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The crossing number of $K_{1,m,n}$

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Abstract

A longstanding problem of crossing number, Zarankiewicz's conjecture, asserts that the crossing number of the complete bipartite graph $K_{m,n}$ is $\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor$, which is known only for $m \leq 6$. It is natural to generalize Zarankiewicz conjecture and ask: What is the crossing number for the complete multipartite graph? In this paper, we prove the following lower bounds for the crossing number of $K_{1,m,n}$ in terms of the crossing number of the complete bipartite graph:

$$cr(K_{1,m,n}) \geq cr(K_{m+1,n+1}) - \left\lfloor \frac{n}{m} \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m+1}{2} \right\rfloor \right\rfloor;$$
$$cr(K_{1,2M,n}) \geq \frac{1}{2}(cr(K_{2M+1,n+2}) + cr(K_{2M+1,n}) - M(M+n-1)).$$

As a corollary, we show that:

1. $cr(K_{1,m,n}) \geq 0.8594Z(m+1,n+1) - \lfloor\frac{n}{m}\lfloor\frac{m}{2}\rfloor\lfloor\frac{m+1}{2}\rfloor\rfloor$;
2. If Zarankiewicz's conjecture is true for $m = 2M+1$, then $cr(K_{1,2M,n}) = M^2\lfloor\frac{n+1}{2}\rfloor\lfloor\frac{n}{2}\rfloor - M\lfloor\frac{n}{2}\rfloor$;
3. $cr(K_{1,4,n}) = 4\lfloor\frac{n+1}{2}\rfloor\lfloor\frac{n}{2}\rfloor - 2\lfloor\frac{n}{2}\rfloor$.

Keywords: Crossing number; Multipartite graphs; Zarankiewicz's conjecture

1. Introduction

Determining the crossing numbers of graphs is a notorious problem in Graph Theory, as in general it is quite easy to find a drawing of a sufficiently "nice" graph in which the number of crossings can hardly be decreased, but it is very difficult to prove that such a drawing indeed has the smallest possible number of crossings. In fact, computing the crossing number of a graph is NP-complete [4,7], and exact values are known only for very restricted classes of graphs. Bhatt and Leighton [2] showed that the crossing number of a network (graph) is closely related to the minimum layout area required for the implementation of a VLSI circuit for that network. For more about crossing number, see [15] and the references therein.

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One of the conjecture in crossing number states that the crossing number for a complete graph of order n is

$$cr(K_n) = \frac{1}{4}\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor,$$

which is known only for $n \leq 10$ [5]. Recently, Pan and Richter [14] proved that it is true for $n = 11$ and 12. Another problem in crossing number is Zarankiewicz's conjecture, which asserts that the crossing number of the complete bipartite graph $K_{m,n}$ is

$$cr(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor, \tag{1}$$

(we assume $m \leq n$ throughout this paper) which is known only for $m \leq 6$ [13]; and for $7 \leq m \leq 8$ and $n \leq 10$ [16]. In the following, $Z(m, n)$ will denote the right member of (1). Recently, in [3], deKlerk et al. gave a new lower bound for the crossing number of $K_{m,n}$.

It is natural to generalize Zarankiewicz conjecture and ask: What is the crossing number for the complete multipartite graph? In [6], Harboth gave an upper bound for the crossing number of $K_{n_1,n_2,\ldots,n_k}$ for any positive n_i and k. In particular, he proved that

$$cr(K_{1,m,n}) \leq Z(m+1, n+1) - \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor. \tag{2}$$

He also conjectured that equality holds in (2). In [1], Asano showed that the crossing numbers of $K_{1,3,n}$ and $K_{2,3,n}$ are $Z(4, n) + \lfloor \frac{n}{2} \rfloor$ and $Z(5, n) + n$ respectively. Recently, Huang and Zhao have computed the crossing of $K_{1,4,n}$ in [12]. See also [9–11]. In [8], the author computed the crossing numbers of $K_{1,1,1,1,n}$, $K_{1,2,2,n}$, $K_{1,1,1,2,n}$ and $K_{1,4,n}$. The technique the author used in [8] is similar to Asano in [1], that is: If the crossing number of $K_{t_1,\ldots,t_k,n}$ is less than the expected value, then the $K_{t_1,\ldots,t_k}$ in the optimal drawing of $K_{t_1,\ldots,t_k,n}$ must be drawn in some special forms. Then by analyzing each of these drawings of $K_{t_1,\ldots,t_k}$ carefully, one can show that it is impossible to extend these drawings to the optimal drawing of $K_{t_1,\ldots,t_k,n}$.

In this paper, we study the crossing number of the complete tripartite graph $K_{1,m,n}$. We obtain the lower bounds of $cr(K_{1,m,n})$ in terms of the crossing number of the complete bipartite graphs by showing that:

$$cr(K_{1,m,n}) \geq cr(K_{m+1,n+1}) - \left\lfloor \frac{n}{m} \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m+1}{2} \right\rfloor \right\rfloor;$$

$$cr(K_{1,2M,n}) \geq \frac{1}{2}(cr(K_{2M+1,n+2}) + cr(K_{2M+1,n}) - M(M+n-1)).$$

We prove these inequalities by constructing drawings of the complete bipartite graphs from the drawing of $K_{1,m,n}$. As a corollary, we show that:

1. $cr(K_{1,m,n}) \geq 0.8594Z(m+1, n+1) - \lfloor \frac{n}{m} \lfloor \frac{m}{2} \rfloor \lfloor \frac{m+1}{2} \rfloor \rfloor$;
2. If Zarankiewicz's conjecture is true for $m = 2M+1$, then the equality holds in (2) for $m = 2M$, i.e. $cr(K_{1,2M,n}) = M^2 \lfloor \frac{n+1}{2} \rfloor \lfloor \frac{n}{2} \rfloor - M \lfloor \frac{n}{2} \rfloor$;
3. $cr(K_{1,4,n}) = 4\lfloor \frac{n+1}{2} \rfloor \lfloor \frac{n}{2} \rfloor - 2\lfloor \frac{n}{2} \rfloor$.

Here are some definitions. Let G be a graph with edge set E. A *drawing* of a graph G is a mapping from G into the plane. A drawing is *good* if no edge crosses itself; adjacent edges do not cross; two crossing edges cross only once; edges do not cross vertices; and no more than two edges cross at a point. Let A and B be subsets of E. In a drawing ϕ, the number of crossings of edges in A with edges in B is denoted by $cr_\phi(A, B)$. Especially, $cr_\phi(A, A)$ will be denoted by $cr_\phi(A)$. Then the total number of crossings of ϕ is $cr_\phi(E)$. The *crossing number* of a graph G, $cr(G)$, is the minimum of $cr_\phi(E)$ among all good drawings ϕ of G. We note the following formulas, which can be shown easily.

$$cr_\phi(A \cup B) = cr_\phi(A) + cr_\phi(B) + cr_\phi(A, B) \tag{3}$$

$$cr_\phi(A, B \cup C) = cr_\phi(A, B) + cr_\phi(A, C), \tag{4}$$

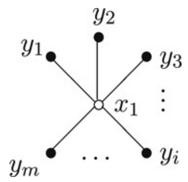

Fig. 1(a). Subgraph induced by $X \cup Y$.

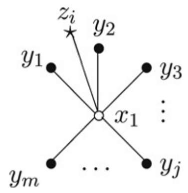

Fig. 1(b). $z_i \in A_1$.

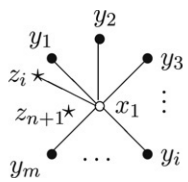

Fig. 1(c). z_{n+1} is drawn.

where A, B and C are mutually disjoint subsets of E. For the complete tripartite graph $K_{1,m,n}$ with the partition (X, Y, Z), where $X = \{x_1\}$, $Y = \{y_1, \ldots, y_m\}$ and $Z = \{z_1, \ldots, z_n\}$ we write E_{XY} for the set of all edges incident to X and Y; and $E(z_i)$ for the set of all edges incident to z_i.

2. Lower bounds for $cr(K_{1,m,n})$

Firstly, we give the following lower bound of the crossing number of $K_{1,m,n}$ in terms of the crossing number of $K_{m+1,n+1}$:

Theorem 2.1. $cr(K_{1,m,n}) \geq cr(K_{m+1,n+1}) - \lfloor \frac{n}{m} \lfloor \frac{m}{2} \rfloor \lfloor \frac{m+1}{2} \rfloor \rfloor$.

Proof. Let ϕ be a good drawing of $K_{1,m,n}$ with crossing number $cr(K_{1,m,n})$. Since ϕ is good, $cr_\phi(E_{XY}) = 0$. By (3) and (4),

$$cr(K_{1,m,n}) = cr_\phi(E) = cr_\phi\left(\bigcup_{i=1}^{n} E(z_i)\right) + \sum_{i=1}^{n} cr_\phi(E_{XY}, E(z_i)). \tag{5}$$

By renaming the vertices of y_j if necessary, we may assume that the subgraph induced by $X \cup Y$ is drawn such that $x_1 y_j$ lies between $x_1 y_{j-1}$ and $x_1 y_{j+1}$ (mod m for $j+1$), as in Fig. 1(a). For $1 \leq j \leq m$, let A_j be the set of z_i where $1 \leq i \leq n$, such that $x_1 z_i$ lies between the edges $x_1 y_j$ and $x_1 y_{j+1}$. See Fig. 1(b) for $z_i \in A_1$.

We are going to obtain a drawing of $K_{m+1,n+1}$ from ϕ. To do this, we draw a new vertex, z_{n+1}, near the vertex x_1 and lying in the region between the edges $x_1 y_m$ and $x_1 y_1$ such that z_{n+1} lies between $x_1 y_m$ and $x_1 z_i$ for all $z_i \in A_m$, as shown in Fig. 1(c). Let $m' = \lfloor m/2 \rfloor$. For $1 \leq j \leq m'$, draw the edge $z_{n+1} y_j$ next to the edge $x_1 y_j$ such that $z_{n+1} y_j$ only crosses $x_1 y_i$ where $1 \leq i \leq j-1$ and does not cross other edges in E_{XY}. For $m'+1 \leq j \leq m$, draw the edge $z_{n+1} y_j$ next to the edge $x_1 y_j$ such that $z_{n+1} y_j$ only crosses $x_1 y_i$ where $j+1 \leq i \leq m$ and does not cross other edges in E_{XY}. Draw the edge $z_{n+1} x_1$ without crossing any edges. Then remove the edges $x_1 y_j$ for $1 \leq j \leq m$. See Fig. 1(d).

Now we have a drawing ϕ' of $K_{m+1,n+1}$ with $\{x_1, y_1, \ldots, y_m\}$ as the partition with $m+1$ vertices and $\{z_1, \ldots, z_{n+1}\}$ as the partition with $n+1$ vertices. Note that if $z_i \in A_j$ where $1 \leq j \leq m'$ (see Fig. 2(a) for

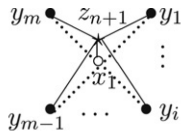

Fig. 1(d). Remove the edges $x_1 y_j$.

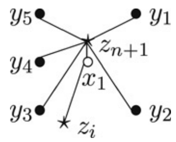

Fig. 2(a). $z_i \in A_2$ for $m = 5$.

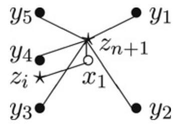

Fig. 2(b). $z_i \in A_3$ for $m = 5$.

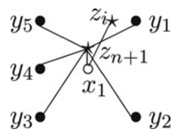

Fig. 2(c). $z_i \in A_5$ for $m = 5$.

$m = 5$ and $j = 2$), then

$$cr_{\phi'}(E(z_i), E(z_{n+1})) = cr_{\phi}(E(z_i), E_{XY}) + m' - j. \quad (6)$$

If $z_i \in A_j$ where $m' + 1 \le j \le m - 1$ (see Fig. 2(b) for $m = 5$ and $j = 3$), then

$$cr_{\phi'}(E(z_i), E(z_{n+1})) = cr_{\phi}(E(z_i), E_{XY}) + j - m'. \quad (7)$$

If $z_i \in A_m$, then by our construction that z_{n+1} lies between $x_1 y_m$ and $x_1 z_i$ for all $z_i \in A_m$ (see Fig. 2(c) for $m = 5$), we have

$$cr_{\phi'}(E(z_i), E(z_{n+1})) = cr_{\phi}(E(z_i), E_{XY}) + m'. \quad (8)$$

Note also that

$$cr_{\phi'}\left(\bigcup_{i=1}^{n} E(z_i)\right) = cr_{\phi}\left(\bigcup_{i=1}^{n} E(z_i)\right). \quad (9)$$

By (3) and (4), the crossing number of ϕ' is

$$cr_{\phi'}\left(\bigcup_{i=1}^{n} E(z_i)\right) + \sum_{i=1}^{n} cr_{\phi'}(E(z_i), E(z_{n+1})). \quad (10)$$

Putting (6)–(9) into (10), we obtain that the crossing number of ϕ' is

$$cr_{\phi}\left(\bigcup_{i=1}^{n} E(z_i)\right) + \sum_{i=1}^{n} cr_{\phi}(E_{XY}, E(z_i)) + \sum_{j=1}^{m'}(m' - j)|A_j| + \sum_{j=m'+1}^{m-1}(j - m')|A_j| + m'|A_m|,$$

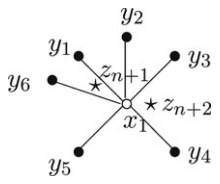

Fig. 3(a). z_{n+1} and z_{n+2} are drawn.

which is at least $cr(K_{m+1,n+1})$. Combining this with (5), we have

$$\sum_{j=1}^{m'}(m'-j)|A_j| + \sum_{j=m'+1}^{m-1}(j-m')|A_j| + m'|A_m| \geq cr(K_{m+1,n+1}) - cr(K_{1,m,n}). \tag{11}$$

If we put z_{n+1} between the edges x_1y_i and x_1y_{i+1} where $1 \leq i \leq m$ in the above construction of $K_{m+1,n+1}$, then by the same arguments, we can show that for $1 \leq i \leq m$,

$$\sum_{j=1}^{m'}(m'-j)|A_{j+i}| + \sum_{j=m'+1}^{m-1}(j-m')|A_{j+i}| + m'|A_{m+i}| \geq cr(K_{m+1,n+1}) - cr(K_{1,m,n}), \tag{12}$$

where the indices of A_{j+i} read modulo m. Summing up (12) for $1 \leq i \leq m$, we get

$$\sum_{i=1}^{m}\left(\sum_{j=1}^{m'}(m'-j)|A_{j+i}| + \sum_{j=m'+1}^{m-1}(j-m')|A_{j+i}| + m'|A_{m+i}|\right) \geq m(cr(K_{m+1,n+1}) - cr(K_{1,m,n})), \tag{13}$$

where the indices of A_{j+i} read modulo m. One can show that the left-hand side of (13) is equal to $\lfloor\frac{m}{2}\rfloor\lfloor\frac{m+1}{2}\rfloor\sum_{j=1}^{m}|A_j|$. Note also that $\sum_{j=1}^{m}|A_j| = n$. Combining all these, (13) becomes $n\lfloor\frac{m}{2}\rfloor\lfloor\frac{m+1}{2}\rfloor \geq m(cr(K_{m+1,n+1}) - cr(K_{1,m,n}))$, as required. □

In [3], deKlerk et al. give the lower bound of the crossing number of $K_{m,n}$ by showing that $cr(K_{m,n}) \geq 0.8594Z(m,n)$. Combining this with Theorem 2.1, we can obtain a numerical lower bound for the crossing number of $K_{1,m,n}$:

Corollary 2.1. $cr(K_{1,m,n}) \geq 0.8594Z(m+1,n+1) - \lfloor\frac{n}{m}\lfloor\frac{m}{2}\rfloor\lfloor\frac{m+1}{2}\rfloor\rfloor$.

Using similar arguments in the proof of Theorem 2.1, we can also prove the following:

Theorem 2.2. $cr(K_{1,2M,n}) \geq \frac{1}{2}(cr(K_{2M+1,n+2}) + cr(K_{2M+1,n}) - M(M+n-1))$.

Proof. Let ϕ be a drawing of $K_{1,2M,n}$ with $cr_\phi(E) = cr(K_{1,2M,n})$. Then (5) still holds for ϕ with $m = 2M$. We are going to obtain a drawing of $K_{2M+1,n+2}$ from ϕ. Following the same arguments in the proof of Theorem 2.1, we draw a new vertex z_{n+1} between x_1z_{2M} and x_1z_1, as in Fig. 1(c). On the other hand, we draw a vertex z_{n+2} between x_1z_M and x_1z_{M+1}. See Fig. 3(a) for $M = 3$.

Then draw the edges $z_{n+1}x_1$ and $z_{n+1}y_j$ where $1 \leq j \leq 2M$ as in the proof of Theorem 2.1. Moreover, for $1 \leq j \leq M$, draw the edge $z_{n+2}y_j$ next to the edge x_1y_j such that $z_{n+2}y_j$ only crosses x_1y_i where $j+1 \leq i \leq M$ and does not cross other edges in E_{XY}. For $M+1 \leq j \leq 2M$, draw the edge $z_{n+2}y_j$ next to the edge x_1y_j such that $z_{n+2}y_j$ only crosses x_1y_i where $M+1 \leq i \leq j-1$ and does not cross other edges in E_{XY}. Draw the edge $z_{n+2}x_1$ without crossing any edges. Finally remove the edges x_1y_j for $1 \leq j \leq 2M$. See Fig. 3(b) for $M = 3$.

Therefore we obtain a drawing ϕ'' of $K_{2M+1,n+2}$ with $\{x_1, y_1, \ldots, y_{2M}\}$ as the partition with $2M+1$ vertices and $\{z_1, \ldots, z_n, z_{n+1}, z_{n+2}\}$ as the partition with $n+2$ vertices. Using the notion of A_j defined in the proof of Theorem 2.1, one can show that if $z_i \in A_j$

$$cr_{\phi''}(E(z_i), E(z_{n+1})) = cr_\phi(E(z_i), E_{XY}) + M - j \quad \text{if } 1 \leq j \leq M; \tag{14}$$

$$cr_{\phi''}(E(z_i), E(z_{n+1})) = cr_\phi(E(z_i), E_{XY}) + j - M \quad \text{if } M+1 \leq j \leq 2M. \tag{15}$$

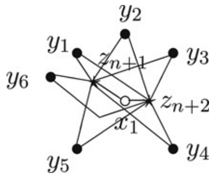

Fig. 3(b). The result drawing.

Also, one can show that if $z_i \in A_j$

$$cr_{\phi''}(E(z_i), E(z_{n+2})) = cr_{\phi}(E(z_i), E_{XY}) + j \quad \text{if } 1 \le j \le M; \tag{16}$$

$$cr_{\phi''}(E(z_i), E(z_{n+2})) = cr_{\phi}(E(z_i), E_{XY}) + 2M - j \quad \text{if } M + 1 \le j \le 2M. \tag{17}$$

On the other hand, we have

$$cr_{\phi''}(E(z_{n+1}), E(z_{n+2})) = M(M-1). \tag{18}$$

Note also that

$$cr_{\phi''}\left(\bigcup_{i=1}^{n} E(z_i)\right) = cr_{\phi}\left(\bigcup_{i=1}^{n} E(z_i)\right). \tag{19}$$

By (3) and (4), the crossing number of ϕ'' is

$$cr_{\phi''}\left(\bigcup_{i=1}^{n} E(z_i)\right) + cr_{\phi''}(E(z_{n+1}), E(z_{n+2})) + \sum_{i=1}^{n}\left(cr_{\phi''}(E(z_i), E(z_{n+1})) + cr_{\phi''}(E(z_i), E(z_{n+2}))\right). \tag{20}$$

Hence, by putting (14)–(19) into (20), and by the fact that $\sum_{i=1}^{2M} |A_i| = n$, we obtain that the crossing number of ϕ'' is

$$\begin{aligned} &cr_{\phi}\left(\bigcup_{i=1}^{n} E(z_i)\right) + M(M-1) + 2\sum_{i=1}^{n} cr_{\phi}(E_{XY}, E(z_i)) + Mn \\ &= 2cr(K_{1,2M,n}) - cr_{\phi}\left(\bigcup_{i=1}^{n} E(z_i)\right) + M(M-1) + Mn \\ &\le 2cr(K_{1,2M,n}) - cr(K_{2M+1,n}) + M(M+n-1), \end{aligned}$$

where the first equality follows from (5) with $m = 2M$; and the second inequality follows from the fact that the graph induced by $\bigcup_{i=1}^{n} E(z_i)$ is $K_{2M+1,n}$. Note also that the crossing number of ϕ'' is at least $cr(K_{2M+1,n+2})$. Combining all these, we obtain $2cr(K_{1,2M,n}) - cr(K_{2M+1,n}) + M(M+n-1) \ge cr(K_{2M+1,n+2})$ as required. □

From Theorems 2.1 and 2.2, we can derive the following:

Theorem 2.3. *If Zarankiewicz's conjecture is true for $m = 2M + 1$, then*

$$cr(K_{1,2M,n}) = Z(2M+1, n+1) - M\left\lfloor \frac{n}{2} \right\rfloor.$$

Proof. From (2), it suffices to prove

$$cr(K_{1,2M,n}) \ge Z(2M+1, n+1) - M\left\lfloor \frac{n}{2} \right\rfloor. \tag{21}$$

If Zarankiewicz's conjecture is true for $m = 2M + 1$, then $cr(K_{2M+1,n}) = Z(2M+1, n)$. Then (21) follows from Theorem 2.1 for n is even, and from Theorem 2.2 for n is odd. □

Therefore if Zarankiewicz's conjecture is true for $m = 2M + 1$, then equality holds in (2) for $m = 2M$. Since Zarankiewicz's conjecture is true for $m = 5$ [13], by putting $M = 2$ in Theorem 2.3, we have the following result appeared in [8,12]:

Corollary 2.2. $cr(K_{1,4,n}) = Z(5,n) + 2\lfloor \frac{n}{2} \rfloor$.

By putting $M = 3, 4$ in Theorem 2.3, we have the following results appeared in [10,11]:

Corollary 2.3. *The crossing number of $K_{1,6,n}$ (and $K_{1,8,n}$ respectively) is $Z(7,n) + 6\lfloor \frac{n}{2} \rfloor$ (and $Z(9,n) + 12\lfloor \frac{n}{2} \rfloor$ respectively) provided that Zarankiewicz's conjecture holds for $m = 7$ (and $m = 9$ respectively).*

To conclude, we state the following:

Conjecture 2.1.

$$cr(K_{1,m,n}) = cr(K_{m+1,n+1}) - \left\lfloor \frac{n}{m} \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m+1}{2} \right\rfloor \right\rfloor;$$
$$cr(K_{1,2M,n}) = \frac{1}{2}(cr(K_{2M+1,n+2}) + cr(K_{2M+1,n}) - M(M+n-1)).$$

Theorems 2.1 and 2.2 provide some evidences supporting Conjecture 2.1.

References

[1] K. Asano, The crossing number of $K_{1,3,n}$ and $K_{2,3,n}$, J. Graph Theory 10 (1986) 1–8.
[2] S.N. Bhatt, F.T. Leighton, A framework for solving VLSI graph layout problems, J. Comput. System Sci. 28 (1984) 300–343.
[3] E. deKlerk, D.V. Pasechnik, A. Schrijver, Reduction of symmetric semidefinite programs using the regular $*$-representation, Math. Program. 109 (2007) 613–624.
[4] M.R. Garey, D.S. Johnson, Crossing number is NP-complete, SIAM J. Algebraic Discrete Methods 4 (1983) 312–316.
[5] R.K. Guy, Latest results on crossing numbers, in: Recent Trends in Graph Theory, Springer, NY, 1971, pp. 143–156.
[6] H. Harborth, Über die Kreuzungszahl vollständiger, n-geteilter Graphen, Math. Nachr. 48 (1971) 179–188.
[7] P. Hliněný, Crossing number is hard for cubic graphs, J. Combin. Theory Ser. B 96 (2006) 455–471.
[8] P.T. Ho, On the crossing number of some complete multipartite graphs, Util. Math. (in press).
[9] P.T. Ho, The crossing number of $K_{1,5,n}$, $K_{2,4,n}$ and $K_{3,3,n}$, Int. J. Pure Appl. Math. 17 (2004) 491–515.
[10] Y. Huang, T. Zhao, On the crossing number of the complete tripartite $K_{1,6,n}$, Acta Math. Appl. Sin. 29 (2006) 1046–1053.
[11] Y. Huang, T. Zhao, On the crossing number of the complete tripartite graph $K_{1,8,n}$, Acta Math. Sci. Ser. A Chin. Ed. 26 (2006) 1115–1122.
[12] Y. Huang, T. Zhao, The crossing number of $K_{1,4,n}$, Discrete Math. (in press).
[13] D.J. Kleitman, The crossing number of $K_{5,n}$, J. Combin. Theory 9 (1970) 315–323.
[14] S. Pan, R.B. Richter, The crossing number of K_{11} is 100, J. Graph Theory 56 (2007) 128–134.
[15] L.A. Székely, A successful concept for measuring non-planarity of graphs: The crossing number, Discrete Math. 276 (2004) 331–352.
[16] D.R. Woodall, Cyclic-order graphs and Zarankiewicz's crossing number conjecture, J. Graph Theory 17 (1993) 657–671.