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J. Math. Anal. Appl. 329 (2007) 879–890

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

A fixed point approach to the stability of isometries

Soon-Mo Jung

Mathematics Section, College of Science and Technology, Hong-Ik University, 339-701 Chochiwon, Republic of Korea

Received 21 May 2006

Available online 8 August 2006

Submitted by G. Jungck

Abstract

In this paper, we apply a fixed point theorem to the proof of Hyers–Ulam–Rassias stability property for isometries from a normed space into a Banach space, in which the parallelogram law holds.

Keywords: Fixed point; Fixed point method; Hyers–Ulam–Rassias stability; Isometry; Contractive operator

1. Introduction

Let (E, d_E) and (F, d_F) be metric spaces. A mapping $I : E \to F$ is called an isometry if I satisfies

$$d_F\big(I(x), I(y)\big) = d_E(x, y)$$

for all $x, y \in E$.

Extending the definition by D.H. Hyers and S.M. Ulam [13], we may call a mapping $f : E \to F$ an ε-isometry if f satisfies the inequality

$$\big|d_F\big(f(x), f(y)\big) - d_E(x, y)\big| \leqslant \varepsilon \tag{$*$}$$

for all $x, y \in E$. The question is whether ε being small implies the existence of an isometry I such that $\operatorname{dist}(f, I) = \sup\{d_F(f(x), I(x))\}$ is small. If the answer to this question is affirmative, then we may say that the stability problem for isometries between (E, d_E) and (F, d_F) is stable in the sense of Ulam.

E-mail address: smjung@hongik.ac.kr.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2006.06.098

In [13], Hyers and Ulam proved the stability property for surjective isometries between real Hilbert spaces. Indeed, they proved that if a surjective mapping $f : E \to E$, where E is a real Hilbert space, satisfies $f(0) = 0$ as well as inequality $(*)$ for some $\varepsilon \geqslant 0$ and for all $x, y \in E$, then there exists a surjective isometry $I : E \to E$ such that $\|f(x) - I(x)\| \leqslant 10\varepsilon$ for all $x \in E$.

This result was further generalized by D.G. Bourgin [4], who proved the following theorem: Assume that E is a Banach space and that F belongs to a class of uniformly convex spaces which includes the spaces $L_p(0, 1)$ for $1 < p < \infty$. If a mapping $f : E \to F$ satisfies $f(0) = 0$ as well as inequality $(*)$ for some $\varepsilon \geqslant 0$ and for all $x, y \in E$, then there exists a linear isometry $I : E \to F$ such that $\|f(x) - I(x)\| \leqslant 12\varepsilon$ for each $x \in E$.

Subsequently, Hyers and Ulam [14] studied a stability problem for spaces of continuous mappings: Let S_1 and S_2 be compact metric spaces and $C(S_i)$ denote the space of real-valued continuous mappings on S_i equipped with the metric topology with $\|\cdot\|_\infty$. If a homeomorphism $T : C(S_1) \to C(S_2)$ satisfies the inequality

$$\left| \left\| T(f) - T(g) \right\|_\infty - \|f - g\|_\infty \right| \leqslant \varepsilon \tag{$**$}$$

for some $\varepsilon \geqslant 0$ and for all $f, g \in C(S_1)$, then there exists an isometry $U : C(S_1) \to C(S_2)$ such that $\|T(f) - U(f)\|_\infty \leqslant 21\varepsilon$ for every $f \in C(S_1)$.

This result of Hyers and Ulam was significantly generalized by D.G. Bourgin again (see [5]): Let S_1 and S_2 be completely regular Hausdorff spaces and let $T : C(S_1) \to C(S_2)$ be a surjective mapping satisfying inequality $(**)$ for some $\varepsilon \geqslant 0$ and for all $f, g \in C(S_1)$. Then there exists a linear isometry $U : C(S_1) \to C(S_2)$ such that $\|T(f) - U(f)\|_\infty \leqslant 10\varepsilon$ for any $f \in C(S_1)$.

The study of stability problems for isometries on finite-dimensional Banach spaces was continued by R.D. Bourgin [6].

In 1978, P.M. Gruber [12] obtained an elegant result as follows: Let E and F be real normed spaces. Suppose that $f : E \to F$ is a surjective mapping and it satisfies inequality $(*)$ for some $\varepsilon \geqslant 0$ and for all $x, y \in E$. Furthermore, assume that $I : E \to F$ is an isometry with $f(p) = I(p)$ for some $p \in E$. If $\|f(x) - I(x)\| = o(\|x\|)$ as $\|x\| \to \infty$ uniformly, then I is a surjective linear isometry and $\|f(x) - I(x)\| \leqslant 5\varepsilon$ for all $x \in E$. If in addition f is continuous, then $\|f(x) - I(x)\| \leqslant 3\varepsilon$ for all $x \in E$.

J. Gevirtz [11] established the stability property for isometries between arbitrary Banach spaces: Given real Banach spaces E and F, let $f : E \to F$ be a surjective mapping satisfying inequality $(*)$ for some $\varepsilon \geqslant 0$ and for all $x, y \in E$. Then there exists a surjective isometry $I : E \to F$ such that $\|f(x) - I(x)\| \leqslant 5\varepsilon$ for each $x \in E$.

Later, M. Omladič and P. Šemrl [21] obtained a sharp stability result for ε-isometries. Indeed, they proved that if a surjective mapping $f : E \to F$ satisfies $f(0) = 0$ as well as inequality $(*)$ for all $x, y \in E$, where E and F are real Banach spaces, then there exists a unique surjective linear isometry $I : E \to F$ such that $\|f(x) - I(x)\| \leqslant 2\varepsilon$ for any $x \in E$. The upper bound 2ε is sharp even in the case that $E = F$ is an n-dimensional real Hilbert space, $n = 1, 2, \ldots.$

On the other hand, G. Dolinar [9] proved the superstability property for isometries. In fact, he proved that for $p > 1$ every surjective (ε, p)-isometry $f : E \to F$ between finite-dimensional real Banach spaces is an isometry, where a mapping $f : E \to F$ is called an (ε, p)-isometry if f satisfies the inequality

$$\left| \left\| f(x) - f(y) \right\| - \|x - y\| \right| \leqslant \varepsilon \|x - y\|^p$$

for some $\varepsilon \geqslant 0$ and for all $x, y \in E$. (One may refer to [9,26] for more exact definition of (ε, p)-isometry.)

For more general information on the stability property for isometries and related topics, refer to [1–3,10,15–17,19,23–25,27–30].

Recently, L. Cădariu and V. Radu [8] applied the fixed point method to the investigation of the Cauchy additive functional equation (refs. [7,22]). Using such an elegant idea, they could present a short and simple proof for the Hyers–Ulam stability of that equation.

In this paper, we apply the fixed point method to the proof of Hyers–Ulam–Rassias stability property for isometries whose domain is a normed space and range is a Banach space in which the parallelogram law holds.

2. Preliminaries

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if and only if d satisfies

(M_1) $d(x, y) = 0$ if and only if $x = y$;
(M_2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(M_3) $d(x, z) \leqslant d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Note that the distinction between the generalized metric and the usual metric is that the range of the former is permitted to include the infinity.

Let (X, d) be a generalized metric space. An operator $\Lambda : X \to X$ satisfies a Lipschitz condition with Lipschitz constant L if there exists a constant $L \geqslant 0$ such that

$$d(\Lambda x, \Lambda y) \leqslant L d(x, y)$$

for all $x, y \in X$. If the Lipschitz constant L is less than 1, then the operator Λ is called a strictly contractive operator.

We now introduce one of fundamental results of fixed point theory. For the proof, we refer to [20].

Theorem 2.1. *Let (X, d) be a generalized complete metric space. Assume that $\Lambda : X \to X$ is a strictly contractive operator with the Lipschitz constant $L < 1$. If there exists a nonnegative integer k such that $d(\Lambda^{k+1}x, \Lambda^k x) < \infty$ for some $x \in X$, then the following holds true*:

(a) *the sequence $\{\Lambda^n x\}$ converges to a fixed point x^* of Λ;*
(b) *x^* is the unique fixed point of Λ in*

$$X^* = \{ y \in X \mid d(\Lambda^k x, y) < \infty \};$$

(c) *if $y \in X^*$, then*

$$d(y, x^*) \leqslant \frac{1}{1 - L} d(\Lambda y, y).$$

3. Main results

Throughout this section, by $\mathbb{K}$ we denote either $\mathbb{R}$ or $\mathbb{C}$, where $\mathbb{R}$ is the field of real numbers and $\mathbb{C}$ is the field of complex numbers.

In the following theorem, by applying the fixed point method (Theorem 2.1), we will prove the Hyers–Ulam–Rassias stability property for isometries from a normed space over $\mathbb{K}$ into a Banach space over $\mathbb{K}$ in which the parallelogram law holds.

Theorem 3.1. *Let E_1 be a normed space over $\mathbb{K}$ and let E_2 be a Banach space over $\mathbb{K}$ in which the parallelogram law holds. Assume that $\delta : E_1 \to [0,\infty)$ is an even function such that there exists a constant L $(0 < L < 1)$ with*

$$\begin{cases} \delta(2x) \leqslant 2L^2\delta(x) & \text{for all } x \in E_1, \\ \|x\| \leqslant L^2\delta(x) & \text{for all } x \in E_1 \text{ with } \|x\| < 1, \\ \|x\| \geqslant L^2\delta(x) & \text{for all } x \in E_1 \text{ with } \|x\| \geqslant 1, \end{cases} \tag{1}$$

and that

$$\lim_{n\to\infty} \frac{1}{2^n}\delta\bigl(2^n x\bigr) = 0 \tag{2}$$

for all $x \in E_1$. If a function $f : E_1 \to E_2$ satisfies

$$\bigl|\bigl\|f(x) - f(y)\bigr\| - \|x - y\|\bigr| \leqslant \delta(x - y) \tag{3}$$

for all $x, y \in E_1$, then there exists an isometry $I : E_1 \to E_2$ such that

$$\bigl\|f(x) - f(0) - I(x)\bigr\| \leqslant \frac{1 + L^2}{1 - L}\psi(x) \tag{4}$$

for all $x \in E_1$, where

$$\psi(x) = \begin{cases} \frac{1}{\sqrt{1+2L^2}}\{\|x\| + \delta(x)\} & \text{for } \|x\| < 1, \\ \frac{1}{L}\sqrt{\|x\|\delta(x)} & \text{for } \|x\| \geqslant 1. \end{cases}$$

The I is the unique isometry satisfying both (4) and $I(2x) = 2I(x)$ for every $x \in E_1$.

Proof. First, we will prove that

$$\psi(2x) \leqslant 2L\psi(x) \tag{5}$$

for all $x \in E_1$. If $\|x\| \geqslant 1$, then (5) immediately follows from the first condition of (1). If $\|x\| < \frac{1}{2}$, then the first and second conditions of (1) imply that

$$2L\psi(x) - \psi(2x) \geqslant \frac{2(1-L)}{\sqrt{1+2L^2}}\bigl(L\delta(x) - \|x\|\bigr) \geqslant \frac{2(1-L)}{\sqrt{1+2L^2}}\bigl(L^2\delta(x) - \|x\|\bigr) \geqslant 0.$$

Now, let $\frac{1}{2} \leqslant \|x\| < 1$, i.e., $\|2x\| \geqslant 1$. It then follows from the first two conditions of (1) that

$$\begin{aligned} \frac{1}{4L^2}\psi(2x)^2 &= \frac{1}{4L^2}\frac{1}{L^2}\|2x\|\delta(2x) \\ &\leqslant \frac{1}{L^2}\|x\|\delta(x) \\ &= \frac{1}{L^2(1+2L^2)}\|x\|\delta(x) + \frac{2}{1+2L^2}\|x\|\delta(x) \\ &\leqslant \frac{1}{1+2L^2}\delta(x)^2 + \frac{2}{1+2L^2}\|x\|\delta(x) \\ &\leqslant \frac{1}{1+2L^2}\bigl\{\|x\| + \delta(x)\bigr\}^2 \\ &= \psi(x)^2 \end{aligned}$$

for all $x \in E_1$.

Let us define

$$X = \{h : E_1 \to E_2 \mid h(0) = 0\}$$

and introduce a generalized metric on X as follows:

$$d(h_1, h_2) = \inf\{C \in [0, \infty] \mid \|h_1(x) - h_2(x)\| \leqslant C\psi(x) \text{ for all } x \in E_1\}.$$

Then, it is easy to show that (X, d) is a generalized complete metric space (see the proof of [18, Theorem 3.1] or [8, Theorem 2.5]). We now define an operator $\Lambda : X \to X$ by

$$(\Lambda h)(x) = \frac{1}{2}h(2x) \quad \text{for all } h \in X \text{ and } x \in E_1.$$

We assert that Λ is a strictly contractive operator. Given $h_1, h_2 \in X$, let $C \in [0, \infty]$ be an arbitrary constant with $d(h_1, h_2) \leqslant C$. From the definition of d, it follows that

$$\|h_1(x) - h_2(x)\| \leqslant C\psi(x)$$

for each $x \in E_1$. By the last inequality and (5), we have

$$\|(\Lambda h_1)(x) - (\Lambda h_2)(x)\| = \frac{1}{2}\|h_1(2x) - h_2(2x)\| \leqslant \frac{1}{2}C\psi(2x) \leqslant CL\psi(x)$$

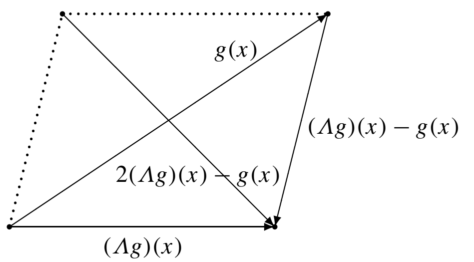
for all $x \in E_1$. Hence, it holds that $d(\Lambda h_1, \Lambda h_2) \leqslant CL$, i.e., $d(\Lambda h_1, \Lambda h_2) \leqslant Ld(h_1, h_2)$ for any $h_1, h_2 \in X$.

If we set $g(x) = f(x) - f(0)$ for any $x \in E_1$, then it follows from (3) that

$$\big|\|g(x) - g(y)\| - \|x - y\|\big| \leqslant \delta(x - y) \tag{6}$$

for all $x, y \in E_1$.

Now, we can apply the parallelogram law to the following parallelogram



and we conclude that

$$2\|(\Lambda g)(x) - g(x)\|^2 + 2\|(\Lambda g)(x)\|^2 = \|2(\Lambda g)(x) - g(x)\|^2 + \|g(x)\|^2$$

for any $x \in E_1$. Thus, since $(\Lambda g)(x) = \frac{1}{2}g(2x)$, it follows from the last equality and (6) that

$$\begin{aligned} 2\|(\Lambda g)(x) - g(x)\|^2 &= \|g(2x) - g(x)\|^2 - \frac{1}{2}\|g(2x)\|^2 + \|g(x)\|^2 \\ &\leqslant 2\{\|x\| + \delta(x)\}^2 - \frac{1}{2}\|g(2x)\|^2 \end{aligned} \tag{7}$$

for each $x \in E_1$.

If $\|x\| < 1$, then it follows from (7) that

$$2\|(\Lambda g)(x) - g(x)\|^2 \leqslant 2\{\|x\| + \delta(x)\}^2$$

or equivalently,

$$\big\|(\Lambda g)(x)-g(x)\big\| \leqslant \sqrt{1+2L^2}\,\psi(x) \tag{8}$$

for all $x \in E_1$ with $\|x\| < 1$. If $\|x\| \geqslant 1$, then it follows from (1), (6) and (7) that

$$\begin{aligned}
2\big\|(\Lambda g)(x)-g(x)\big\|^2 &\leqslant 2\big\{\|x\|+\delta(x)\big\}^2 - \frac{1}{2}\big\|g(2x)\big\|^2 \\
&\leqslant 2\big\{\|x\|+\delta(x)\big\}^2 - \frac{1}{2}\big\{\|2x\|-\delta(2x)\big\}^2 \\
&\leqslant 2\big\{\|x\|+\delta(x)\big\}^2 - \frac{1}{2}\big\{2\|x\|-2L^2\delta(x)\big\}^2 \\
&= 4\big(1+L^2\big)\|x\|\delta(x) + 2\big(1-L^4\big)\delta(x)^2 \\
&\leqslant 4\big(1+L^2\big)\|x\|\delta(x) + 2\frac{1-L^4}{L^2}\|x\|\delta(x) \\
&= 2\bigg(\frac{1+L^2}{L}\bigg)^2\|x\|\delta(x).
\end{aligned}$$

Hence, it holds that

$$\big\|(\Lambda g)(x)-g(x)\big\| \leqslant \big(1+L^2\big)\psi(x) \tag{9}$$

for all $x \in E_1$ with $\|x\| \geqslant 1$. In view of (8) and (9), we conclude that

$$d(\Lambda g, g) \leqslant 1+L^2. \tag{10}$$

According to Theorem 2.1(a), the sequence $\{\Lambda^n g\}$ converges to a fixed point I of Λ, i.e., if we define a function $I : E_1 \to E_2$ by

$$I(x) = \lim_{n\to\infty}\big(\Lambda^n g\big)(x) = \lim_{n\to\infty}\frac{1}{2^n}g\big(2^n x\big) \tag{11}$$

for all $x \in E_1$, then I belongs to X and I satisfies

$$I(2x) = 2I(x) \tag{12}$$

for any $x \in E_1$. Moreover, it follows from Theorem 2.1(c) and (10) that

$$d(g, I) \leqslant \frac{1}{1-L}d(\Lambda g, g) \leqslant \frac{1+L^2}{1-L},$$

i.e., inequality (4) holds true for every $x \in E_1$.

If we replace x by $2^n x$ and y by $2^n y$ in (6), if we divide by 2^n both sides of the resulting inequality, and if we let n go to infinity, then it follows from (2) and (11) that I is an isometry.

Finally, it remains to prove the uniqueness of I. Let J be another isometry satisfying (4) and (12) in place of I. If we substitute g, I, and 0 for x, x^*, and k in Theorem 2.1, respectively, then (4) implies that $d(\Lambda^k g, J) = d(g, J) \leqslant \frac{1+L^2}{1-L} < \infty$. Hence, $J \in X^*$ (see Theorem 2.1 for X^*). By (12), we further have $J(x) = \frac{1}{2}J(2x) = (\Lambda J)(x)$ for all $x \in E_1$, i.e., J is a 'fixed point' of Λ. Therefore, Theorem 2.1(b) implies that $J = I$. □

We notice that the parallelogram law is specifically true for norms derived from inner products. It is also known that every isometry from a real normed space into a real Hilbert space is affine (see [1]). Since the isometry I satisfies $I(0) = 0$ (see the proof of Theorem 3.1), the following corollary is a consequence of Theorem 3.1.

Corollary 3.2. *Let E_1 be a real normed space and let E_2 be a real Hilbert space. Given any $0 \leqslant p < 1$, choose a constant ε with $1 < \varepsilon \leqslant 2^{1-p}$ and define a function $\delta : E_1 \to [0, \infty)$ by*

$$\delta(x) = \varepsilon \|x\|^p$$

for all $x \in E_1$. If a function $f : E_1 \to E_2$ satisfies inequality (3) *for all $x, y \in E_1$, then there exists a unique linear isometry $I : E_1 \to E_2$ such that*

$$\big\| f(x) - f(0) - I(x) \big\| \leqslant \frac{1+\varepsilon}{\varepsilon - \sqrt{\varepsilon}} \psi(x)$$

for all $x \in E_1$, where

$$\psi(x) = \begin{cases} \frac{\sqrt{\varepsilon}}{\sqrt{\varepsilon+2}} \{\|x\| + \varepsilon\|x\|^p\} & \text{for} \|x\| < 1, \\ \varepsilon\|x\|^{\frac{1+p}{2}} & \text{for } \|x\| \geqslant 1. \end{cases}$$

Proof. From the assumption $1 < \varepsilon \leqslant 2^{1-p}$, it follows that $2^p - \frac{2}{\varepsilon} \leqslant 0$. If we set $L = \frac{1}{\sqrt{\varepsilon}}$, then we get

$$\delta(2x) - 2L^2\delta(x) = \left(2^p - \frac{2}{\varepsilon}\right)\varepsilon\|x\|^p \leqslant 0$$

and

$$L^2\delta(x) = \|x\|^p \begin{cases} > \|x\| & \text{for } \|x\| < 1, \\ \leqslant \|x\| & \text{for } \|x\| \geqslant 1. \end{cases}$$

That is, δ satisfies all the conditions in (1). Moreover, it holds that

$$\lim_{n\to\infty} \frac{1}{2^n}\delta\big(2^n x\big) = \lim_{n\to\infty} \frac{\varepsilon}{2^{n(1-p)}}\|x\|^p = 0$$

for all $x \in E_1$.

In view of Theorem 3.1 and the argument given just above this corollary, we conclude that our assertion is true. □

As we did in the proof of Theorem 3.1, we also apply Theorem 2.1 for proving the following theorem.

Theorem 3.3. *Let E_1 be a normed space over $\mathbb{K}$ and let E_2 be a Banach space over $\mathbb{K}$ in which the parallelogram law holds. Assume that $\delta : E_1 \to [0, \infty)$ is an even function such that there exists a constant L $(0 < L < 1)$ with*

$$\begin{cases} 2\delta(x) \leqslant L^2\delta(2x) & \text{for all } x \in E_1, \\ \|x\| \geqslant \delta(x) & \text{for all } x \in E_1 \text{ with } \|x\| < 1, \\ \|x\| \leqslant L^2\delta(x) & \text{for all } x \in E_1 \text{ with } \|x\| \geqslant 1, \end{cases} \tag{13}$$

and

$$\lim_{n\to\infty} 2^n\delta\left(\frac{1}{2^n}x\right) = 0$$

for all $x \in E_1$. If a function $f : E_1 \to E_2$ satisfies inequality (3) *for all $x, y \in E_1$, then there exists an isometry $I : E_1 \to E_2$ such that inequality* (4) *holds true for all $x \in E_1$, where*

$$\psi(x) = \begin{cases} \frac{1}{L}\sqrt{\|x\|\delta(x)} & \textit{for } \|x\| < 1, \\ \frac{1}{\sqrt{1+2L^2}}\{\|x\| + \delta(x)\} & \textit{for } \|x\| \geqslant 1. \end{cases}$$

The I is the unique isometry satisfying both (4) *and $I(2x) = 2I(x)$ for every $x \in E_1$.*

Proof. If $\|x\| \geqslant 1$, then it follows from the first and third conditions of (13) that

$$\begin{aligned} \frac{2}{L}\psi(x) &= \frac{1}{\sqrt{1+2L^2}}\left\{\frac{2}{L}\|x\| + \frac{2}{L}\delta(x)\right\} \\ &\leqslant \frac{1}{\sqrt{1+2L^2}}\left\{\frac{1}{L}\|2x\| + L\delta(2x)\right\} \\ &= \frac{1}{\sqrt{1+2L^2}}\left[\big\{\|2x\| + \delta(2x)\big\} + (1-L)\left\{\frac{1}{L}\|2x\| - \delta(2x)\right\}\right] \\ &\leqslant \psi(2x) + \frac{1-L}{\sqrt{1+2L^2}}\big\{L\delta(2x) - \delta(2x)\big\} \\ &\leqslant \psi(2x). \end{aligned}$$

If $\|x\| < \frac{1}{2}$, then it follows from the first condition of (13) that

$$\frac{2}{L}\psi(x) = \frac{2}{L^2}\sqrt{\|x\|\delta(x)} \leqslant \frac{2}{L^2}\sqrt{\|x\|\frac{L^2}{2}\delta(2x)} = \psi(2x).$$

Now, assume that $\frac{1}{2} \leqslant \|x\| < 1$. By (13), we have

$$\begin{aligned} \frac{4}{L^2}\psi(x)^2 &= \frac{4}{L^2}\frac{1}{L^2}\|x\|\delta(x) \\ &\leqslant \frac{1}{L^2}\|2x\|\delta(2x) \\ &= \frac{1}{L^2(1+2L^2)}\|2x\|\delta(2x) + \frac{2}{1+2L^2}\|2x\|\delta(2x) \\ &\leqslant \frac{1}{1+2L^2}\delta(2x)^2 + \frac{2}{1+2L^2}\|2x\|\delta(2x) \\ &\leqslant \frac{1}{1+2L^2}\big\{\|2x\| + \delta(2x)\big\}^2 \\ &= \psi(2x)^2. \end{aligned}$$

Consequently, we have proved that

$$\psi(x) \leqslant \frac{L}{2}\psi(2x) \tag{14}$$

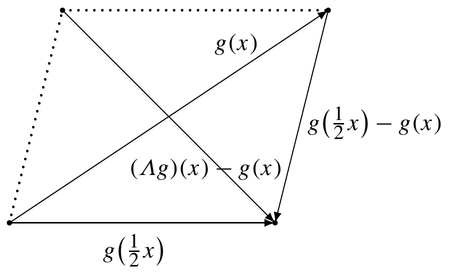
for each $x \in E_1$.

We introduce the same definitions for X and d as in the proof of Theorem 3.1 such that (X, d) becomes a generalized complete metric space. Let us define an operator $\Lambda : X \to X$ by

$$(\Lambda h)(x) = 2h\left(\frac{1}{2}x\right) \quad \text{for all } h \in X \text{ and } x \in E_1.$$

As we did in the proof of Theorem 3.1, we can similarly show that Λ is a strictly contractive operator. More precisely, it holds that $d(\Lambda h_1, \Lambda h_2) \leqslant Ld(h_1, h_2)$ for any $h_1, h_2 \in X$.

If we set $g(x) = f(x) - f(0)$ for all $x \in E_1$, then we get inequality (6) for any $x, y \in E_1$. Let us apply the parallelogram law to the parallelogram



to get

$$\big\|(\Lambda g)(x) - g(x)\big\|^2 + \big\|g(x)\big\|^2 = 2\left\|g\left(\frac{1}{2}x\right) - g(x)\right\|^2 + 2\left\|g\left(\frac{1}{2}x\right)\right\|^2$$

for any $x \in E_1$. Using the last equality and (6), we have

$$\begin{aligned}\frac{1}{2}\big\|(\Lambda g)(x) - g(x)\big\|^2 &= \left\|g\left(\frac{1}{2}x\right) - g(x)\right\|^2 + \left\|g\left(\frac{1}{2}x\right)\right\|^2 - \frac{1}{2}\big\|g(x)\big\|^2 \\ &\leqslant 2\left\{\frac{1}{2}\|x\| + \delta\left(\frac{1}{2}x\right)\right\}^2 - \frac{1}{2}\big\|g(x)\big\|^2 \qquad (15)\end{aligned}$$

for every $x \in E_1$. If $\|x\| < 1$, then it follows from (6), (13) and (15) that

$$\begin{aligned}\frac{1}{2}\big\|(\Lambda g)(x) - g(x)\big\|^2 &\leqslant 2\left\{\frac{1}{2}\|x\| + \delta\left(\frac{1}{2}x\right)\right\}^2 - \frac{1}{2}\big\{\|x\| - \delta(x)\big\}^2 \\ &\leqslant 2\left\{\frac{1}{2}\|x\| + \frac{L^2}{2}\delta(x)\right\}^2 - \frac{1}{2}\big\{\|x\| - \delta(x)\big\}^2 \\ &= \big(1 + L^2\big)\|x\|\delta(x) + \frac{1}{2}\big(L^4 - 1\big)\delta(x)^2 \\ &\leqslant \big(1 + L^2\big)\|x\|\delta(x) \\ &\leqslant 2\|x\|\delta(x).\end{aligned}$$

Thus, we conclude that

$$\big\|(\Lambda g)(x) - g(x)\big\| \leqslant 2L\psi(x) \qquad (16)$$

for all $x \in E_1$ with $\|x\| < 1$.

On the other hand, if $\|x\| \geqslant 1$, then it follows from (15) that

$$\frac{1}{2}\big\|(\Lambda g)(x) - g(x)\big\|^2 \leqslant 2\left\{\frac{1}{2}\|x\| + \delta\left(\frac{1}{2}x\right)\right\}^2.$$

Hence, by the above inequality and the first condition of (13), we obtain

$$\big\|(\Lambda g)(x) - g(x)\big\| \leqslant \|x\| + 2\delta\left(\frac{1}{2}x\right) \leqslant \|x\| + \delta(x) = \sqrt{1 + 2L^2}\,\psi(x) \qquad (17)$$

for any $x \in E_1$ with $\|x\| \geqslant 1$. Combining (16) and (17), we conclude that

$$d(\Lambda g, g) \leqslant \max\bigl\{2L, \sqrt{1+2L^2}\bigr\} \leqslant 1 + L^2. \tag{18}$$

Due to Theorem 2.1(a), the sequence $\{\Lambda^n g\}$ converges to a fixed point I of Λ, i.e., if we define a function $I : E_1 \to E_2$ by

$$I(x) = \lim_{n\to\infty} \bigl(\Lambda^n g\bigr)(x) = \lim_{n\to\infty} 2^n g\left(\frac{1}{2^n}x\right)$$

for all $x \in E_1$, then I belongs to X and I satisfies equality (12) for any $x \in E_1$. By Theorem 2.1(c) and (18), we get

$$d(g, I) \leqslant \frac{1}{1-L} d(\Lambda g, g) \leqslant \frac{1+L^2}{1-L},$$

which implies the validity of (4) for all $x \in E_1$.

Analogously to the proof of Theorem 3.1, we can easily show that I is a unique isometry with the properties (4) and (12). □

The following corollary is a consequence of Theorem 3.3.

Corollary 3.4. *Let E_1 be a real normed space and let E_2 be a real Hilbert space. For a given $p > 1$, choose constants ε_1 and ε_2 with $0 < \varepsilon_1 \leqslant 1$, $1 < \varepsilon_2 \leqslant 2^{p-1}$ and define a function $\delta : E_1 \to [0, \infty)$ by*

$$\delta(x) = \begin{cases} \varepsilon_1 \|x\|^p & \text{for } \|x\| < 1, \\ \varepsilon_2 \|x\|^p & \text{for } \|x\| \geqslant 1, \end{cases}$$

for all $x \in E_1$. If a function $f : E_1 \to E_2$ satisfies inequality (3) for all $x, y \in E_1$, then there exists a unique linear isometry $I : E_1 \to E_2$ such that

$$\bigl\| f(x) - f(0) - I(x) \bigr\| \leqslant \frac{1+\varepsilon_2}{\varepsilon_2 - \sqrt{\varepsilon_2}} \psi(x)$$

for all $x \in E_1$, where

$$\psi(x) = \begin{cases} \sqrt{\varepsilon_1 \varepsilon_2}\, \|x\|^{\frac{1+p}{2}} & \text{for } \|x\| < 1, \\ \frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_2 + 2}} \{\|x\| + \varepsilon_2 \|x\|^p\} & \text{for } \|x\| \geqslant 1. \end{cases}$$

Proof. First, we set $L = \frac{1}{\sqrt{\varepsilon_2}}$. Since $1 < \varepsilon_2 \leqslant 2^{p-1}$, it holds that $\frac{2^p}{\varepsilon_2} \geqslant 2$. If $x \in E_1$ is given with $\|x\| < \frac{1}{2}$, then

$$L^2 \delta(2x) - 2\delta(x) = \left(\frac{2^p}{\varepsilon_2} - 2\right) \varepsilon_1 \|x\|^p \geqslant 0.$$

If $x \in E_1$ satisfies $\|x\| \geqslant 1$, then we get

$$L^2 \delta(2x) - 2\delta(x) = \left(\frac{2^p}{\varepsilon_2} - 2\right) \varepsilon_2 \|x\|^p \geqslant 0.$$

For $\frac{1}{2} \leqslant \|x\| < 1$, we obtain

$$L^2 \delta(2x) - 2\delta(x) = \bigl(2^p - 2\varepsilon_1\bigr) \|x\|^p \geqslant \bigl(2^p - 2\bigr) \|x\|^p > 0.$$

Hence, δ satisfies the first condition of (13).

Besides, δ satisfies

$$\delta(x) = \begin{cases} \varepsilon_1 \|x\|^p \leqslant \|x\| & \text{for } \|x\| < 1, \\ \varepsilon_2 \|x\|^p \geqslant \frac{1}{L^2}\|x\| & \text{for } \|x\| \geqslant 1. \end{cases}$$

Therefore, δ satisfies all the conditions in (13).

Moreover, we see

$$\lim_{n\to\infty} 2^n \delta\left(\frac{1}{2^n} x\right) = \lim_{n\to\infty} \frac{\varepsilon_1}{2^{n(p-1)}} \|x\|^p = 0$$

for any $x \in E_1$.

Subsequently, we can now apply Theorem 3.3 to the verification of the corollary. In particular, the uniqueness of I follows from the fact that every isometry I from a real normed space into a real Hilbert space with $I(0) = 0$ is linear (see the statement between Theorem 3.1 and Corollary 3.2). □

Acknowledgment

The author thanks the referee for his/her valuable suggestions.

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