith *multiplicity of* Y.

Since $A_0,\ldots,A_D$ and $E_0,\ldots,E_D$ are both bases for M, there exist complex scalars $p_i(j),q_i(j)$ $(0\leqslant i,j\leqslant D)$ which satisfy

$$A_i=\sum_{j=0}^{D}p_i(j)E_j\qquad (0\leqslant i\leqslant D), \tag{2}$$

$$E_i=|X|^{-1}\sum_{j=0}^{D}q_i(j)A_j\qquad (0\leqslant i\leqslant D). \tag{3}$$

By Bannai and Ito [2, pp. 59, 63], the $p_i(j),q_i(j)$ are real. We refer to $p_i(j)$ (resp. $q_i(j)$) as the jth *eigenvalue* (resp. jth *dual eigenvalue*) associated with A_i (resp. E_i). By Bannai and Ito [2, p. 63], the eigenvalues and dual eigenvalues satisfy

$$\frac{p_i(j)}{k_i}=\frac{q_j(i)}{m_j}\qquad (0\leqslant i,j\leqslant D).$$

We now recall the Krein parameters of Y. Observe that $A_i\circ A_j=\delta_{ij}A_i$ $(0\leqslant i,j\leqslant D)$, where $\circ$ denotes the entry-wise matrix product. It follows that M is closed under $\circ$, so there exist complex scalars q^h_{ij} satisfying $E_i\circ E_j=|X|^{-1}\sum_{h=0}^{D}q^h_{ij}E_h$ $(0\leqslant i,j\leqslant D)$. The constants q^h_{ij} are called the *Krein parameters* of Y. By Bannai and Ito [2, pp. 67–69], the Krein parameters are real, and $q^0_{ij}=\delta_{ij}m_i$ $(0\leqslant i,j\leqslant D)$.

We now recall the dual Bose–Mesner algebra of Y. For the rest of this section, fix any $x \in X$. For each integer i $(0 \leqslant i \leqslant D)$, let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with yy-entry

$$(E_i^*)_{yy} = \begin{cases} 1 & \text{if } xy \in R_i \\ 0 & \text{if } xy \notin R_i \end{cases} \quad (y \in X). \tag{4}$$

We refer to E_i^* as the ith *dual idempotent of Y with respect to x*. For convenience, set $E_i^* := 0$ if $i > D$ or $i < 0$. From the definition, the dual idempotents satisfy: (i) $\overline{E_i^*}^{\mathrm{t}} = E_i^*$ $(0 \leqslant i \leqslant D)$; (ii) $E_i^* E_j^* = \delta_{ij} E_i^*$ $(0 \leqslant i, j \leqslant D)$; (iii) $E_0^* + E_1^* + \cdots + E_D^* = I$.

It follows that the matrices $E_0^*, \ldots, E_D^*$ form a basis for a subalgebra $M^* = M^*(x)$ of $\mathrm{Mat}_X(\mathbb{C})$. M^* is known as the *dual Bose–Mesner algebra of Y with respect to x*. Observe that M^* is commutative since the dual idempotents are diagonal.

For each integer i $(0 \leqslant i \leqslant D)$, let $A_i^* = A_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with yy-entry

$$(A_i^*)_{yy} = |X|(E_i)_{xy} \quad (y \in X). \tag{5}$$

We refer to A_i^* as the ith *dual associate matrix of Y with respect to x*. Combining (2), (3) with (4), (5),

$$A_i^* = \sum_{j=0}^{D} q_i(j) E_j^* \quad (0 \leqslant i \leqslant D), \tag{6}$$

$$E_i^* = |X|^{-1} \sum_{j=0}^{D} p_i(j) A_j^* \quad (0 \leqslant i \leqslant D). \tag{7}$$

It follows that $A_0^*, \ldots, A_D^*$ form a second basis for M^*.

From the definitions, the dual associate matrices satisfy: (i) $A_0^* = I$; (ii) $\overline{A_i^*}^{\mathrm{t}} = A_i^*$ $(0 \leqslant i \leqslant D)$; (iii) $A_i^* A_j^* = \sum_{h=0}^{D} q_{ij}^h A_h^*$ $(0 \leqslant i, j \leqslant D)$; (iv) $A_0^* + A_1^* + \cdots + A_D^* = |X| E_0^*$.

3. The Terwilliger algebra and its modules

Let $Y = (X, \{R_i\}_{0 \leqslant i \leqslant D})$ denote a scheme. Fix any $x \in X$, and write $M^* = M^*(x)$. Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by M and M^*. We call T the *Terwilliger algebra of Y with respect to x*.

In [23, Lemma 3.2], it is shown that for all integers h, i, j $(0 \leqslant h, i, j \leqslant D)$,

$$p_{ij}^h = 0 \quad \text{if and only if} \quad E_i^* A_j E_h^* = 0, \tag{8}$$

$$q_{ij}^h = 0 \quad \text{if and only if} \quad E_i A_j^* E_h = 0, \tag{9}$$

where $A_i^* = A_i^*(x)$, $E_i^* = E_i^*(x)$ $(0 \leqslant i \leqslant D)$.

Let V denote the vector space $\mathbb{C}^X$ (column vectors), where the coordinates are indexed by X. Then $\mathrm{Mat}_X(\mathbb{C})$ acts on V by left multiplication. We endow V with the inner product

$\langle\, , \rangle$ satisfying $\langle u, v\rangle := u^{\mathrm{t}}\overline{v}$ for all $u, v \in V$. Observe that $V = \sum_{i=0}^{D} E_i V$ (orthogonal direct sum). Similarly, we have the decomposition $V = \sum_{i=0}^{D} E_i^* V$ (orthogonal direct sum).

By a *T-module*, we mean a subspace W of V such that $TW \subseteq W$. We refer to V itself as the *standard module* for T. Let W, W' denote T-modules. By a *T-module isomorphism* from W to W', we mean an isomorphism of vector spaces $\phi : W \to W'$ such that

$$(B\phi - \phi B)W = 0 \quad (\forall B \in T).$$

W, W' are said to be *T-isomorphic* whenever there exists a T-module isomorphism from W to W'. A T-module W is said to be *irreducible* whenever $W \neq 0$ and W contains no T-modules other than 0 and W.

Because T is closed under the conjugate-transpose map, T is semisimple. It follows that for any T-module W and any T-module $U \subseteq W$ there exists a unique T-module $U' \subseteq W$ such that

$$W = U + U' \quad \text{(orthogonal direct sum)}.$$

Moreover, W is an orthogonal direct sum of irreducible T-modules.

Now let W denote an irreducible T-module. Observe that

$$W = \sum E_i^* W \quad \text{(orthogonal direct sum)}, \tag{10}$$

where the sum is taken over all the indices i $(0 \leqslant i \leqslant D)$ such that $E_i^* W \neq 0$. We set $d := |\{i \mid E_i^* W \neq 0\}| - 1$, and observe that the dimension of W is at least $d + 1$. We refer to d as the *diameter* of W. W is said to be *thin* whenever $\dim(E_i^* W) \leqslant 1$ $(0 \leqslant i \leqslant D)$. Note that W is thin if and only if the diameter of W equals $\dim(W) - 1$.

Similarly,

$$W = \sum E_i W \quad \text{(orthogonal direct sum)},$$

where the sum is taken over all the indices i $(0 \leqslant i \leqslant D)$ such that $E_i W \neq 0$. We set $d^* := |\{i \mid E_i W \neq 0\}| - 1$, and observe that the dimension of W is at least $d^* + 1$. We refer to d^* as the *dual diameter* of W. W is said to be *dual thin* whenever $\dim(E_i W) \leqslant 1$ $(0 \leqslant i \leqslant D)$. Note that W is dual thin if and only if the dual diameter of W equals $\dim(W) - 1$.

We wish to emphasize the following point, which follows immediately from the above discussion.

Lemma 3.1. *Let $Y = (X, \{R_i\}_{0 \leqslant i \leqslant D})$ denote a scheme. Fix any $x \in X$, and write $T = T(x)$. Let W denote an irreducible T-module that is both thin and dual thin. Then the diameter and dual diameter of W are equal.*

4. The P-polynomial property

Let $Y = (X, \{R_i\}_{0 \leqslant i \leqslant D})$ denote a scheme. We say that Y is *P-polynomial* (with respect to the ordering $R_0, \ldots, R_D$ of the associate classes) whenever for all integers h, i, j $(0 \leqslant h, i, j \leqslant D)$,

$$p_{ij}^h = 0 \text{ if one of } h, i, j \text{ is greater than the sum of the other two}, \tag{11}$$

$$p^h_{ij} \neq 0 \text{ if one of } h, i, j \text{ equals the sum of the other two.} \tag{12}$$

For the rest of this section, assume Y is P-polynomial. We abbreviate $c_i := p^i_{1i-1}$ $(1 \leqslant i \leqslant D)$, $a_i := p^i_{1i}$ $(0 \leqslant i \leqslant D)$, $b_i := p^i_{1i+1}$ $(0 \leqslant i \leqslant D-1)$, and define $c_0 := 0$, $b_D := 0$. We note that $a_0 = 0$ and $c_1 = 1$. By Bannai and Ito [2, Proposition III.1.2],

$$k_i = \frac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i} \quad (0 \leqslant i \leqslant D).$$

Of particular interest are the matrix $A := A_1$ and the scalars $\theta_i := p_1(i)$ $(0 \leqslant i \leqslant D)$. It follows from (2) that

$$AE_i = \theta_i E_i \quad (0 \leqslant i \leqslant D). \tag{13}$$

It is shown in [2, p. 190] that

$$\theta_i \neq \theta_j \quad \text{if } i \neq j \quad (0 \leqslant i, j \leqslant D), \tag{14}$$

and also that $A_i = v_i(A)$ $(0 \leqslant i \leqslant D)$, where v_i is a polynomial with real coefficients and degree exactly i. In particular, A multiplicatively generates M, the Bose–Mesner algebra. By (2), it follows that

$$p_i(j) = v_i(\theta_j) \quad (0 \leqslant i, j \leqslant D).$$

We now recall the raising, lowering, and flat matrices of Y. Fix any $x \in X$, and write $E^*_i = E^*_i(x)$ $(0 \leqslant i \leqslant D)$. Define matrices $R = R(x)$, $F = F(x)$, $L = L(x)$ by

$$R := \sum_{i=0}^{D} E^*_{i+1} A E^*_i, \quad F := \sum_{i=0}^{D} E^*_i A E^*_i, \quad L := \sum_{i=0}^{D} E^*_{i-1} A E^*_i. \tag{15}$$

Note that R, F, and L have real entries by (1) and (4). Also, observe that F is symmetric and $R = L^{\mathrm{t}}$. By (8) and (11),

$$A = R + F + L. \tag{16}$$

Using (15) and recalling $E^*_{-1} = 0$, $E^*_{D+1} = 0$, we find

$$\begin{aligned} &RE^*_i = E^*_{i+1} R \ \ (-1 \leqslant i \leqslant D), \quad FE^*_i = E^*_i F \ \ (0 \leqslant i \leqslant D), \\ &LE^*_i = E^*_{i-1} L \ \ (0 \leqslant i \leqslant D+1). \end{aligned} \tag{17}$$

5. The T-modules of P-polynomial schemes

In this section, we describe the irreducible T-modules of P-polynomial schemes.

Let $Y = (X, \{R_i\}_{0 \leqslant i \leqslant D})$ denote a scheme which is P-polynomial with respect to the ordering $R_0, \ldots, R_D$ of the associate classes. Fix any $x \in X$ and write $T = T(x)$. Let W denote an irreducible T-module. We define the *endpoint* r of W by

$$r := \min\{i \mid 0 \leqslant i \leqslant D, E^*_i W \neq 0\}.$$

We observe that $0 \leqslant r \leqslant D - d$, where d denotes the diameter of W.

In [23, Lemma 3.9], it was shown that $RE_i^*W \neq 0$ $(r \leqslant i < r+d)$, $LE_i^*W \neq 0$ $(r < i \leqslant r+d)$, and also that

$$E_i^*W \neq 0 \quad \text{iff} \quad r \leqslant i \leqslant r+d \quad (0 \leqslant i \leqslant D). \tag{18}$$

By Caughman IV [5, Lemma 5.1], we have

$$2r + d^* \geqslant D,$$

where d^* denotes the dual diameter of W.

Lemma 5.1 (*Terwilliger [23, Lemma 3.9]*). *Let $Y = (X, \{R_i\}_{0 \leqslant i \leqslant D})$ denote a scheme which is P-polynomial with respect to the ordering $R_0, \ldots, R_D$ of the associate classes. Fix any $x \in X$, and write $E_i^* = E_i^*(x)$ $(0 \leqslant i \leqslant D)$, $T = T(x)$. Let W denote a thin, irreducible T-module with endpoint r. Then*

(i) $W = ME_r^*W$.
(ii) $E_iW = E_iE_r^*W$ $(0 \leqslant i \leqslant D)$.
(iii) *W is dual thin.*

6. The Q-polynomial property

Let $Y = (X, \{R_i\}_{0 \leqslant i \leqslant D})$ denote a scheme. We say that Y is *Q-polynomial* (with respect to the given ordering $E_0, E_1, \ldots, E_D$ of the primitive idempotents) whenever for all integers h, i, j $(0 \leqslant h, i, j \leqslant D)$, the Krein parameters satisfy

$$q_{ij}^h = 0 \textit{ if one of } h, i, j \textit{ is greater than the sum of the other two},$$

$$q_{ij}^h \neq 0 \textit{ if one of } h, i, j \textit{ equals the sum of the other two}.$$

For the rest of this section, assume Y is Q-polynomial with respect to the ordering $E_0, \ldots, E_D$. We abbreviate $c_i^* := q_{1i-1}^i$ $(1 \leqslant i \leqslant D)$, $a_i^* := q_{1i}^i$ $(0 \leqslant i \leqslant D)$, $b_i^* := q_{1i+1}^i$ $(0 \leqslant i \leqslant D-1)$, and define $c_0^* := 0$, $b_D^* := 0$. We note that $a_0^* = 0$ and $c_1^* = 1$ [2, Proposition II.3.7]. By Bannai and Ito [2, p. 196],

$$m_i = \frac{b_0^* b_1^* \cdots b_{i-1}^*}{c_1^* c_2^* \cdots c_i^*} \quad (0 \leqslant i \leqslant D).$$

Fix any $x \in X$ and write $E_i^* = E_i^*(x)$, $A_i^* = A_i^*(x)$ $(0 \leqslant i \leqslant D)$. Of particular interest are the matrix $A^* := A_1^*(x)$ and the scalars $\theta_i^* := q_1(i)$ $(0 \leqslant i \leqslant D)$. By (6),

$$A^*E_i^* = \theta_i^*E_i^* \quad (0 \leqslant i \leqslant D). \tag{19}$$

It is shown in [2, p. 193] that

$$\theta_i^* \neq \theta_j^* \quad \text{if } i \neq j \quad (0 \leqslant i, j \leqslant D), \tag{20}$$

and also that $A_i^* = v_i^*(A^*)$ $(0 \leqslant i \leqslant D)$, where v_i^* is a polynomial with real coefficients and degree exactly i. In particular, A^* generates the dual Bose–Mesner algebra $M^* = M^*(x)$. By (6), it follows that

$$q_i(j) = v_i^*(\theta_j^*) \quad (0 \leqslant i, j \leqslant D).$$

We now recall the dual raising, lowering, and flat matrices of Y. Define the matrices $R^* = R^*(x)$, $F^* = F^*(x)$, $L^* = L^*(x)$ by

$$R^* := \sum_{i=0}^{D} E_{i+1} A^* E_i, \quad F^* := \sum_{i=0}^{D} E_i A^* E_i, \quad L^* := \sum_{i=0}^{D} E_{i-1} A^* E_i. \tag{21}$$

Note that R^*, F^*, and L^* have real entries by (6), and since the $q_i(j)$ are real. Also, observe that F^* is symmetric and $R^* = L^{*t}$. Moreover,

$$A^* = R^* + F^* + L^*. \tag{22}$$

Using (21) and recalling $E_{-1} = 0$, $E_{D+1} = 0$, we find

$$\begin{aligned} &R^* E_i = E_{i+1} R^* \ (-1 \leqslant i \leqslant D), \quad F^* E_i = E_i F^* \ (0 \leqslant i \leqslant D), \\ &L^* E_i = E_{i-1} L^* \ (0 \leqslant i \leqslant D+1). \end{aligned} \tag{23}$$

7. The T-modules of Q-polynomial schemes

In this section, we describe the irreducible T-modules of Q-polynomial schemes.

Let $Y = (X, \{R_i\}_{0 \leqslant i \leqslant D})$ denote a scheme which is Q-polynomial with respect to the ordering $E_0, \ldots, E_D$ of the primitive idempotents. Fix any $x \in X$ and write $T = T(x)$. Let W denote an irreducible T-module. We define the *dual endpoint* t of W by

$$t := \min\{i \mid 0 \leqslant i \leqslant D, \ \ E_i W \neq 0\}.$$

We observe that $0 \leqslant t \leqslant D - d^*$, where d^* denotes the dual diameter of W.

In [23, Lemma 3.12], it was shown that $R^* E_i W \neq 0$ $(t \leqslant i < t + d^*)$, $L^* E_i W \neq 0$ $(t < i \leqslant t + d^*)$, and also that

$$E_i W \neq 0 \quad \text{iff} \quad t \leqslant i \leqslant t + d^* \quad (0 \leqslant i \leqslant D). \tag{24}$$

By Caughman IV [5, Lemma 7.1], we have

$$2t + d \geqslant D, \tag{25}$$

where d denotes the diameter of W.

Lemma 7.1 (*Terwilliger [23, Lemma 3.12]*). *Let $Y = (X, \{R_i\}_{0 \leqslant i \leqslant D})$ denote a scheme which is Q-polynomial with respect to the ordering $E_0, \ldots, E_D$ of the primitive idempotents. Fix any $x \in X$, and write $E_i^* = E_i^*(x)$ $(0 \leqslant i \leqslant D)$, $M^* = M^*(x)$, $T = T(x)$. Let W*

denote a dual thin, irreducible T-module with dual endpoint t. Then (i)–(iii) *hold below.*

(i) $W = M^* E_t W$.
(ii) $E_i^* W = E_i^* E_t W \ (0 \leqslant i \leqslant D)$.
(iii) *W is thin.*

8. The T-modules of P- and Q-polynomial schemes

Let $Y = (X, \{R_i\}_{0 \leqslant i \leqslant D})$ denote a scheme which is P-polynomial with respect to the ordering $R_0, \ldots, R_D$ of the associate classes, and Q-polynomial with respect to the ordering $E_0, \ldots, E_D$ of the primitive idempotents. Fix any $x \in X$ and write $T = T(x)$. Let W denote a thin irreducible T-module. Observe W is dual thin by Lemma 5.1, and the diameter and dual diameter of W coincide by Lemma 3.1. We now present two bases for W, one of which diagonalizes A and the other diagonalizes A^*. We then consider the action of T on these bases.

Theorem 8.1. *Let $Y = (X, \{R_i\}_{0 \leqslant i \leqslant D})$ denote a scheme which is P-polynomial with respect to the ordering $R_0, \ldots, R_D$ of the associate classes, and Q-polynomial with respect to the ordering $E_0, \ldots, E_D$ of the primitive idempotents. Fix any $x \in X$, and write $E_i^* = E_i^*(x)$ $(0 \leqslant i \leqslant D)$, $T = T(x)$. Let W denote a thin irreducible T-module with endpoint r, dual endpoint t, and diameter d.*

(i) *For all nonzero $v \in E_t W$, the vector $E_i^* v$ is a basis for $E_i^* W$ for $r \leqslant i \leqslant r+d$. Moreover, $E_r^* v, E_{r+1}^* v, \ldots, E_{r+d}^* v$ is a basis for W.*
(ii) *For all nonzero $v \in E_r^* W$, the vector $E_i v$ is a basis for $E_i W$ for $t \leqslant i \leqslant t+d$. Moreover, $E_t v, E_{t+1} v, \ldots, E_{t+d} v$ is a basis for W.*

Proof. (i) Recall W is dual thin by Lemma 5.1 so v spans $E_t W$. Fix any $i \ (r \leqslant i \leqslant r+d)$, and observe $E_i^* W \neq 0$ by (18). Also $E_i^* v$ spans $E_i^* W$, since by Lemma 7.1 and the construction,

$$\begin{aligned} E_i^* W &= E_i^* E_t W \\ &= \mathrm{span}(E_i^* v). \end{aligned}$$

We have now shown that $E_i^* v$ is a basis for $E_i^* W$. Applying (10), (18), we find $E_r^* v, \ldots, E_{r+d}^* v$ is a basis for W.

(ii) Similar to the proof of (i). □

Definition 8.2. Let $Y = (X, \{R_i\}_{0 \leqslant i \leqslant D})$ denote a scheme which is P-polynomial with respect to the ordering $R_0, \ldots, R_D$ of the associate classes, and Q-polynomial with respect to the ordering $E_0, \ldots, E_D$ of the primitive idempotents. Fix any $x \in X$, and write $E_i^* = E_i^*(x) \ (0 \leqslant i \leqslant D)$, $T = T(x)$. Let W denote a thin irreducible T-module with endpoint r, dual endpoint t, and diameter d. For all $i \ (0 \leqslant i \leqslant d)$, let $c_i(W), a_i(W), b_i(W)$ denote the complex scalars such that

$$R E_{r+i-1}^* v = c_i(W) E_{r+i}^* v, \tag{26}$$

$$F E_{r+i}^* v = a_i(W) E_{r+i}^* v, \tag{27}$$

$$L E_{r+i+1}^* v = b_i(W) E_{r+i}^* v, \tag{28}$$

where v is any nonzero vector in E_tW. Since E_tW has dimension 1, we see $c_i(W)$, $a_i(W)$, $b_i(W)$ are independent of the choice of v. We refer to the $c_i(W)$, $a_i(W)$, $b_i(W)$ as the *intersection numbers* of W. We observe $c_0(W)=0$, $b_d(W)=0$. By the *intersection matrix* of W, we mean the tridiagonal matrix

$$B(W) := \begin{pmatrix} a_0(W) & b_0(W) & & & & \mathbf{0} \\ c_1(W) & a_1(W) & b_1(W) & & & \\ & c_2(W) & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & b_{d-1}(W) \\ \mathbf{0} & & & & c_d(W) & a_d(W) \end{pmatrix}.$$

Definition 8.3. Let $Y=(X,\{R_i\}_{0\leqslant i\leqslant D})$ denote a scheme which is P-polynomial with respect to the ordering $R_0,\ldots,R_D$ of the associate classes, and Q-polynomial with respect to the ordering $E_0,\ldots,E_D$ of the primitive idempotents. Fix any $x\in X$, and write $E_i^*=E_i^*(x)$ $(0\leqslant i\leqslant D)$, $T=T(x)$. Let W denote a thin irreducible T-module with endpoint r, dual endpoint t, and diameter d. For all i $(0\leqslant i\leqslant d)$, let $c_i^*(W)$, $a_i^*(W)$, $b_i^*(W)$ denote the complex scalars such that

$$R^*E_{t+i-1}v = c_i^*(W)E_{t+i}v, \tag{29}$$

$$F^*E_{t+i}v = a_i^*(W)E_{t+i}v, \tag{30}$$

$$L^*E_{t+i+1}v = b_i^*(W)E_{t+i}v, \tag{31}$$

where v is any nonzero vector in E_r^*W. Since E_r^*W has dimension 1, we see $c_i^*(W)$, $a_i^*(W)$, $b_i^*(W)$ are independent of the choice of v. We refer to the $c_i^*(W)$, $a_i^*(W)$, $b_i^*(W)$ as the *dual intersection numbers* of W. We observe $c_0^*(W)=0$, $b_d^*(W)=0$. By the *dual intersection matrix* of W, we mean the tridiagonal matrix

$$B^*(W) := \begin{pmatrix} a_0^*(W) & b_0^*(W) & & & & \mathbf{0} \\ c_1^*(W) & a_1^*(W) & b_1^*(W) & & & \\ & c_2^*(W) & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & b_{d-1}^*(W) \\ \mathbf{0} & & & & c_d^*(W) & a_d^*(W) \end{pmatrix}.$$

Lemma 8.4. *Let $Y=(X,\{R_i\}_{0\leqslant i\leqslant D})$ denote a scheme which is P-polynomial with respect to the ordering $R_0,\ldots,R_D$ of the associate classes, and Q-polynomial with respect to the ordering $E_0,\ldots,E_D$ of the primitive idempotents. Fix any $x\in X$, and write $E_i^*=E_i^*(x)$ $(0\leqslant i\leqslant D)$, $T=T(x)$. Let W denote a thin irreducible T-module with endpoint r, dual endpoint t, and diameter d.*

(i) *$B(W)$ is the matrix representing multiplication by A with respect to the basis E_r^*v, $E_{r+1}^*v,\ldots,E_{r+d}^*v$, where v is any nonzero vector in E_tW.*
(ii) *$\mathrm{Diag}(\theta_r^*,\theta_{r+1}^*,\ldots,\theta_{r+d}^*)$ is the matrix representing multiplication by A^* with respect to the basis E_r^*v, $E_{r+1}^*v,\ldots,E_{r+d}^*v$, where v is any nonzero vector in E_tW.*
(iii) *$B^*(W)$ is the matrix representing multiplication by A^* with respect to the basis E_tv, $E_{t+1}v,\ldots,E_{t+d}v$, where v is any nonzero vector in E_r^*W.*

(iv) $\mathrm{Diag}(\theta_t, \theta_{t+1}, \ldots, \theta_{t+d})$ *is the matrix representing multiplication by A withrespect to the basis* $E_t v, E_{t+1} v, \ldots, E_{t+d} v$, *where* v *is any nonzero vector in* $E_r^* W$.

Proof. (i) Immediate from (16) and Definition 8.2.
(ii) Immediate from (19).
(iii) Similar to the proof of (i).
(iv) Similar to the proof of (ii). □

Corollary 8.5. *Let* $Y = (X, \{R_i\}_{0 \leqslant i \leqslant D})$ *denote a scheme which is P-polynomial with respect to the ordering* $R_0, \ldots, R_D$ *of the associate classes, and Q-polynomial with respect to the ordering* $E_0, \ldots, E_D$ *of the primitive idempotents. Fix any* $x \in X$, *and write* $T = T(x)$. *Let W denote a thin irreducible T-module with endpoint r, dual endpoint t, and diameter* d.

(i) *The eigenvalues of* $B(W)$ *are* $\theta_t, \theta_{t+1}, \ldots, \theta_{t+d}$.
(ii) *The eigenvalues of* $B^*(W)$ *are* $\theta_r^*, \theta_{r+1}^*, \ldots, \theta_{r+d}^*$.

Corollary 8.6. *Let* $Y = (X, \{R_i\}_{0 \leqslant i \leqslant D})$ *denote a scheme which is P-polynomial with respect to the ordering* $R_0, \ldots, R_D$ *of the associate classes, and Q-polynomial with respect to the ordering* $E_0, \ldots, E_D$ *of the primitive idempotents. Fix any* $x \in X$, *and write* $E_i^* = E_i^*(x)$ $(0 \leqslant i \leqslant D)$, $T = T(x)$. *Let W denote a thin irreducible T-module with endpoint* r, *dual endpoint* t, *and diameter* d. *Then* (i) *and* (ii) *hold below.*

(i) $\sum_{i=0}^{d} a_i(W) = \sum_{i=t}^{t+d} \theta_i$.
(ii) $\sum_{i=0}^{d} a_i^*(W) = \sum_{i=r}^{r+d} \theta_i^*$.

Proof. (i) By Corollary 8.5(i), both sides of the equation in (i) equal the trace of $B(W)$.
(ii) By Corollary 8.5(ii), both sides of the equation in (ii) equal the trace of $B^*(W)$. □

9. Almost-bipartite P- and Q-polynomial schemes

Let $Y = (X, \{R_i\}_{0 \leqslant i \leqslant D})$ denote a scheme which is P-polynomial with respect to the ordering $R_0, \ldots, R_D$ of the associate classes. We say Y is *almost-bipartite* (with respect to the P-polynomial ordering) whenever $a_i = 0$ for $0 \leqslant i \leqslant D-1$ and $a_D \neq 0$.

For the remainder of this article, we shall be concerned with P- and Q-polynomial schemes for which the P-polynomial structure is almost-bipartite. We thus make the following definition.

Definition 9.1. Let $Y = (X, \{R_i\}_{0 \leqslant i \leqslant D})$ denote a scheme with $D \geqslant 3$ which is almost-bipartite P-polynomial with respect to the ordering $R_0, \ldots, R_D$ of the associate classes, and Q-polynomial with respect to the ordering $E_0, \ldots, E_D$ of the primitive idempotents. Fix any $x \in X$, and write $T = T(x)$ to denote the Terwilliger algebra of Y with respect to x. (Where the context allows, we will also suppress the reference to x for the individual matrices in T, e.g., $E_0^* = E_0^*(x)$, $R = R(x)$, etc.).

Lemma 9.2. *Let Y be as in Definition* 9.1. *Then* (i)–(iii) *hold below*.

(i) $E^*_i A E^*_i = 0 \ (0 \leqslant i \leqslant D-1)$, *and* $E^*_D A E^*_D \neq 0$.
(ii) $F = E^*_D A E^*_D$, *where F is the matrix from* (15).
(iii) $F E^*_i = 0 \ (0 \leqslant i \leqslant D-1)$.

Proof. (i) Immediate from (8) and the fact that $a_i = 0 \ (0 \leqslant i \leqslant D-1)$, $a_D \neq 0$.
(ii), (iii) Immediate using (i). □

Theorem 9.3 (*Collins [7, Theorem 14.3]*). *With reference to Definition* 9.1, *let W denote an irreducible T-module with endpoint r and diameter d. Then* $r + d = D$.

10. Each irreducible T-module is thin and dual thin

Let Y be as in Definition 9.1, and let W denote an irreducible T-module. In this section, we show that W is both thin and dual thin.

Lemma 10.1. *With reference to Definition* 9.1, *let W denote an irreducible T-module with dual endpoint t, and fix any nonzero* $v \in E_t W$. *Then*

(i) $R E^*_{i-1} v + L E^*_{i+1} v = \theta_t E^*_i v \ (0 \leqslant i \leqslant D-1)$,
(ii) $R E^*_{D-1} v + F E^*_D v = \theta_t E^*_D v$.

Proof. Observe that $Av = \theta_t v$. Fix an integer $i \ (0 \leqslant i \leqslant D)$. Now by (16), (17), we have

$$\begin{aligned} R E^*_{i-1} v + F E^*_i v + L E^*_{i+1} v &= E^*_i A v \\ &= \theta_t E^*_i v. \end{aligned}$$

Assertion (i) follows since $F E^*_i = 0$ for $0 \leqslant i \leqslant D-1$. Assertion (ii) similarly follows since $E^*_{D+1} = 0$. □

Lemma 10.2. *With reference to Definition* 9.1, *let W denote an irreducible T-module with dual endpoint t, and fix any nonzero* $v \in E_t W$. *Suppose* v *is an eigenvector for* F^* *with eigenvalue* α. *Then*

(i) $\theta^*_{i-1} R E^*_{i-1} v + \theta^*_{i+1} L E^*_{i+1} v = (\theta_{t+1}\theta^*_i - \alpha\theta_{t+1} + \alpha\theta_t) E^*_i v \ (0 \leqslant i \leqslant D-1)$,
(ii) $\theta^*_{D-1} R E^*_{D-1} v + \theta^*_D F E^*_D v = (\theta_{t+1}\theta^*_D - \alpha\theta_{t+1} + \alpha\theta_t) E^*_D v$,

where θ_{D+1}, θ^*_{-1} *are indeterminates*.

Proof. Observe $L^* v = 0$ by (23), and $F^* v = \alpha v$ by assumption, so $R^* v = (A^* - \alpha I) v$ in view of (22). Since $R^* v \in E_{t+1} W$ by (23),

$$A(A^* - \alpha I) v = \theta_{t+1} (A^* - \alpha I) v. \tag{32}$$

Fix an integer i $(0 \leqslant i \leqslant D)$. We may now argue that

$$\begin{aligned}
&\theta^*_{i-1} RE^*_{i-1}v + \theta^*_i FE^*_i v + \theta^*_{i+1} LE^*_{i+1} v \\
&\quad = (RE^*_{i-1} + FE^*_i v + LE^*_{i+1})A^* v \quad \text{(by (19))} \\
&\quad = E^*_i AA^* v \quad \text{(by (16),(17))} \\
&\quad = E^*_i A(A^* - \alpha I)v + \alpha E^*_i Av \\
&\quad = \theta_{t+1} E^*_i (A^* - \alpha I)v + \theta_t \alpha E^*_i v \quad \text{(by (32))} \\
&\quad = (\theta_{t+1}(\theta^*_i - \alpha) + \alpha\theta_t)E^*_i v \quad \text{(by (19))},
\end{aligned}$$

where $\theta_{D+1}, \theta^*_{-1}, \theta^*_{D+1}$ are indeterminates. Assertion (i) follows since $FE^*_i = 0$ for $0 \leqslant i \leqslant D-1$. Assertion (ii) similarly follows since $E^*_{D+1} = 0$. $\square$

Lemma 10.3. *With reference to Definition* 9.1, *let W denote an irreducible T-module. Then W is thin and dual thin*.

Proof. Let t denote the dual endpoint of W. Since $F^* E_t W \subseteq E_t W$, the space $E_t W$ contains a nonzero eigenvector v for F^*. By Lemma 10.1,

$$RE^*_{i-1}v + LE^*_{i+1}v \in \operatorname{span}(E^*_i v) \quad (0 \leqslant i \leqslant D-1), \tag{33}$$

$$RE^*_{D-1}v + FE^*_D v \in \operatorname{span}(E^*_D v). \tag{34}$$

By Lemma 10.2,

$$\theta^*_{i-1}RE^*_{i-1}v + \theta^*_{i+1}LE^*_{i+1}v \in \operatorname{span}(E^*_i v) \quad (0 \leqslant i \leqslant D-1), \tag{35}$$

$$\theta^*_{D-1}RE^*_{D-1}v + \theta^*_D FE^*_D v \in \operatorname{span}(E^*_D v), \tag{36}$$

where θ^*_{-1} is indeterminate. By (33), (35), and (20), we find

$$RE^*_i v \in \operatorname{span}(E^*_{i+1}v) \ (0 \leqslant i \leqslant D-2), \quad LE^*_i v \in \operatorname{span}(E^*_{i-1}v) \ (1 \leqslant i \leqslant D).$$

By (34), (36), and (20), we find $RE^*_{D-1}v \in \operatorname{span}(E^*_D v)$ and $FE^*_D v \in \operatorname{span}(E^*_D v)$. Combining the above information with Lemma 9.2(iii) and recalling that $RE^*_D = 0$, $LE^*_0 = 0$, we see

$$RE^*_i v \in \operatorname{span}(E^*_{i+1}v) \quad (0 \leqslant i \leqslant D), \tag{37}$$

$$FE^*_i v \in \operatorname{span}(E^*_i v) \quad (0 \leqslant i \leqslant D), \tag{38}$$

$$LE^*_i v \in \operatorname{span}(E^*_{i-1}v) \quad (0 \leqslant i \leqslant D). \tag{39}$$

We claim that

$$W = \operatorname{span}\{E^*_0 v, E^*_1 v, \ldots, E^*_D v\}. \tag{40}$$

To see this, let W' denote the right side of (40). Certainly $W' \subseteq W$; to prove that $W' = W$, we show that W' is a nonzero T-module. Observe that $v = \sum_{i=0}^{D} E^*_i v \in W'$, so $W' \neq 0$. Observe that $M^* W' \subseteq W'$ by the construction. Observe that $RW' \subseteq W'$, $FW' \subseteq W'$, and $LW' \subseteq W'$ by (37)–(39). Recall that $A = R + F + L$ generates M, so $MW' \subseteq W'$. Since

M, M^* generate T, we now have that $TW' \subseteq W'$, so W' is a T-module. It follows that $W' = W$ by the irreducibility of W. We now have (40), which implies that W is thin. By Lemma 5.1, W is dual thin. □

We conclude this section with a comment.

Theorem 10.4. *With reference to Definition* 9.1, *let W denote an irreducible T-module with diameter* $d \geqslant 1$, *endpoint* r, *and dual endpoint t. Then*

$$a_0^*(W) = \frac{\theta_{r+1}^*\theta_t - \theta_{t+1}\theta_r^*}{\theta_t - \theta_{t+1}}. \tag{41}$$

Proof. Fix any nonzero $v \in E_t W$. Setting $i = r$ in the equation in Lemma 10.1(i), we find that

$$LE_{r+1}^* v = \theta_t E_r^* v. \tag{42}$$

By Lemma 10.3 and (30), v is an eigenvector for F^*, with eigenvalue $a_0^*(W)$. Setting $i = r$, $\alpha = a_0^*(W)$ in the equation in Lemma 10.2(i), we find

$$\theta_{r+1}^* LE_{r+1}^* v = (\theta_{t+1}\theta_r^* - a_0^*(W)(\theta_{t+1} - \theta_t))E_r^* v. \tag{43}$$

Eliminating $LE_{r+1}^* v$ in (43) using (42), and since $E_r^* v \neq 0$, we obtain

$$\theta_{r+1}^*\theta_t - \theta_{t+1}\theta_r^* = a_0^*(W)(\theta_t - \theta_{t+1}),$$

and (41) follows. □

11. Computation of $c_i(W), a_i(W), b_i(W)$

Let Y be as in Definition 9.1, and let W denote an irreducible T-module with diameter d. In this section, we compute the parameters $c_i(W)$, $a_i(W)$, $b_i(W)$ $(0 \leqslant i \leqslant d)$. We begin with $a_i(W)$.

Lemma 11.1. *With reference to Definition* 9.1, *let W denote an irreducible T-module with diameter d. Then*

(i) $a_i(W) = 0$ $(0 \leqslant i \leqslant d-1)$,
(ii) $a_d(W) \neq 0$.

Proof. (i) Immediate from (27) and Lemma 9.2(iii).

(ii) Immediate from Theorem 9.3 and [7, Theorem 15.2]. □

Lemma 11.2. *With reference to Definition* 9.1, *let W denote an irreducible T-module with dual endpoint t and diameter* d. *Then*

(i) $c_i(W) + b_i(W) = \theta_t$ $(0 \leqslant i \leqslant d-1)$,
(ii) $c_d(W) + a_d(W) = \theta_t$.

Proof. Fix any nonzero $v \in E_t W$, and fix an integer i $(0 \leqslant i \leqslant d)$. Using (26)–(28), Lemma 10.1, and Lemma 9.2(iii), we find

$$\begin{aligned}(c_i(W) + a_i(W) + b_i(W))E^*_{r+i}v &= (RE^*_{r+i-1} + FE^*_{r+i} + LE^*_{r+i+1})v \\ &= \theta_t E^*_{r+i}v.\end{aligned}$$

Since $E^*_{r+i}v \neq 0$ by Theorem 8.1(i), we have

$$c_i(W) + a_i(W) + b_i(W) = \theta_t \quad (0 \leqslant i \leqslant d).$$

We observe $a_i(W) = 0$ $(0 \leqslant i \leqslant d-1)$ by Lemma 11.1 and $b_d(W) = 0$ by Definition 8.2. The result follows. □

Lemma 11.3. *With reference to Definition* 9.1, *let W denote an irreducible T-module with endpoint r, dual endpoint t, and diameter d. Suppose* $d \geqslant 1$. *Then*

(i) $\theta^*_{r+i-1}c_i(W) + \theta^*_{r+i+1}b_i(W) = \theta_t\theta^*_{r+1} + \theta_{t+1}\theta^*_{r+i} - \theta_{t+1}\theta^*_r$ $(0 \leqslant i \leqslant d-1)$,
(ii) $\theta^*_{r+d-1}c_d(W) + \theta^*_{r+d}a_d(W) = \theta_t\theta^*_{r+1} + \theta_{t+1}\theta^*_{r+d} - \theta_{t+1}\theta^*_r$,

where θ^*_{-1} *is indeterminate*.

Proof. Fix any nonzero $v \in E_t W$, and fix any integer i $(0 \leqslant i \leqslant d)$. By (30), v is an eigenvector for F^* with eigenvalue $a^*_0(W)$. Using (26)–(28), Lemmas 9.2(iii), 10.2, and (41), we find

$$\begin{aligned}&(\theta^*_{r+i-1}c_i(W) + \theta^*_{r+i}a_i(W) + \theta^*_{r+i+1}b_i(W))E^*_{r+i}v \\ &\quad = (\theta^*_{r+i-1}RE^*_{r+i-1} + \theta^*_{r+i}FE^*_{r+i} + \theta^*_{r+i+1}LE^*_{r+i+1})v \\ &\quad = (\theta_{t+1}\theta^*_{r+i} - a^*_0(W)\theta_{t+1} + a^*_0(W)\theta_t)E^*_{r+i}v \\ &\quad = (\theta_t\theta^*_{r+1} + \theta_{t+1}\theta^*_{r+i} - \theta_{t+1}\theta^*_r)E^*_{r+i}v,\end{aligned}$$

where $\theta^*_{-1}, \theta^*_{D+1}$ are indeterminates. Recall $E^*_{r+i}v \neq 0$ by Theorem 8.1(i). We observe $a_i(W) = 0$ $(0 \leqslant i \leqslant d-1)$ by Lemma 11.1(i) and $b_d(W) = 0$ by Definition 8.2. The result follows. □

Theorem 11.4. *With reference to Definition* 9.1, *let W denote an irreducible T-module with endpoint r, dual endpoint t, and diameter d. First assume* $d = 0$. *Then* $c_0(W) = 0$, $a_0(W) = \theta_t$, *and* $b_0(W) = 0$. *Now assume* $d \geqslant 1$. *Then*

$$c_0(W) = 0, \tag{44}$$

$$c_i(W) = \frac{\theta_t(\theta^*_{r+i+1} - \theta^*_{r+1}) - \theta_{t+1}(\theta^*_{r+i} - \theta^*_r)}{\theta^*_{r+i+1} - \theta^*_{r+i-1}} \quad (1 \leqslant i \leqslant d-1), \tag{45}$$

$$c_d(W) = \frac{\theta_t(\theta^*_{r+d} - \theta^*_{r+1}) - \theta_{t+1}(\theta^*_{r+d} - \theta^*_r)}{\theta^*_{r+d} - \theta^*_{r+d-1}}, \tag{46}$$

$$a_i(W) = 0 \quad (0 \leqslant i \leqslant d-1), \tag{47}$$

$$a_d(W) = \frac{\theta_t(\theta^*_{r+d-1} - \theta^*_{r+1}) - \theta_{t+1}(\theta^*_{r+d} - \theta^*_r)}{\theta^*_{r+d-1} - \theta^*_{r+d}}, \tag{48}$$

$$b_0(W) = \theta_t, \tag{49}$$

$$b_i(W) = \frac{\theta_t(\theta^*_{r+i-1} - \theta^*_{r+1}) - \theta_{t+1}(\theta^*_{r+i} - \theta^*_r)}{\theta^*_{r+i-1} - \theta^*_{r+i+1}} \quad (1 \leqslant i \leqslant d-1), \tag{50}$$

$$b_d(W) = 0. \tag{51}$$

In particular, $c_i(W), a_i(W), b_i(W)$ *are real for* $0 \leqslant i \leqslant d$.

Proof. First assume $d=0$. We find $a_0(W)=\theta_t$ by setting $d=0$ in the equation in Corollary 8.6(i). By Definition 8.2, we find $c_0(W)=0$, $b_0(W)=0$.

Now assume $d \geqslant 1$. Eqs. (44), (51) are immediate from Definition 8.2, and Eq. (47) follows from Lemma 11.1. Eq. (49) follows from Lemmas 11.2(i) and (44). To obtain (45) and (50), solve the linear system determined by the equations in Lemmas 11.2(i) and 11.3(i) for the variables $c_i(W), b_i(W)$. We observe that the coefficient matrix of this system is nonsingular since $\theta^*_0, \ldots, \theta^*_D$ are distinct. To obtain (46) and (48), solve the linear system determined by the equations in Lemmas 11.2(ii) and 11.3(ii) for the variables $c_d(W), a_d(W)$. We observe that the coefficient matrix of this system is nonsingular since $\theta^*_0, \ldots, \theta^*_D$ are distinct. $\square$

12. Computation of $c^*_i(W), a^*_i(W), b^*_i(W)$

Let Y be as in Definition 9.1, and let W denote an irreducible T-module with diameter d. In this section, we compute the parameters $c^*_i(W), a^*_i(W), b^*_i(W)$ $(0 \leqslant i \leqslant d)$.

Lemma 12.1. *With reference to Definition* 9.1, *let W denote an irreducible T-module with endpoint r and diameter d. Fix any integer i* $(0 \leqslant i \leqslant d)$. *Then*

$$c^*_i(W) + a^*_i(W) + b^*_i(W) = \theta^*_r. \tag{52}$$

Proof. Let t denote the dual endpoint of W. Pick any nonzero $v \in E^*_r W$, and observe that $A^* v = \theta^*_r v$. We may now argue that

$$\begin{aligned}
&(c^*_i(W) + a^*_i(W) + b^*_i(W))E_{t+i}v \\
&\quad = (R^* E_{t+i-1} + F^* E_{t+i} + L^* E_{t+i+1})v \quad \text{(by Def. 8.3)} \\
&\quad = E_{t+i} A^* v \quad \text{(by (22), (23))} \\
&\quad = \theta^*_r E_{t+i} v.
\end{aligned}$$

The result now follows, since $E_{t+i}v \neq 0$ by Theorem 8.1(ii). $\square$

Lemma 12.2. *With reference to Definition* 9.1, *let W denote an irreducible T-module with endpoint r, dual endpoint t, and diameter d. Suppose* $d \geqslant 1$, *and fix any integer i* $(0 \leqslant i \leqslant d)$. *Then*

$$\theta_{t+i-1}c^*_i(W) + \theta_{t+i}a^*_i(W) + \theta_{t+i+1}b^*_i(W) = \theta^*_{r+1}\theta_{t+i}, \tag{53}$$

where $\theta_{-1}, \theta_{D+1}$ *are indeterminates.*

Proof. Pick any nonzero $v \in E_r^* W$. Observe $r < D$ since $d \geqslant 1$, so $Fv = 0$ by Lemma 9.2(iii). Hence $Av = Rv \in E_{r+1}^* W$. It follows that

$$A^* A v = \theta_{r+1}^* A v. \tag{54}$$

We may now argue that

$$\begin{aligned}
&(\theta_{t+i-1} c_i^*(W) + \theta_{t+i} a_i^*(W) + \theta_{t+i+1} b_i^*(W)) E_{t+i} v \\
&\quad = (\theta_{t+i-1} R^* E_{t+i-1} + \theta_{t+i} F^* E_{t+i} + \theta_{t+i+1} L^* E_{t+i+1}) v \quad \text{(by Def. 8.3)} \\
&\quad = (R^* E_{t+i-1} + F^* E_{t+i} + L^* E_{t+i+1}) A v \quad \text{(by (13))} \\
&\quad = E_{t+i} A^* A v \quad \text{(by (22), (23))} \\
&\quad = \theta_{r+1}^* E_{t+i} A v \quad \text{(by (54))} \\
&\quad = \theta_{r+1}^* \theta_{t+i} E_{t+i} v \quad \text{(by (13))}.
\end{aligned}$$

The result now follows, since $E_{t+i} v \neq 0$ by Theorem 8.1(ii). $\square$

Lemma 12.3. *With reference to Definition* 9.1, *let W denote an irreducible T-module with endpoint r, dual endpoint t, and diameter d. Suppose $d \geqslant 2$, and fix any integer i $(0 \leqslant i \leqslant d)$. Then*

$$\begin{aligned}
&\theta_{t+i-1}^2 c_i^*(W) + \theta_{t+i}^2 a_i^*(W) + \theta_{t+i+1}^2 b_i^*(W) \\
&\quad = \theta_{r+2}^* \theta_{t+i}^2 + b_0(W) c_1(W) (\theta_r^* - \theta_{r+2}^*),
\end{aligned} \tag{55}$$

where θ_{-1}, θ_{D+1} are indeterminates.

Proof. For notational convenience, set $\alpha := b_0(W) c_1(W)$. Pick any nonzero $v \in E_r^* W$. We first claim that $LRv = \alpha v$. To see this, observe by Theorem 8.1(i), there exists a nonzero $z \in E_t W$ such that $v = E_r^* z$. Applying Definition 8.2,

$$\begin{aligned}
LRv &= LRE_r^* z \\
&= c_1(W) L E_{r+1}^* z \\
&= b_0(W) c_1(W) E_r^* z,
\end{aligned}$$

and the claim follows.

Observe $r+1 < D$ since $d \geqslant 2$, so $Fv = 0$, $FRv = 0$ by Lemma 9.2(iii). By these remarks, the above claim, Eq. (16), and since $Lv = 0$,

$$\begin{aligned}
R^2 v &= (R + F + L)^2 v - LRv \\
&= (A^2 - \alpha I) v.
\end{aligned} \tag{56}$$

By (56), and since $R^2 v \in E_{r+2}^* V$ by (17),

$$A^* (A^2 - \alpha I) v = \theta_{r+2}^* (A^2 - \alpha I) v. \tag{57}$$

We may now argue that

$$
\begin{aligned}
&(\theta^2_{t+i-1}c^*_i(W) + \theta^2_{t+i}a^*_i(W) + \theta^2_{t+i+1}b^*_i(W))E_{t+i}v \\
&\quad = (\theta^2_{t+i-1}R^*E_{t+i-1} + \theta^2_{t+i}F^*E_{t+i} + \theta^2_{t+i+1}L^*E_{t+i+1})v \quad \text{(by Def. 8.3)} \\
&\quad = (R^*E_{t+i-1} + F^*E_{t+i} + L^*E_{t+i+1})A^2v \quad \text{(by (13))} \\
&\quad = E_{t+i}A^*A^2v \quad \text{(by (22),(23))} \\
&\quad = E_{t+i}A^*(A^2 - \alpha I)v + \alpha E_{t+i}A^*v \\
&\quad = \theta^*_{r+2}(A^2 - \alpha I)E_{t+i}v + \alpha\theta^*_r E_{t+i}v \quad \text{(by (57))} \\
&\quad = (\theta^*_{r+2}(\theta^2_{t+i} - \alpha) + \alpha\theta^*_r)E_{t+i}v \quad \text{(by (13))}.
\end{aligned}
$$

The result now follows, since $E_{t+i}v \neq 0$ by Theorem 8.1(ii). □

Theorem 12.4. *With reference to Definition* 9.1, *let W denote an irreducible T-module with endpoint r, dual endpoint t, and diameter d. First assume $d=0$. Then $c^*_0(W)=0$, $a^*_0(W)=\theta^*_r$, $b^*_0(W)=0$. Now assume $d \geqslant 1$. Then*

$$c^*_0(W) = 0, \tag{58}$$

$$c^*_i(W) = \frac{(\theta^2_{t+i} - \theta^2_t)(\theta^*_{r+2} - \theta^*_{r+1}) + (\theta_t\theta_{t+1} - \theta_{t+i}\theta_{t+i+1})(\theta^*_{r+1} - \theta^*_r)}{(\theta_{t+i-1} - \theta_{t+i})(\theta_{t+i-1} - \theta_{t+i+1})} \quad (1 \leqslant i \leqslant d-1), \tag{59}$$

$$c^*_d(W) = \frac{\theta_{t+d}(\theta^*_{r+1} - \theta^*_r)}{\theta_{t+d-1} - \theta_{t+d}}, \tag{60}$$

$$b^*_0(W) = \frac{\theta_t(\theta^*_r - \theta^*_{r+1})}{\theta_t - \theta_{t+1}}, \tag{61}$$

$$b^*_i(W) = \frac{(\theta^2_{t+i} - \theta^2_t)(\theta^*_{r+2} - \theta^*_{r+1}) + (\theta_t\theta_{t+1} - \theta_{t+i}\theta_{t+i-1})(\theta^*_{r+1} - \theta^*_r)}{(\theta_{t+i+1} - \theta_{t+i})(\theta_{t+i+1} - \theta_{t+i-1})} \quad (1 \leqslant i \leqslant d-1), \tag{62}$$

$$b^*_d(W) = 0, \tag{63}$$

$$a^*_i(W) = \theta^*_r - b^*_i(W) - c^*_i(W) \quad (0 \leqslant i \leqslant d). \tag{64}$$

In particular, $c^*_i(W)$, $a^*_i(W)$, $b^*_i(W)$ are real for $0 \leqslant i \leqslant d$.

Proof. First assume $d=0$. We find $a^*_0(W)=\theta^*_r$ by setting $d=0$ in the equation in Corollary 8.6(ii). By Definition 8.3, we find $c^*_0(W)=0$, $b^*_0(W)=0$.

Now assume $d \geqslant 1$. Eqs. (58), (63) follow from Definition 8.3. Eq. (64) follows from (52). We obtain (60) by setting $i=d$ in (52), (53), and solving for $c^*_d(W)$, using (63). We now have (60), and Eq. (61) is obtained similarly.

It remains to prove (59) and (62). Assume $d \geqslant 2$ and fix any i $(1 \leqslant i \leqslant d-1)$. Observe (52), (53), and (55) form a system of three linear equations in the three variables $c^*_i(W)$, $a^*_i(W)$, and $b^*_i(W)$. Observe the coefficient matrix is Vandermonde, hence nonsingular, since $\theta_0, \ldots, \theta_D$ are distinct. Solving this system, we obtain (59) and (62). □

13. Some comments on the intersection numbers

Let Y be as in Definition 9.1, and let W denote an irreducible T-module with diameter d. In this section, we show the expressions $b_{i-1}(W)c_i(W)$ and $b^*_{i-1}(W)c^*_i(W)$ are positive for $1 \leqslant i \leqslant d$.

Lemma 13.1. *With reference to Definition* 9.1, *let W denote an irreducible T-module with endpoint r, dual endpoint t, and diameter d.*

(i) *For any nonzero* $v \in E_t W$,

$$c_i(W)\|E^*_{r+i}v\|^2 = b_{i-1}(W)\|E^*_{r+i-1}v\|^2 \quad (1 \leqslant i \leqslant d).$$

(ii) *For any nonzero* $v \in E^*_r W$,

$$c^*_i(W)\|E_{t+i}v\|^2 = b^*_{i-1}(W)\|E_{t+i-1}v\|^2 \quad (1 \leqslant i \leqslant d).$$

Proof. (i) By (26), (28), and since $R = \overline{L}^{\mathrm{t}}$,

$$\begin{aligned} c_i(W)\|E^*_{r+i}v\|^2 &= \langle RE^*_{r+i-1}v, E^*_{r+i}v\rangle \\ &= \langle E^*_{r+i-1}v, LE^*_{r+i}v\rangle \\ &= \overline{b_{i-1}(W)}\|E^*_{r+i-1}v\|^2. \end{aligned}$$

Recall that $b_{i-1}(W)$ is real by Theorem 11.4, so the result follows.

(ii) Similar to the proof of (i). □

Corollary 13.2. *With reference to Definition* 9.1, *let W denote an irreducible T-module with diameter d. Then*

(i) $b_{i-1}(W)c_i(W) > 0$ $(1 \leqslant i \leqslant d)$,
(ii) $b^*_{i-1}(W)c^*_i(W) > 0$ $(1 \leqslant i \leqslant d)$.

Proof. (i) Let r denote the endpoint of W. The product $b_{i-1}(W)c_i(W)$ is nonnegative by Lemma 13.1(i), and since $\|E^*_{r+i}v\|^2$ and $\|E^*_{r+i-1}v\|^2$ are positive. Observe $b_{i-1}(W) \neq 0$ by (28) and the previously mentioned fact that $LE^*_iW \neq 0$ $(r < i \leqslant r+d)$. Similarly, $c_i(W) \neq 0$ by (26) and the fact that $RE^*_iW \neq 0$ $(r \leqslant i < r+d)$.

(ii) Similar to the proof of (i). □

14. The isomorphism classes of irreducible T-modules

With reference to Definition 9.1, in this section we prove that the isomorphism class of any given irreducible T-module is determined by its dual endpoint and diameter.

Theorem 14.1. *With reference to Definition* 9.1, *let* W, W' *denote irreducible T-modules with endpoints* r, r', *dual endpoints* t, t', *and diameters* d, d', *respectively. Then the following* *are equivalent*:

(i) W *and* W' *are isomorphic as T-modules*.
(ii) $t = t'$ *and* $d = d'$.
(iii) $t = t'$ *and* $r = r'$.
(iv) $B(W) = B(W')$.
(v) $t = t'$ *and* $B^*(W) = B^*(W')$.

Proof. (i) → (ii) Let ϕ denote a T-isomorphism from W to W'. Then for any integer i $(0 \leqslant i \leqslant D)$,

$$E_i W = 0 \quad \Leftrightarrow \quad \phi(E_i W) = 0 \quad \Leftrightarrow \quad E_i \phi(W) = 0 \quad \Leftrightarrow \quad E_i W' = 0.$$

Now (ii) follows by (24) and since $d = d^*$.

(ii) ↔ (iii) Immediate by Theorem 9.3.

(ii), (iii) → (v) By Theorem 11.4, the entries in the intersection matrix are determined by the endpoint, dual endpoint, and diameter.

(ii), (iii) → (v) By Theorem 12.4, the entries in the dual intersection matrix are determined by the endpoint, dual endpoint, and diameter.

(iv) → (ii) $B(W)$ is a $d+1$ by $d+1$ matrix, and $B(W')$ is $d'+1$ by $d'+1$, so $d = d'$. By Lemmas 11.1 and 11.2, the sum of the entries in each row of $B(W)$ equals θ_t, and the sum of the entries in each row of $B(W')$ equals $\theta_{t'}$. Hence $\theta_t = \theta_{t'}$, so $t = t'$ in view of (14).

(v) → (i) $B^*(W)$ is a $d+1$ by $d+1$ matrix, and $B^*(W')$ is $d'+1$ by $d'+1$, so $d = d'$. Now $r = r'$ by Theorem 9.3. Pick a nonzero $v \in E_r^* W$ and recall by Theorem 8.1(ii) that $E_t v, \ldots, E_{t+d} v$ is a basis for W. Similarly, pick a nonzero $v' \in E_r^* W'$, and observe $E_t v', \ldots, E_{t+d} v'$ is a basis for W'. By linear algebra, there exists an isomorphism of vector spaces $\phi : W \to W'$ such that

$$\phi : E_i v \mapsto E_i v' \quad (t \leqslant i \leqslant t+d). \tag{65}$$

We show ϕ is an isomorphism of T-modules. Since $A^*, E_0, \ldots, E_D$ generate T, and since $A^* = R^* + F^* + L^*$, it suffices to show

$$(R^*\phi - \phi R^*)W = 0, \tag{66}$$

$$(F^*\phi - \phi F^*)W = 0, \tag{67}$$

$$(L^*\phi - \phi L^*)W = 0, \tag{68}$$

$$(E_j\phi - \phi E_j)W = 0 \quad (0 \leqslant j \leqslant D). \tag{69}$$

Eq. (69) is immediate from the construction. To see (66), observe that $c_i^*(W) = c_i^*(W')$ $(0 \leqslant i \leqslant d)$. Now by (29), $R^*\phi - \phi R^*$ vanishes on $E_{t+i} v$ $(0 \leqslant i \leqslant d)$. Eq. (66) follows. Eqs. (67), (68) are proven similarly. □

We conclude this section with a few comments.

Lemma 14.2 (*Terwilliger [25, Theorem 4.1]*). *With reference to Definition* 9.1, *there exists a unique irreducible T-module* W_0 *with diameter D. The endpoint and dual endpoint of* W_0 *are both zero. We refer to* W_0 *as the trivial T-module*.

Lemma 14.3 (*Terwilliger [25, Theorem 4.1]*). *With reference to Definition* 9.1, *let* W_0 *denote the trivial module for Y. Then*

(i) $c_i(W_0) = c_i, \; a_i(W_0) = a_i, \; b_i(W_0) = b_i \; (0 \leqslant i \leqslant D)$,
(ii) $c_i^*(W_0) = c_i^*, \; a_i^*(W_0) = a_i^*, \; b_i^*(W_0) = b_i^* \; (0 \leqslant i \leqslant D)$.

15. The parameters in terms of q and s

With reference to Definition 9.1, we now explicitly compute the intersection matrices and dual intersection matrices of the irreducible T-modules. For convenience, we exclude a small class of examples.

Let D denote any integer at least 3. Let O_{D+1} denote the odd graph with diameter D, and let $\square_{2D+1}$ denote the folded $(2D+1)$-cube [4, pp. 259,264]. It is well-known that O_{D+1}, $\square_{2D+1}$ are almost-bipartite P- and Q-polynomial schemes with diameter D. In [20], it was shown that O_{D+1} is uniquely determined by its intersection numbers. By Brouwer et al. [4, p. 264], $\square_{2D+1}$ is uniquely determined by its intersection numbers. For more information on the structure of the irreducible T-modules for these schemes, see [26].

Lemma 15.1. *With reference to Definition* 9.1, *suppose Y is not one of* $O_{D+1}, \square_{2D+1}$. *Then there exist scalars* $q, s, h, h^* \in \mathbb{C}$, *with* q, h, h^* *nonzero, such that*

$$\theta_i = \theta_0 + h(1-q^i)(1-sq^{i+1})q^{-i} \quad (0 \leqslant i \leqslant D), \tag{70}$$

$$\theta_i^* = \theta_0^* + h^*(1-q^i)(1-q^{-2D-1+i})q^{-i} \quad (0 \leqslant i \leqslant D). \tag{71}$$

Proof. By Brouwer et al. [4, pp. 237, 240], there exist scalars $\beta, \gamma, \gamma^* \in \mathbb{R}$ such that

$$\theta_{i-1} - \beta\theta_i + \theta_{i+1} = \gamma \quad (1 \leqslant i \leqslant D-1), \tag{72}$$

$$\theta_{i-1}^* - \beta\theta_i^* + \theta_{i+1}^* = \gamma^* \quad (1 \leqslant i \leqslant D-1). \tag{73}$$

We show $\beta \neq 2$, $\beta \neq -2$. First, suppose $\beta = 2$. By Brouwer et al. [4, Theorem 1.11.1], there exists a bipartite P-polynomial scheme Y' whose quotient scheme is Y. Applying [19, Theorems 10.4,15.2] to Y', we find Y is $\square_{2D+1}$, a contradiction. Therefore $\beta \neq 2$.

Now suppose $\beta = -2$. Then by Terwilliger [22, Theorem 2], Y must be O_{D+1} or $\square_{2D+1}$, a contradiction. Therefore $\beta \neq -2$.

Since $\beta \neq 2$ and $\beta \neq -2$, there exists $q \in \mathbb{C}$ such that $q \notin \{1, 0, -1\}$, and such that $\beta = q + q^{-1}$. Solving the recurrence in (72), we obtain

$$\theta_i = hq^{-i} + h'q^i + h'' \quad (0 \leqslant i \leqslant D) \tag{74}$$

for some $h, h', h'' \in \mathbb{C}$. By (14), h, h' are not both zero. Replacing q by q^{-1} if necessary, we may assume $h \neq 0$. Now there exists $s \in \mathbb{C}$ such that $h' = sqh$. Eliminating h' in (74) using this, we find

$$\theta_i - \theta_0 = h(1-q^i)(1-sq^{i+1})q^{-i} \quad (0 \leqslant i \leqslant D),$$

and (70) follows.

Solving (73), we obtain

$$\theta^*_i = h^* q^{-i} + h^{*\prime} q^i + h^{*\prime\prime} \quad (0 \leqslant i \leqslant D) \tag{75}$$

for some $h^*, h^{*\prime}, h^{*\prime\prime} \in \mathbb{C}$. By Terwilliger [24, Theorem 2] we have $h^{*\prime} = h^* q s^*$, where $s^* = q^{-2D-2}$. Eliminating $h^{*\prime}$ in (75) using this, we obtain (71). Observe h^* is nonzero by (20). □

Corollary 15.2. *With reference to Definition* 9.1, *suppose* Y *is not one of* $O_{D+1}, \Box_{2D+1}$. *Let* q, s *be as in Lemma* 15.1. *Then*

$$q^i \neq 1 \quad (1 \leqslant i \leqslant 2D), \tag{76}$$

$$sq^i \neq 1 \quad (2 \leqslant i \leqslant 2D). \tag{77}$$

Proof. Evaluate (14) and (20) in terms of q, s using Lemma 15.1. □

Lemma 15.3. *With reference to Definition* 9.1, *suppose* Y *is not one of* $O_{D+1}, \Box_{2D+1}$. *Let* q, s, h, h^* *be as in Lemma* 15.1. *Then*

$$\theta_0 = h(1 + sq), \tag{78}$$

$$\theta^*_0 = \frac{h^*(q^{2D} - 1)(1 + sq)}{q^{2D}(1 - sq^2)}. \tag{79}$$

Proof. We apply (55) with $W = W_0$, $r = 0$, $t = 0$, $d = D$, and $i = D$. Evaluating the result using Theorem 11.4, Theorem 12.4, and Lemma 15.1, we obtain (78).

Recall that $a^*_0 = 0$, so by (41) (with $r = 0$, $t = 0$),

$$\theta^*_1 \theta_0 = \theta_1 \theta^*_0.$$

Solving this equation for θ^*_0 using (70), (71), (78), we obtain (79). □

Corollary 15.4. *With reference to Definition* 9.1, *suppose* Y *is not one of* $O_{D+1}, \Box_{2D+1}$. *Let* q, s, h, h^* *be as in Lemma* 15.1. *Then*

$$\theta_i = hq^{-i}(1 + sq^{2i+1}) \quad (0 \leqslant i \leqslant D). \tag{80}$$

Proof. Routine using (70), (78). □

Theorem 15.5. *With reference to Definition* 9.1, *suppose* Y *is not one of* $O_{D+1}, \Box_{2D+1}$. *Let* W *denote an irreducible T-module with dual endpoint* t *and diameter* d. *First assume* $d = 0$. *Then* $c_0(W) = 0$, $a_0(W) = hq^{-t}(1 + sq^{2t+1})$, $b_0(W) = 0$, *where* q, s, h *are as in Lemma* 15.1. *Now assume* $d \geqslant 1$. *Then*

$$c_0(W) = 0, \tag{81}$$

$$c_i(W) = \frac{h(1 - q^i)(1 + sq^{2+2d+2t-i})}{q^{t+i}(q^{2d-2i+1} - 1)} \quad (1 \leqslant i \leqslant d - 1), \tag{82}$$

$$c_d(W) = \frac{h(1-q^d)(1+sq^{2+d+2t})}{q^{t+d}(q-1)}, \tag{83}$$

$$a_i(W) = 0 \quad (0 \leqslant i \leqslant d-1), \tag{84}$$

$$a_d(W) = \frac{h(q^{d+1}-1)(1+sq^{1+d+2t})}{q^{t+d}(q-1)}, \tag{85}$$

$$b_0(W) = hq^{-t}(sq^{2t+1}+1), \tag{86}$$

$$b_i(W) = \frac{h(q^{2d+1-i}-1)(1+sq^{2t+i+1})}{q^{t+i}(q^{2d-2i+1}-1)} \quad (1 \leqslant i \leqslant d-1), \tag{87}$$

$$b_d(W) = 0, \tag{88}$$

where q, s, h *are from Lemma* 15.1.

Proof. Routine using Theorems 11.4, 9.3, (71), (79), and (80). □

Theorem 15.6. *With reference to Definition* 9.1, *suppose Y is not one of* $O_{D+1}, \square_{2D+1}$. *Let W denote an irreducible T-module with endpoint r, dual endpoint t, and diameter d. Then*

$$c_0^*(W) = 0, \tag{89}$$

$$c_i^*(W) = \frac{h^*(1-q^{2i})(1-s^2q^{2+2d+4t+2i})}{q^{D+d+1}(1-sq^{2i+2t})(1-sq^{1+2i+2t})} \quad (1 \leqslant i \leqslant d-1), \tag{90}$$

$$c_d^*(W) = \frac{h^*(1-q^{2d})(1+sq^{2t+2d+1})}{q^{D+d+1}(1-sq^{2t+2d})}, \tag{91}$$

$$b_0^*(W) = \frac{h^*(q^{2d}-1)(1+sq^{2t+1})}{q^{D+d}(1-sq^{2+2t})}, \tag{92}$$

$$b_i^*(W) = \frac{h^*(q^{2d-2i}-1)(1-s^2q^{2+2i+4t})}{q^{D+d-2i}(1-sq^{2+2i+2t})(1-sq^{1+2i+2t})} \quad (1 \leqslant i \leqslant d-1), \tag{93}$$

$$b_d^*(W) = 0, \tag{94}$$

$$a_i^*(W) = \theta_r^* - b_i^*(W) - c_i^*(W) \quad (0 \leqslant i \leqslant d), \tag{95}$$

where q, s, h^* *are from Lemma* 15.1.

Proof. Routine using Theorems 12.4, 9.3, (71), (79), and (80). □

Corollary 15.7. *With reference to Definition* 9.1, *suppose Y is not one of* $O_{D+1}, \square_{2D+1}$. *Let* q, s *be as in Lemma* 15.1. *Then*

$$sq^i \neq -1 \quad (1 \leqslant i \leqslant 2D+1).$$

Proof. Recall the trivial module W_0 has dual endpoint $t=0$ and diameter $d=D$. By Lemma 14.3, the intersection numbers and dual intersection numbers of W_0 in (82), (85)–(87), (90) are nonzero. Using this information, we obtain the desired result. □

Corollary 15.8. *With reference to Definition* 9.1, *suppose Y is not one of* O_{D+1}, $\square_{2D+1}$. *Then*

$$h = \frac{q - q^{2D}}{(q-1)(1+sq^{2D+1})}, \tag{96}$$

$$h^* = \frac{q^{2D+1}(1-sq^2)(1-sq^3)}{(1-q^2)(1-s^2q^{2D+4})}, \tag{97}$$

where q, s, h, h^* *are from Lemma* 15.1. *We remark the denominators in* (96), (97) *are nonzero by Corollaries* 15.2 *and* 15.7.

Proof. Concerning (96), recall the trivial module W_0 has endpoint $r=0$, dual endpoint $t=0$, and diameter $d=D$. Also $c_1(W_0)=1$ by Lemma 14.3 and since $c_1=1$. Evaluating (82) using this data we find

$$1 = \frac{h(q-1)(1+sq^{2D+1})}{q-q^{2D}},$$

and Eq. (96) follows. Eq. (97) is proven similarly. □

16. Multiplicities of the irreducible T-modules

With reference to Definition 9.1, in this section we compute the multiplicities with which the irreducible T-modules appear in the standard module V.

Definition 16.1. With reference to Definition 9.1, fix a decomposition of the standard module V into an orthogonal direct sum of irreducible T-modules. For any integers t, d $(0 \leqslant t, d \leqslant D)$, we define $\mathrm{mult}(t,d)$ to be the number of irreducible modules in this decomposition which have dual endpoint t and diameter d. It is well-known that $\mathrm{mult}(t,d)$ is independent of the decomposition (cf. [10]).

Definition 16.2. With reference to Definition 9.1, define a set Υ by

$$\Upsilon := \{(t,d) \in \mathbb{Z}^2 \mid 0 \leqslant d \leqslant D,\ \tfrac{1}{2}(D-d) \leqslant t \leqslant D-d\}.$$

By (25), $\mathrm{mult}(t,d)=0$ for all integers t, d such that $(t,d) \notin \Upsilon$. We define a partial order $\preccurlyeq$ on Υ by

$$(t,d) \preccurlyeq (t',d') \quad \text{if and only if} \quad t \leqslant t' \text{and } t'+d' \leqslant t+d.$$

Example 16.3. With reference to Definition 9.1, suppose $D=7$. In Fig. 1, we represent each element $(t,d) \in \Upsilon$ by a line segment beginning in the tth column and having length d. For any elements $a \in \Upsilon$, $b \in \Upsilon$, observe that $a \preccurlyeq b$ if and only if the line segment representing a extends the line segment representing b.

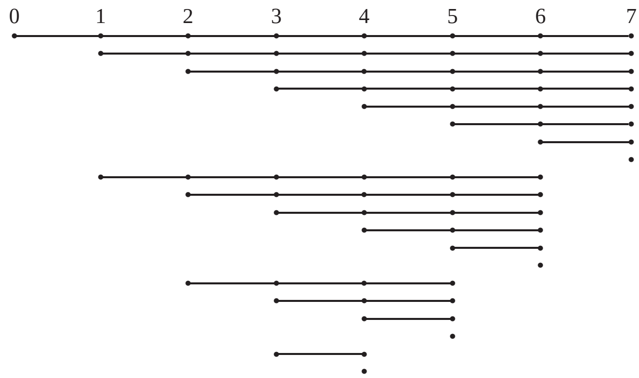

Fig. 1. The set Υ when $D=7$.

Lemma 16.4. *With reference to Definition* 9.1 *and Definition* 16.2, *fix any* $(t,d)\in\Upsilon$. *Then*

$$\operatorname{trace}(E_t L^{*d} R^{*d} E_t) = m_t \prod_{h=t}^{t+d-1} b_h^* c_{t+d-h}^*. \tag{98}$$

Proof. By Dickie and Terwilliger [12, Lemma 4.1], we find

$$\operatorname{trace}(E_t A_d^* E_{t+d} A_d^* E_t) = m_t q_{d,t+d}^t. \tag{99}$$

By Bannai and Ito [2, p. 276] and since Γ is Q-polynomial, we find

$$q_{d,t+d}^t = \prod_{h=t}^{t+d-1} \frac{b_h^*}{c_{t+d-h}^*}. \tag{100}$$

We claim

$$E_{t+d} A_d^* E_t = \frac{1}{c_1^* c_2^* \cdots c_d^*} R^{*d} E_t. \tag{101}$$

To see this, recall that for $0 \leqslant i \leqslant D$, A_i^* is a polynomial in A^* with degree i and leading coefficient $(c_1^* c_2^* \cdots c_i^*)^{-1}$. Combining this and (9) we find $E_{t+d} A^{*i} E_t = 0$ for $0 \leqslant i \leqslant d-1$. We may now argue

$$\begin{aligned} c_1^* c_2^* \cdots c_d^* E_{t+d} A_d^* E_t &= E_{t+d} A^{*d} E_t \\ &= E_{t+d} A^* E_{t+d-1} A^* \cdots E_{t+1} A^* E_t \\ &= R^{*d} E_t. \end{aligned}$$

We now have (101). Taking the transpose of (101) we get

$$E_t A_d^* E_{t+d} = \frac{1}{c_1^* c_2^* \cdots c_d^*} E_t L^{*d}. \tag{102}$$

Evaluating (99) using (100)–(102), we get the desired result. $\square$

Lemma 16.5. *With reference to Definition* 9.1 *and Definition* 16.2, *fix* $(i, j) \in \Upsilon$ *and* $(t, d) \in \Upsilon$. *Then for any irreducible T-module W with dual endpoint i and diameter j,*

$$\mathrm{trace}(E_t L^{*d} R^{*d} E_t)|_W = \prod_{h=t-i}^{t-i+d-1} b_h^*(W) c_{h+1}^*(W) \tag{103}$$

if $(i, j) \preccurlyeq (t, d)$, *and* $\mathrm{trace}(E_t L^{*d} R^{*d} E_t)|_W = 0$ *if* $(i, j) \not\preccurlyeq (t, d)$.

Proof. Let r denote the endpoint of W and pick any nonzero $v \in E_r^* W$. By Theorem 8.1(ii), $B := \{E_h v \mid i \leqslant h \leqslant i + j\}$ is a basis for W. We consider the action of $E_t L^{*d} R^{*d} E_t$ on B. First assume $(i, j) \preccurlyeq (t, d)$. Then $E_t L^{*d} R^{*d} E_t$ vanishes on every element of B except $E_t v$, and

$$E_t L^{*d} R^{*d} E_t (E_t v) = \prod_{h=t-i}^{t-i+d-1} b_h^*(W) c_{h+1}^*(W) E_t v$$

by (29), (31). Eq. (103) follows. Next assume $(i, j) \not\preccurlyeq (t, d)$. Then $E_t L^{*d} R^{*d} E_t$ vanishes on each element of B, and so its trace on W is zero. □

To state our next theorem, we need a bit of notation. Let Y be as in Definition 9.1 and let Υ be as in Definition 16.2. Select $(t, d) \in \Upsilon$. We define $c_0^*(t, d) := 0$, $b_d^*(t, d) := 0$. For $d \geqslant 1$ we further define

$$c_i^*(t, d) := \frac{(\theta_{t+i}^2 - \theta_t^2)(\theta_{r+2}^* - \theta_{r+1}^*) + (\theta_t \theta_{t+1} - \theta_{t+i} \theta_{t+i+1})(\theta_{r+1}^* - \theta_r^*)}{(\theta_{t+i-1} - \theta_{t+i})(\theta_{t+i-1} - \theta_{t+i+1})}$$
$$(1 \leqslant i \leqslant d - 1),$$

$$c_d^*(t, d) := \frac{\theta_{t+d}(\theta_{r+1}^* - \theta_r^*)}{\theta_{t+d-1} - \theta_{t+d}},$$

$$b_0^*(t, d) := \frac{\theta_t(\theta_r^* - \theta_{r+1}^*)}{\theta_t - \theta_{t+1}},$$

$$b_i^*(t, d) := \frac{(\theta_{t+i}^2 - \theta_t^2)(\theta_{r+2}^* - \theta_{r+1}^*) + (\theta_t \theta_{t+1} - \theta_{t+i} \theta_{t+i-1})(\theta_{r+1}^* - \theta_r^*)}{(\theta_{t+i+1} - \theta_{t+i})(\theta_{t+i+1} - \theta_{t+i-1})}$$
$$(1 \leqslant i \leqslant d - 1),$$

where $r := D - d$. Observe that if W is any irreducible T-module with dual endpoint t and diameter d, then $(t, d) \in \Upsilon$ and $c_i^*(t, d) = c_i^*(W)$, $b_i^*(t, d) = b_i^*(W)$ $(0 \leqslant i \leqslant d)$. However, such a module need not exist.

We now give a recurrence which will enable us to compute the multiplicities of the irreducible T-modules.

Theorem 16.6. *With reference to Definition* 9.1, *fix any* $(t, d) \in \Upsilon$. *Then*

$$m_t \prod_{h=t}^{t+d-1} b_h^* c_{t+d-h}^* = \sum_{\substack{(i,j)\in\Upsilon \\ (i,j)\preccurlyeq(t,d)}} \mathrm{mult}(i, j) \prod_{h=t-i}^{t-i+d-1} b_h^*(i, j) c_{h+1}^*(i, j). \tag{104}$$

Proof. Since T is semisimple, we may decompose the standard module V as

$$V = W_1 + W_2 + \cdots + W_s \quad \text{(orthogonal direct sum)},$$

where $W_1, W_2, \ldots, W_s$ are irreducible T-modules. It follows that

$$\mathrm{trace}(E_t L^{*d} R^{*d} E_t) = \sum_{i=1}^{s} \mathrm{trace}(E_t L^{*d} R^{*d} E_t)|_{W_i}. \tag{105}$$

Evaluating (105) using (98), (103) we obtain (104). □

Remark 16.7. With reference to Definition 9.1, we can use Theorem 16.6 to recursively compute the multiplicities $\{\mathrm{mult}(t, d) \mid (t, d) \in \Upsilon\}$. Indeed, pick any $(t, d) \in \Upsilon$. Then (104) gives a linear equation in the variables $\{\mathrm{mult}(i, j) \mid (i, j) \in \Upsilon, (i, j) \preccurlyeq (t, d)\}$. In this equation, the coefficient of $\mathrm{mult}(t, d)$ is

$$\prod_{h=0}^{d-1} b_h^*(t, d) c_{h+1}^*(t, d). \tag{106}$$

Suppose the coefficient (106) is nonzero. Then we can divide both sides of Eq. (104) by it, and obtain $\mathrm{mult}(t, d)$ in terms of $\{\mathrm{mult}(i, j) \mid (i, j) \prec (t, d)\}$. Suppose the coefficient (106) equals 0. By Corollary 13.2(ii), there is no module with dual endpoint t and diameter d, so $\mathrm{mult}(t, d) = 0$.

We now illustrate Theorem 16.6 with an example.

Example 16.8. With reference to Definition 9.1, suppose Y is not one of $O_{D+1}, \square_{2D+1}$. For all $(t, d) \in \Upsilon$ such that $d \geqslant D - 3$, the scalar $\mathrm{mult}(t, d)$ is given below.

(i) $\mathrm{mult}(0, D) = 1$.

(ii) $\mathrm{mult}(1, D-1) = \frac{(q^{2D}-1)(1+sq^2)}{(1-q)(1+sq^{2D+1})}$.

(iii) $\mathrm{mult}(1, D-2) = \frac{(q^{2D}-q^2)(1+sq)(1+sq^2)(sq^{2D+2}-1)}{(q^2-1)(s^2q^{2D+4}-1)(1+sq^{2D+1})}$.

(iv) $\mathrm{mult}(2, D-2) = \frac{(q^{2D}-1)(q^{2D}-q^2)(1+sq)(1+sq^4)(s^2q^{2D+3}-1)}{q(q+1)(q-1)^2(s^2q^{2D+4}-1)(1+sq^{2D})(1+sq^{2D+1})}$.

(v) $\mathrm{mult}(2, D-3) = \frac{(q^{2D}-1)(q^{2D}-q^4)(1+sq)(1+sq^2)(1+sq^4)(1-sq^{2D+2})}{q(q-1)(q^2-1)(1+sq^{2D+1})(sq^{D+3}-1)(q+sq^{2D})(1+sq^{D+3})}$.

(vi) $\mathrm{mult}(3, D-3) = \frac{(q^{2D}-1)(q^{2D}-q^2)(q^{2D}-q^4)(1+sq)(1+sq^2)(1+sq^6)(1-s^2q^{2D+3})}{q^2(q-1)(q^2-1)(q^3-1)(1+sq^{D+3})(sq^{D+3}-1)(q+sq^{2D})(1+sq^{2D})(1+sq^{2D+1})}$.

The scalars q, s are from Lemma 15.1. We remark the denominators in (i)–(vi) are nonzero by Corollaries 15.2 and 15.7.

Proof. Solve (104) recursively for the multiplicities, as outlined in Remark 16.7. Evaluate the results using Lemmas 15.1, 15.3 and Corollaries 15.4, 15.8. □

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