The crossing number of $K_{1,m,n}$

Pak Tung Ho

Department of Mathematics, Purdue University, West Lafayette 47907, USA

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Abstract

A longstanding problem of crossing number, Zarankiewicz’s conjecture, asserts that the crossing number of the complete bipartite graph $K_{m,n}$ is $\left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$, which is known only for $m \leq 6$. It is natural to generalize Zarankiewicz conjecture and ask: What is the crossing number for the complete multipartite graph? In this paper, we prove the following lower bounds for the crossing number of $K_{1,m,n}$ in terms of the crossing number of the complete bipartite graph:

$$cr(K_{1,m,n}) \geq cr(K_{m+1,n+1}) - \left\lfloor \frac{n}{m+2} \right\rfloor \left\lfloor \frac{m+1}{2} \right\rfloor;$$

$$cr(K_{1,2M,n}) \geq \frac{1}{2}(cr(K_{2M+1,n+2}) + cr(K_{2M+1,n}) - M(M+n-1)).$$

As a corollary, we show that:

1. $cr(K_{1,m,n}) \geq 0.8594Z(m+1,n+1) - \left\lfloor \frac{n}{m} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor$;
2. If Zarankiewicz’s conjecture is true for $m = 2M + 1$, then $cr(K_{1,2M,n}) = M^2 \left\lfloor \frac{n+1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor - M \left\lfloor \frac{n}{2} \right\rfloor$;
3. $cr(K_{1,4,n}) = 4 \left\lfloor \frac{n+1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor - 2 \left\lfloor \frac{n}{2} \right\rfloor$.

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1. Introduction

Determining the crossing numbers of graphs is a notorious problem in Graph Theory, as in general it is quite easy to find a drawing of a sufficiently “nice” graph in which the number of crossings can hardly be decreased, but it is very difficult to prove that such a drawing indeed has the smallest possible number of crossings. In fact, computing the crossing number of a graph is NP-complete [4,7], and exact values are known only for very restricted classes of graphs. Bhatt and Leighton [2] showed that the crossing number of a network (graph) is closely related to the minimum layout area required for the implementation of a VLSI circuit for that network. For more about crossing number, see [15] and the references therein.
One of the conjecture in crossing number states that the crossing number for a complete graph of order \( n \) is

\[
\text{cr}(K_n) = \frac{1}{4} n \left[ \left\lfloor \frac{n-1}{2} \right\rfloor \right] \left[ \left\lfloor \frac{n-2}{2} \right\rfloor \right] \left[ \left\lfloor \frac{n-3}{2} \right\rfloor \right],
\]

which is known only for \( n \leq 10 \) [5]. Recently, Pan and Richter [14] proved that it is true for \( n = 11 \) and 12. Another problem in crossing number is Zarankiewicz’s conjecture, which asserts that the crossing number of the complete bipartite graph \( K_{m,n} \) is

\[
\text{cr}(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor, \tag{1}
\]

(we assume \( m \leq n \) throughout this paper) which is known only for \( m \leq 6 \) [13]; and for \( 7 \leq m \leq 8 \) and \( n \leq 10 \) [16].

In the following, \( Z(m, n) \) will denote the right member of (1). Recently, in [3], deKlerk et al. gave a new lower bound for the crossing number of \( K_{m,n} \).

It is natural to generalize Zarankiewicz conjecture and ask: What is the crossing number for the complete multipartite graph? In [6], Harboth gave an upper bound for the crossing number of \( K_{n_1,n_2,...,n_k} \) for any positive \( n_i \) and \( k \). In particular, he proved that

\[
\text{cr}(K_{1,m,n}) \leq Z(m+1, n+1) - \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor. \tag{2}
\]

He also conjectured that equality holds in (2). In [1], Asano showed that the crossing numbers of \( K_{1,3,n} \) and \( K_{2,3,n} \) are \( Z(4, n) + \left\lfloor \frac{n}{2} \right\rfloor \) and \( Z(5, n) + n \) respectively. Recently, Huang and Zhao have computed the crossing of \( K_{1,4,n} \) in [12]. See also [9–11]. In [8], the author computed the crossing numbers of \( K_{1,1,1,n} \), \( K_{1,2,2,n} \), \( K_{1,1,1,2,n} \) and \( K_{1,4,n} \). The technique the author used in [8] is similar to Asano in [1], that is: If the crossing number of \( K_{t_1,...,t_k,n} \) is less than the expected value, then the \( K_{t_1,...,t_k} \) in the optimal drawing of \( K_{t_1,...,t_k,n} \) must be drawn in some special forms. Then by analyzing each of these drawings of \( K_{t_1,...,t_k} \) carefully, one can show that it is impossible to extend these drawings to the optimal drawing of \( K_{t_1,...,t_k,n} \).

In this paper, we study the crossing number of the complete tripartite graph \( K_{1,m,n} \). We obtain the lower bounds of \( \text{cr}(K_{1,m,n}) \) in terms of the crossing number of the complete bipartite graphs by showing that:

\[
\text{cr}(K_{1,m,n}) \geq \text{cr}(K_{m+1,n+1}) - \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m+1}{2} \right\rfloor;
\]

\[
\text{cr}(K_{1,2M,n}) \geq \frac{1}{2}(\text{cr}(K_{2M+1,n+2}) + \text{cr}(K_{2M+1,n}) - M(M + n - 1)).
\]

We prove these inequalities by constructing drawings of the complete bipartite graphs from the drawing of \( K_{1,m,n} \). As a corollary, we show that:

1. \( \text{cr}(K_{1,m,n}) \geq 0.8594Z(m+1, n+1) - \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{m+1}{2} \right\rfloor; \)
2. If Zarankiewicz’s conjecture is true for \( m = 2M+1 \), then the equality holds in (2) for \( m = 2M \), i.e. \( \text{cr}(K_{1,2M,n}) = M^2 \left\lfloor \frac{n+1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor - M \left\lfloor \frac{n}{2} \right\rfloor; \)
3. \( \text{cr}(K_{1,4,n}) = 4 \left\lfloor \frac{n+1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor - 2 \left\lfloor \frac{n}{2} \right\rfloor. \)

Here are some definitions. Let \( G \) be a graph with edge set \( E \). A drawing of a graph \( G \) is a mapping from \( G \) into the plane. A drawing is good if no edge crosses itself; adjacent edges do not cross; two crossing edges cross only once; edges do not cross vertices; and no more than two edges cross at a point. Let \( A \) and \( B \) be subsets of \( E \). In a drawing \( \phi \), the number of crossings of edges in \( A \) with edges in \( B \) is denoted by \( \text{cr}_\phi(A, B) \). Especially, \( \text{cr}_\phi(A, A) \) will be denoted by \( \text{cr}_\phi(A) \). Then the total number of crossings of \( \phi \) is \( \text{cr}_\phi(E) \). The crossing number of a graph \( G, \text{cr}(G) \), is the minimum of \( \text{cr}_\phi(E) \) among all good drawings \( \phi \) of \( G \). We note the following formulas, which can be shown easily.

\[
\text{cr}_\phi(A \cup B) = \text{cr}_\phi(A) + \text{cr}_\phi(B) + \text{cr}_\phi(A, B) \tag{3}
\]

\[
\text{cr}_\phi(A, B \cup C) = \text{cr}_\phi(A, B) + \text{cr}_\phi(A, C). \tag{4}
\]
where $A$, $B$ and $C$ are mutually disjoint subsets of $E$. For the complete tripartite graph $K_{1,m,n}$ with the partition $(X, Y, Z)$, where $X = \{x_1\}$, $Y = \{y_1, \ldots, y_m\}$ and $Z = \{z_1, \ldots, z_n\}$ we write $E_{XY}$ for the set of all edges incident to $X$ and $Y$; and $E(z_i)$ for the set of all edges incident to $z_i$.

2. Lower bounds for $cr(K_{1,m,n})$

Firstly, we give the following lower bound of the crossing number of $K_{m+1,n+1}$:

**Theorem 2.1.** $cr(K_{1,m,n}) \geq cr(K_{m+1,n+1}) - \lfloor \frac{n}{m} \rfloor \lfloor \frac{m+1}{2} \rfloor$.

**Proof.** Let $\phi$ be a good drawing of $K_{1,m,n}$ with crossing number $cr(K_{1,m,n})$. Since $\phi$ is good, $cr_\phi(E_{XY}) = 0$. By (3) and (4),

$$cr(K_{1,m,n}) = cr_\phi(E) = cr_\phi\left(\bigcup_{i=1}^{n} E(z_i)\right) + \sum_{i=1}^{n} cr_\phi(E_{XY}, E(z_i)).$$

(5)

By renaming the vertices of $y_j$ if necessary, we may assume that the subgraph induced by $X \cup Y$ is drawn such that $x_1y_j$ lies between $x_1y_{j-1}$ and $x_1y_{j+1}$ (mod $m$ for $j + 1$), as in Fig. 1(a). For $1 \leq j \leq m$, let $A_j$ be the set of $z_i$ where $1 \leq i \leq n$, such that $x_1z_i$ lies between the edges $x_1y_j$ and $x_1y_{j+1}$. See Fig. 1(b) for $z_i \in A_1$.

We are going to obtain a drawing of $K_{m+1,n+1}$ from $\phi$. To do this, we draw a new vertex, $z_{n+1}$, near the vertex $x_1$ and lying in the region between the edges $x_1y_m$ and $x_1y_1$ such that $z_{n+1}$ lies between $x_1y_m$ and $x_1z_i$ for all $z_i \in A_m$, as shown in Fig. 1(c). Let $m' = \lfloor m/2 \rfloor$. For $1 \leq j \leq m'$, draw the edge $z_{n+1}y_j$ next to the edge $x_1y_j$ such that $z_{n+1}y_j$ only crosses $x_1y_i$ where $1 \leq i \leq j - 1$ and does not cross other edges in $E_{XY}$. For $m' + 1 \leq j \leq m$, draw the edge $z_{n+1}y_j$ next to the edge $x_1y_j$ such that $z_{n+1}y_j$ only crosses $x_1y_j$ where $j + 1 \leq i \leq m$ and does not cross other edges in $E_{XY}$. Draw the edge $z_{n+1}x_1$ without crossing any edges. Then remove the edges $x_1y_j$ for $1 \leq j \leq m$. See Fig. 1(d).

Now we have a drawing $\phi'$ of $K_{m+1,n+1}$ with $\{x_1, y_1, \ldots, y_m\}$ as the partition with $m + 1$ vertices and $\{z_1, \ldots, z_{n+1}\}$ as the partition with $n + 1$ vertices. Note that if $z_i \in A_j$ where $1 \leq j \leq m'$ (see Fig. 2(a) for...
Fig. 1(d). Remove the edges $x_1y_j$.

Fig. 2(a). $z_i \in A_2$ for $m = 5$.

Fig. 2(b). $z_i \in A_3$ for $m = 5$.

Fig. 2(c). $z_i \in A_5$ for $m = 5$.

$m = 5$ and $j = 2$, then

$$cr_{\phi'}(E(z_i), E(z_{n+1})) = cr_\phi(E(z_i), E_{XY}) + m' - j.$$  \hspace{1cm} (6)

If $z_i \in A_j$ where $m' + 1 \leq j \leq m - 1$ (see Fig. 2(b) for $m = 5$ and $j = 3$), then

$$cr_{\phi'}(E(z_i), E(z_{n+1})) = cr_\phi(E(z_i), E_{XY}) + j - m'.$$  \hspace{1cm} (7)

If $z_i \in A_m$, then by our construction that $z_{n+1}$ lies between $x_1y_m$ and $x_1z_i$ for all $z_i \in A_m$ (see Fig. 2(c) for $m = 5$), we have

$$cr_{\phi'}(E(z_i), E(z_{n+1})) = cr_\phi(E(z_i), E_{XY}) + m'.$$  \hspace{1cm} (8)

Note also that

$$cr_{\phi'} \left( \bigcup_{i=1}^{n} E(z_i) \right) = cr_{\phi} \left( \bigcup_{i=1}^{n} E(z_i) \right).$$  \hspace{1cm} (9)

By (3) and (4), the crossing number of $\phi'$ is

$$cr_{\phi'} \left( \bigcup_{i=1}^{n} E(z_i) \right) + \sum_{i=1}^{n} cr_{\phi'}(E(z_i), E(z_{n+1})).$$  \hspace{1cm} (10)

Putting (6)–(9) into (10), we obtain that the crossing number of $\phi'$ is

$$cr_{\phi} \left( \bigcup_{i=1}^{n} E(z_i) \right) + \sum_{i=1}^{n} cr_{\phi}(E_{XY}, E(z_i)) + \sum_{j=1}^{m'} (m' - j) |A_j| + \sum_{j=m'+1}^{m-1} (j - m') |A_j| + m'|A_m|,$$
which is at least \( cr(K_{m+1,n+1}) \). Combining this with (5), we have
\[
\sum_{j=1}^{m} (m' - j)|A_j| + \sum_{j=m+1}^{m} (j - m')|A_j| + m'|A_m| \geq cr(K_{m+1,n+1}) - cr(K_{1,m,n}).
\]

(11)

If we put \( z_{n+1} \) between the edges \( x_1y_i \) and \( x_1y_{i+1} \) where \( 1 \leq i \leq m \) in the above construction of \( K_{m+1,n+1} \), then by the same arguments, we can show that for \( 1 \leq i \leq m, \)
\[
\sum_{j=1}^{m'} (m' - j)|A_{j+i}| + \sum_{j=m'+1}^{m} (j - m')|A_{j+i}| + m'|A_{m+i}| \geq cr(K_{m+1,n+1}) - cr(K_{1,m,n}).
\]

(12)

where the indices of \( A_{j+i} \) read modulo \( m \). Summing up (12) for \( 1 \leq i \leq m \), we get
\[
\sum_{i=1}^{m} \left( \sum_{j=1}^{m'} (m' - j)|A_{j+i}| + \sum_{j=m'+1}^{m} (j - m')|A_{j+i}| + m'|A_{m+i}| \right) \geq m(\text{cr}(K_{m+1,n+1}) - \text{cr}(K_{1,m,n}))
\]

(13)

where the indices of \( A_{j+i} \) read modulo \( m \). One can show that the left-hand side of (13) is equal to \( \left\lfloor \frac{n}{2} \right\rfloor \frac{m+1}{2} \sum_{j=1}^{m} |A_j| \). Note also that \( \sum_{j=1}^{m} |A_j| = n \). Combining all these, (13) becomes \( n\left\lfloor \frac{n}{2} \right\rfloor \frac{m+1}{2} \geq m(\text{cr}(K_{m+1,n+1}) - \text{cr}(K_{1,m,n})) \), as required. \( \square \)

In [3], deKlerk et al. give the lower bound of the crossing number of \( K_{m,n} \) by showing that \( cr(K_{m,n}) \geq 0.8594Z(m,n) \). Combining this with Theorem 2.1, we can obtain a numerical lower bound for the crossing number of \( K_{1,m,n} \):

**Corollary 2.1.** \( cr(K_{1,m,n}) \geq 0.8594Z(m+1,n+1) - \left\lfloor \frac{n}{m} \right\rfloor \left\lfloor \frac{m+1}{2} \right\rfloor \).}

Using similar arguments in the proof of Theorem 2.1, we can also prove the following:

**Theorem 2.2.** \( cr(K_{1,2M,n}) \geq \frac{1}{2} (cr(K_{2M+1,n+2}) + cr(K_{2M+1,n}) - M(M + n - 1)) \).

**Proof.** Let \( \phi \) be a drawing of \( K_{1,2M,n} \) with \( cr_\phi(E) = cr(K_{1,2M,n}) \). Then (5) still holds for \( \phi \) with \( m = 2M \). We are going to obtain a drawing of \( K_{2M+1,n+2} \) from \( \phi \). Following the same arguments in the proof of Theorem 2.1, we draw a new vertex \( z_{n+1} \) between \( x_1z_{2M} \) and \( x_1z_1 \), as in Fig. 1(c). On the other hand, we draw a vertex \( z_{n+2} \) between \( x_1z_{M} \) and \( x_1z_{M+1} \). See Fig. 3(a) for \( M = 3 \).

Then draw the edges \( z_{n+1}x_1 \) and \( z_{n+1}y_j \) where \( 1 \leq j \leq 2M \) as in the proof of Theorem 2.1. Moreover, for \( 1 \leq j \leq M \), draw the edge \( z_{n+2}y_j \) next to the edge \( x_1y_j \) such that \( z_{n+2}y_j \) only crosses \( x_1y_i \) where \( j+1 \leq i \leq M \) and does not cross other edges in \( E_{XY} \). For \( M+1 \leq j \leq 2M \), draw the edge \( z_{n+2}y_j \) next to the edge \( x_1y_j \) such that \( z_{n+2}y_j \) only crosses \( x_1y_i \) where \( M+1 \leq i \leq j-1 \) and does not cross other edges in \( E_{XY} \). Draw the edge \( z_{n+2}x_1 \) without crossing any edges. Finally remove the edges \( x_1y_j \) for \( 1 \leq j \leq 2M \). See Fig. 3(b) for \( M = 3 \).

Therefore we obtain a drawing \( \phi'' \) of \( K_{2M+1,n+2} \) with \( \{x_1,y_1,\ldots,y_{2M}\} \) as the partition with \( 2M+1 \) vertices and \( \{z_1,\ldots,z_n,z_{n+1},z_{n+2}\} \) as the partition with \( n+2 \) vertices. Using the notion of \( A_j \) defined in the proof of Theorem 2.1, one can show that if \( z_i \in A_j \)
\[
\begin{align*}
\cr_{\phi''}(E(z_i), E(z_{n+1})) &= \cr_{\phi}(E(z_i), E_{XY}) + M - j \quad &\text{if } 1 \leq j \leq M; \\
\cr_{\phi''}(E(z_i), E(z_{n+1})) &= \cr_{\phi}(E(z_i), E_{XY}) + j - M \quad &\text{if } M + 1 \leq j \leq 2M.
\end{align*}
\]

(14)

(15)
Also, one can show that if $z_i \in A_j$

\[
\begin{align*}
    cr_{\phi''}(E(z_i), E(z_{n+2})) &= cr_{\phi}(E(z_i), E_{XY}) + j & \text{if } 1 \leq j \leq M; \\
    cr_{\phi''}(E(z_i), E(z_{n+2})) &= cr_{\phi}(E(z_i), E_{XY}) + 2M - j & \text{if } M + 1 \leq j \leq 2M.
\end{align*}
\]  

(16) and (17)

On the other hand, we have

\[
    cr_{\phi''}(E(z_{n+1}), E(z_{n+2})) = M(M - 1).
\]  

(18)

Note also that

\[
    cr_{\phi''} \left( \bigcup_{i=1}^{n} E(z_i) \right) = cr_{\phi} \left( \bigcup_{i=1}^{n} E(z_i) \right).
\]  

(19)

By (3) and (4), the crossing number of $\phi''$ is

\[
    cr_{\phi''} \left( \bigcup_{i=1}^{n} E(z_i) \right) + cr_{\phi''}(E(z_{n+1}), E(z_{n+2})) + \sum_{i=1}^{n} (cr_{\phi''}(E(z_i), E(z_{n+1})) + cr_{\phi''}(E(z_i), E(z_{n+2}))).
\]  

(20)

Hence, by putting (14)–(19) into (20), and by the fact that $\sum_{i=1}^{2M} |A_i| = n$, we obtain that the crossing number of $\phi''$ is

\[
    cr_{\phi} \left( \bigcup_{i=1}^{n} E(z_i) \right) + M(M - 1) + 2 \sum_{i=1}^{n} cr_{\phi}(E_{XY}, E(z_i)) + Mn
\]

\[
    = 2cr(K_{1,2M,n}) - cr_{\phi} \left( \bigcup_{i=1}^{n} E(z_i) \right) + M(M - 1) + Mn
\]

\[
    \leq 2cr(K_{1,2M,n}) - cr(K_{2M+1,n}) + M(M + n - 1),
\]

where the first equality follows from (5) with $m = 2M$; and the second inequality follows from the fact that the graph induced by $\bigcup_{i=1}^{n} E(z_i)$ is $K_{2M+1,n}$. Note also that the crossing number of $\phi''$ is at least $cr(K_{2M+1,n+2})$. Combining all these, we obtain $2cr(K_{1,2M,n}) - cr(K_{2M+1,n}) + M(M + n - 1) \geq cr(K_{2M+1,n+2})$ as required. □

From Theorems 2.1 and 2.2, we can derive the following:

**Theorem 2.3.** If Zarankiewicz’s conjecture is true for $m = 2M + 1$, then

\[
    cr(K_{1,2M,n}) = Z(2M + 1, n + 1) - M \left\lfloor \frac{n}{2} \right\rfloor.
\]

**Proof.** From (2), it suffices to prove

\[
    cr(K_{1,2M,n}) \geq Z(2M + 1, n + 1) - M \left\lfloor \frac{n}{2} \right\rfloor.
\]  

(21)

If Zarankiewicz’s conjecture is true for $m = 2M + 1$, then $cr(K_{2M+1,n}) = Z(2M + 1, n)$. Then (21) follows from Theorem 2.1 for $n$ is even, and from Theorem 2.2 for $n$ is odd. □

Therefore if Zarankiewicz’s conjecture is true for $m = 2M + 1$, then equality holds in (2) for $m = 2M$. Since Zarankiewicz’s conjecture is true for $m = 5$ [13], by putting $M = 2$ in Theorem 2.3, we have the following result appeared in [8,12]:

![Fig. 3(b). The result drawing.](image-url)
Corollary 2.2. \( cr(K_{1,4,n}) = Z(5, n) + 2\left\lfloor \frac{n}{4} \right\rfloor \).

By putting \( M = 3, 4 \) in Theorem 2.3, we have the following results appeared in [10,11]:

**Corollary 2.3.** The crossing number of \( K_{1,6,n} \) (and \( K_{1,8,n} \) respectively) is \( Z(7, n) + 6\left\lfloor \frac{n}{6} \right\rfloor \) (and \( Z(9, n) + 12\left\lfloor \frac{n}{9} \right\rfloor \) respectively) provided that Zarankiewicz’s conjecture holds for \( m = 7 \) (and \( m = 9 \) respectively).

To conclude, we state the following:

**Conjecture 2.1.**

\[

\begin{align*}
cr(K_{1,m,n}) &= cr(K_{m+1,n+1}) - \left\lfloor \frac{n}{m} \right\rfloor \left\lfloor \frac{m+1}{2} \right\rfloor; \\
2cr(K_{1,2M,n}) &= 1/2 (cr(K_{2M+1,n+2}) + cr(K_{2M+1,n}) - M(M + n - 1)).
\end{align*}

\]

Theorems 2.1 and 2.2 provide some evidences supporting Conjecture 2.1.

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