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Bivariate hypergeometric D-modules

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Abstract

We undertake the study of bivariate Horn systems for generic parameters. We prove that these hypergeometric systems are holonomic, and we provide an explicit formula for their holonomic rank as well as bases of their spaces of complex holomorphic solutions. We also obtain analogous results for the generalized hypergeometric systems arising from lattices of any rank.

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1. Introduction

Classically, there have been two main directions in the study of hypergeometric functions. The first of these is to study the properties of a particular series, analyze

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its convergence, compute its values at some specific points providing combinatorial identities, give integral representations, and find relations with other series of the same kind. Here one could refer to well-known works of Gauss and Euler, for instance [9,11].

The other classical avenue of research is to find a differential equation that our hypergeometric function satisfies, and to study all the solutions of that equation. This approach was pioneered by Kummer, who showed that the Gauss hypergeometric function $f(z) = F[a, b; c; z]$ defined as the power series:

$$1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)}\frac{z^3}{3!} + \cdots$$

satisfies the differential equation

$$z(1-z)\frac{d^2 f}{dz^2} + (c - (1+a+b)z)\frac{df}{dz} - abf = 0.$$

Kummer went on to find all of the solutions of this equation (see [19]). He constructed 24 (Gauss) series that, whenever a, b and c are not integers, provide representations of two linearly independent solutions to the Gauss equation, that are valid in any region of the complex plane. Riemann also had a fundamental influence in this field [22]. For more historical details on hypergeometric functions, and a comprehensive treatment of their classical theory, see [26].

Both of these approaches have been tried for bivariate hypergeometric series. In his article [8], Erdélyi gives a complete set of solutions for the following system of two hypergeometric equations in two variables:

$$\begin{aligned}\big(x(\theta_x + \theta_y + a)(\theta_x + b) - \theta_x(\theta_x + \theta_y + c - 1)\big) f &= 0\ ,\\ \big(y(\theta_x + \theta_y + a)(\theta_y + b') - \theta_y(\theta_x + \theta_y + c - 1)\big) f &= 0\ ,\end{aligned}$$

where $\theta_x = x\frac{\partial}{\partial x}$ and $\theta_y = y\frac{\partial}{\partial y}$. This is the system of equations for Appell's function F_1, and for generic values of the parameters a, b, b' and c, Erdélyi constructs more than 120 fully supported series solutions through contour integration. By a *fully supported series*, we mean a series such that the convex hull of the exponents of the monomials appearing with nonzero coefficient contains a full-dimensional cone. The holonomic rank of this system, that is, the dimension of its space of complex holomorphic solutions around a nonsingular point, is 3.

Another interesting system of two second-order hypergeometric equations in two variables is

$$\begin{aligned}\big(x(2\theta_x - \theta_y + a')(2\theta_x - \theta_y + a' + 1) - (-\theta_x + 2\theta_y + a)\theta_x\big) f &= 0\ ,\\ \big(y(-\theta_x + 2\theta_y + a)(-\theta_x + 2\theta_y + a + 1) - (2\theta_x - \theta_y + a')\theta_y\big) f &= 0\ .\end{aligned}$$

This is the system of equations for Horn's function G_3, and its holonomic rank is 4. Erdélyi notes that, in a neighborhood of a given point, three linearly independent solutions of this system can be obtained through contour integral methods. He also finds a fourth linearly independent solution: the Puiseux monomial $x^{-(a+2a')/3}y^{-(2a+a')/3}$. He remarks that the existence of this elementary solution is puzzling, especially since it cannot be expressed using contour integration, and offers no explanation for its occurrence.

One of the goals of this article is to give a formula for the rank of a system of two hypergeometric equations in two variables when the parameters are generic (cf. Theorem 2.5). We will explain why the system for Appell's F_1 has rank 3 and why the very similar system for Horn's G_3 has rank 4. We will also show that Puiseux polynomial solutions are a commonplace phenomenon. Moreover, we will prove that these systems of hypergeometric equations are holonomic for a generic choice of the parameters.

Our starting point are the ideas of Gel'fand et al. [12] about the Γ-series associated with lattices, and how they relate to Horn series. Note that Γ-series as defined in [12] are fully supported, and they do not account for the Puiseux polynomial solutions of Horn systems.

Holomorphic series solutions to a Horn system are equivalent to solutions of corresponding hypergeometric recursions (see Section 6, specifically Eq. (13)), thus our study of Puiseux polynomial solutions also characterizes the solutions to these recurrences that have finite support.

Finally, since we will be dealing with lattices that are not necessarily saturated, we also need to study the generalized hypergeometric systems associated with lattices (more general than the A-hypergeometric systems of Gel'fand, Kapranov and Zelevinsky). We show that, for generic parameters, these systems are also holonomic, without restriction on the number of variables or rank of the corresponding lattice, and prove the expected formula for their generic holonomic rank.

2. Multivariate hypergeometric systems

In order to accommodate two different sets of variables, we denote by D_n the Weyl algebra with generators $x_1, \ldots, x_n$, $\partial_{x_1}, \ldots, \partial_{x_n}$, and by D_m the Weyl algebra whose generators are $y_1, \ldots, y_m$, $\partial_{y_1}, \ldots, \partial_{y_m}$. We set $\theta_{x_j} = x_j \partial_{x_j}$ for $1 \leqslant j \leqslant n$, and $\theta_{y_i} = y_i \partial_{y_i}$, for $1 \leqslant i \leqslant m$. We also define $\theta_x = (\theta_{x_1}, \ldots, \theta_{x_n})$ and $\theta_y = (\theta_{y_1}, \ldots, \theta_{y_m})$. When the meaning is clear, we will drop many of the subindices to simplify the notation.

We fix a matrix $A = (a_{ij}) \in \mathbb{Z}^{(n-m)\times n}$ of full rank $n-m$ whose first row is the vector $(1, \ldots, 1)$, and a matrix $\mathcal{B} \in \mathbb{Z}^{n\times m} = (b_{ji})$ of full rank m such that $A \cdot \mathcal{B} = 0$. For $1 \leqslant j \leqslant m$, set $b_j = (b_{j1}, \ldots, b_{jm}) \in \mathbb{Z}^m$ the jth row of $\mathcal{B}$. The (positive) greatest common divisor of the maximal minors of the matrix $\mathcal{B}$ is denoted by g.

For $i = 1, \ldots, m$, and a fixed parameter vector $c = (c_1, \ldots, c_n) \in \mathbb{C}^n$, we let

$$\boldsymbol{P}_i = \prod_{b_{ji}<0} \prod_{l=0}^{|b_{ji}|-1} (b_j \cdot \theta_y + c_j - l), \tag{1}$$

$$\boldsymbol{Q}_i = \prod_{b_{ji}>0} \prod_{l=0}^{b_{ji}-1} (b_j \cdot \theta_y + c_j - l), \text{ and} \tag{2}$$

$$H_i = \boldsymbol{Q}_i - y_i \boldsymbol{P}_i, \tag{3}$$

where $b_j \cdot \theta_y = \sum_{k=1}^m b_{jk}\theta_{y_k}$. The operators H_i are the *Horn operators* corresponding to the lattice $L_{\mathcal{B}} = \{\mathcal{B}\cdot z : z \in \mathbb{Z}^m\}$ and the parameter vector c. We call $d_i = \sum_{b_{ij}>0} b_{ij} = -\sum_{b_{ij}<0} b_{ij}$ the order of the operator H_i.

Definition 2.1. The *Horn system* is the following left ideal of D_m:

$$\mathrm{Horn}\,(\mathcal{B}, c) = \langle H_1, \ldots, H_m \rangle \subseteq D_m.$$

Now denote by $b^{(i)}$ the columns of the matrix $\mathcal{B}$. Any vector $u \in \mathbb{R}^n$ can be written as $u = u_+ - u_-$, where $(u_+)_i = \max(u_i, 0)$, and $(u_-)_i = -\min(u_i, 0)$. For $i = 1, \ldots, m$, we let:

$$T_i = \partial_x^{b_+^{(i)}} - \partial_x^{b_-^{(i)}},$$

here we use multi-index notation $\partial_x^v = \partial_{x_1}^{v_1} \cdots \partial_{x_n}^{v_n}$. More generally, for any $u \in L_{\mathcal{B}}$, set

$$T_u = \partial_x^{u_+} - \partial_x^{u_-}.$$

These are the *lattice operators* arising from $L_{\mathcal{B}}$.

Definition 2.2. The *lattice ideal* arising from $L_{\mathcal{B}}$ is

$$I_{\mathcal{B}} = \langle T_u : u \in L_{\mathcal{B}} \rangle \subseteq \mathbb{C}[\partial_{x_1}, \ldots, \partial_{x_n}].$$

Recall that the *toric ideal* corresponding to A is

$$I_A = \langle T_u : u \in \ker_{\mathbb{Z}}(A) \rangle \subseteq \mathbb{C}[\partial_{x_1}, \ldots, \partial_{x_n}].$$

We will also denote

$$I = \langle T_1, \ldots, T_m \rangle \subseteq \mathbb{C}[\partial_{x_1}, \ldots, \partial_{x_n}].$$

The ideal I is called a *lattice basis ideal*. Note that for $m = 2$, I is a complete intersection. This is not necessarily true if $m > 2$.

Lattice ideals and toric ideals have been extensively studied (see, for instance [7,27]). Lattice basis ideals were introduced in [16].

There is a natural system of differential equations arising from a toric ideal I_A and a parameter vector. This system, called the *A-hypergeometric system with parameter* $A \cdot c$, is defined as

$$H_A(A \cdot c) = I_A + \left\langle \sum_{j=1}^{n} a_{ij} x_j \partial_{x_j} - (A \cdot c)_i : i = 1, \ldots, n-m \right\rangle \subseteq D_n.$$

From now on we will use the notation $\langle A \cdot \theta - A \cdot c \rangle$ to mean $\langle \sum_{j=1}^{n} a_{ij} x_j \partial_{x_j} - (A \cdot c)_i : i = 1, \ldots, n-m \rangle$.

A-hypergeometric systems were first defined by Gel'fand et al. [13], and their systematic analysis was started by Gel'fand, Kapranov and Zelevinsky (see, for instance [14]). Saito, Sturmfels and Takayama have used Gröbner deformations in the Weyl algebra to study A-hypergeometric systems (see [25]). In this article, we will extend this approach to the case of Horn systems.

Gel'fand, Graev and Retakh have also considered the *hypergeometric system associated with the lattice* $L_{\mathcal{B}} = \{\mathcal{B} \cdot z : z \in \mathbb{Z}^m\}$, which is defined to be the left D_n-ideal:

$$I_{\mathcal{B}} + \langle A \cdot \theta - A \cdot c \rangle \subseteq D_n.$$

We now introduce the left D_n-ideal $H_{\mathcal{B}}(c)$, that is very closely related to the Horn system $\mathrm{Horn}(\mathcal{B}, c)$:

$$H_{\mathcal{B}}(c) = I + \langle A \cdot \theta - A \cdot c \rangle \subseteq D_n.$$

The results in Section 5 imply that, for generic c, there is a vector space isomorphism between the solution spaces of $\mathrm{Horn}(\mathcal{B}, c)$ and $H_{\mathcal{B}}(c)$. Thus, we have two points of view to study Horn hypergeometric functions. We also call $H_{\mathcal{B}}(c)$ a Horn system, when the context is clear.

Remark 2.3. We have defined the Horn operators using falling factorials because this formulation will make clearer the relationship between $\mathrm{Horn}(\mathcal{B}, c)$ and $H_{\mathcal{B}}(c)$, but it is just as legal to define Horn systems using rising factorials, as it is done in many classical sources. For instance, the Horn and Appell systems from the previous section naturally lend themselves to a rising factorial formulation. This is not really a difficulty, since switching between rising and falling factorials in the definition of Horn systems is a matter of shifting the parameters by integers.

It is a well known result of Adolphson [1] that, for generic parameters $A \cdot c$, the holonomic rank of the A-hypergeometric system equals the normalized volume $\mathrm{vol}(A)$ of the convex hull of the columns of A, which is also the degree of the toric ideal I_A. Our goal is to obtain an explicit expression in this spirit for bivariate Horn systems. Previous work in this direction required very strong assumptions (see [23]).

Definition 2.4. In the case that $m = 2$, we set

$$v_{ij} = \begin{cases} \min(|b_{i1}b_{j2}|, |b_{j1}b_{i2}|) & \text{if } b_i,\ b_j \text{ are in opposite open quadrants of } \mathbb{Z}^2, \\ 0 & \text{otherwise,} \end{cases}$$

for $1 \leqslant i, j \leqslant n$. The number v_{ij} is called the *index* associated to b_i and b_j.

The following is the main result in this article, which follows from Corollary 4.3 and Theorems 8.1, 9.10, and 11.1.

Theorem 2.5. *Let $\mathcal{B}$ be an $n \times 2$ integer matrix of full rank such that its rows $b_1, \ldots, b_n$ satisfy $b_1 + \cdots + b_n = 0$. If $c \in \mathbb{C}^n$ is a generic parameter vector, then the ideals* Horn $(\mathcal{B}, c)$ *and $H_{\mathcal{B}}(c)$ are holonomic. Moreover,*

$$\operatorname{rank}(H_{\mathcal{B}}(c)) = \operatorname{rank}(\operatorname{Horn}(\mathcal{B}, c)) = d_1 d_2 - \sum_{\substack{b_i, b_j \\ \text{dependent}}} v_{ij} = g \cdot \operatorname{vol}(A) + \sum_{\substack{b_i, b_j \\ \text{independent}}} v_{ij},$$

where the first summation runs over linearly dependent pairs b_i, b_j of rows of $\mathcal{B}$ that lie in opposite open quadrants of $\mathbb{Z}^2$, and the second summation runs over linearly independent such pairs.

We can also give an explicit basis for the solution space of Horn $(\mathcal{B}, c)$ (and of $H_{\mathcal{B}}(c)$) (Theorem 10.3), and compute the exact dimension of the subspace of Puiseux polynomial solutions (Theorem 6.6).

3. Some observations about Horn systems

The Horn system Horn $(\mathcal{B}, c)$ is always compatible, even if c is not generic, in the sense that its solution space is always nonempty. First of all, the constant zero function is always a solution of Horn $(\mathcal{B}, c)$, since this system is homogeneous. Moreover, as we will see in Section 5, all the solutions of the A-hypergeometric system $H_A(A \cdot c)$ are solutions of $H_{\mathcal{B}}(c)$, and these can be transformed into solutions of Horn $(\mathcal{B}, c)$ (see Corollary 5.2), so that, under the assumptions that $\mathcal{B}$ is $n \times m$ of full rank m, $n > m$, with all column sums equal to zero, Horn $(\mathcal{B}, c)$ always has nonzero solutions, since $H_A(A \cdot c)$ always has nonzero solutions (its solution space has dimension at least $\deg(I_A) = \operatorname{vol}(A)$, see [25, Theorem 3.5.1]).

It is easy to understand how the Horn system Horn $(\mathcal{B}, c)$ changes if we choose a new parameter vector c', as long as $A \cdot c' = A \cdot c$. As a matter of fact, if $c = c' + \mathcal{B} \cdot z$, for some $z \in \mathbb{C}^m$, then it is easy to see that $f(y)$ is a solution of Horn $(\mathcal{B}, c')$ if and only if $y^z f(y)$ is a solution of Horn $(\mathcal{B}, c)$. Note also that the system $H_{\mathcal{B}}(c)$ depends only on $A \cdot c$, so that $H_{\mathcal{B}}(c) = H_{\mathcal{B}}(c')$ if $A \cdot c = A \cdot c'$.

A change in $A \cdot c$ can, instead, dramatically alter the solution space of Horn $(\mathcal{B}, c)$ (and $H_{\mathcal{B}}(c)$). For instance, it could become infinite dimensional, as the following example shows.

Example 3.1. The Horn system defined by the operators

$$(\theta_{y_1} + \theta_{y_2} + c_1)\theta_{y_i} - y_i(\theta_{y_1} + \theta_{y_2} + c_2)(\theta_{y_1} + \theta_{y_2} + c_3), \quad i = 1, 2 \tag{4}$$

is not holonomic if $(c_1 - c_2)(c_1 - c_3) = 0$. Indeed, a holonomic system of equations can only have a finite-dimensional space of analytic solutions. However, since for $(c_1 - c_2)(c_1 - c_3) = 0$ the operator $\theta_{y_1} + \theta_{y_2} + c_1$ can be factored out of each of the operators in (4), it follows that any function which is annihilated by $\theta_{y_1} + \theta_{y_2} + c_1$ is a solution to (4). Thus for any smooth univariate function u the product $y_2^{-c_1} u(y_1/y_2)$ satisfies (4).

Note that for generic values of the parameters c_1, c_2, c_3 system (4) is holonomic. One of its solutions is given by the Gauss function $F[c_2, c_3; c_1; y_1 + y_2]$. Of course, similar examples can be given in any dimension.

We could also ask what happens if we choose another matrix $\mathcal{B}'$ such that $A \cdot \mathcal{B}' = 0$. Even if $g = g' = 1$, so that $\mathcal{B}$ and $\mathcal{B}'$ are two Gale duals of A, the associated Horn systems could have different holonomic rank, as we see in Example 3.2. The systematic analysis of this question, in the case when $m = 2$ is one of the main objectives of this article.

Example 3.2. We choose

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ -2 & 1 \\ 1 & -2 \\ 0 & 1 \end{pmatrix}, \qquad B' = \begin{pmatrix} 1 & 2 \\ -2 & -3 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then, if c is a generic parameter vector, we have rank (Horn $(\mathcal{B}, c)) = 4$, and rank (Horn $(\mathcal{B}', c)) = 6$, as a consequence of Theorem 2.5. This can be verified for specific values of c using the computer algebra system *Macaulay 2* [15]. However, by Theorem 5.3, these two hypergeometric systems share all fully supported solutions.

Note that the definition of Horn $(\mathcal{B}, c)$ makes sense even if $\mathcal{B}$ is a square matrix, or if the rows of $\mathcal{B}$ do not add up to zero, or even if $\mathcal{B}$ does not have full rank. As a matter of fact, we will need to consider such Horn systems on our way to proving results about the case when $\mathcal{B}$ is $n \times m$ of full rank m, $m < n$, and the rows of $\mathcal{B}$ add up to zero. Many of the examples will also concern Horn systems with $n = m$. We remark that if $\mathcal{B}$ is square and nonsingular, then $H_{\mathcal{B}}(c)$ is a system of differential equations with constant coefficients, not depending on c.

4. Preliminaries on codimension 2 binomial ideals

In this section we collect some results about lattice ideals and lattice basis ideals that will be necessary to study Horn systems. Although this section is about commutative algebra, our indeterminates will be called $\partial_1, \ldots, \partial_n$ for consistency with the notation for differential equations.

Recall that $\mathcal{B} = (b_{ji})$ is an $n \times m$ integer matrix of full rank m with all column sums equal to zero. The following ideal is called a *lattice ideal*:

$$I_{\mathcal{B}} = \langle \partial^{u_+} - \partial^{u_-} : u = u_+ - u_- \in L_{\mathcal{B}} \rangle \subset \mathbb{C}[\partial_1, \ldots, \partial_n],$$

where $L_{\mathcal{B}} = \{\mathcal{B} \cdot z : z \in \mathbb{Z}^m\}$ is the rank-m lattice spanned by the columns of $\mathcal{B}$. For the purpose of this section, we could use any field of characteristic 0 instead of $\mathbb{C}$, but later on, when we talk about complex holomorphic solutions of differential equations, we will need our field to be the complex numbers. We let A be any $(n-m) \times n$ integer matrix such that $A \cdot \mathcal{B} = 0$. Then the saturation of $L_{\mathcal{B}}$ is the lattice $L = \ker_{\mathbb{Z}}(A)$. Note that the order of the group $L/L_{\mathcal{B}}$ is g, the positive greatest common divisor of the maximal minors of $\mathcal{B}$.

The ideal $I_{\mathcal{B}}$ is homogeneous with respect to the usual $\mathbb{Z}$-grading and hence defines a subscheme $X_{\mathcal{B}}$ of $\mathbb{P}^{n-1}$. Moreover, the ideal $I_{\mathcal{B}}$ is always radical and $X_{\mathcal{B}}$ is the equidimensional union of $g = |L/L_{\mathcal{B}}|$ torus translates of the toric variety X_A defined by the reduced scheme associated to L as above. This is deduced from [7] since $(I_{\mathcal{B}} : \langle \partial_1, \ldots, \partial_n \rangle^\infty) = I_{\mathcal{B}}$, that is, no component of $X_{\mathcal{B}}$ is contained in a coordinate hyperplane.

These torus translates can be described in terms of the order g group $G_{\mathcal{B}}$ of all partial characters $\rho : L \to \mathbb{C}^*$ which extend the trivial character $1 : L_{\mathcal{B}} \to \mathbb{C}^*$, i.e., ρ satisfying $\rho(\ell + \ell') = \rho(\ell)\rho(\ell'), \forall\ \ell, \ell' \in L$ and $\rho(\ell) = 1, \forall\ \ell \in L_{\mathcal{B}}$.

Example 4.1. We illustrate the previous decomposition in an example before writing it down in general. Let

$$\mathcal{B} = \begin{pmatrix} -1 & 2 \\ 0 & -3 \\ 3 & 0 \\ -2 & 1 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}.$$

In this case $g = 3$. The scheme X_A is the twisted cubic, that is, the closure of the torus orbit of the point $p_0 = (1 : 1 : 1 : 1) \in \mathbb{P}^3$ under the torus action:

$$\lambda \cdot (\partial_1 : \partial_2 : \partial_3 : \partial_4) = (\lambda^0 \partial_1 : \lambda^1 \partial_2 : \lambda^2 \partial_3 : \lambda^3 \partial_4), \quad \lambda \in \mathbb{C}^*. \tag{5}$$

The group $G_{\mathcal{B}}$ has order 3 and is isomorphic to the group of cubic roots of unity $\{1, \omega, \omega^2\}$, where $\omega = e^{\frac{2\pi i}{3}}$. Set $p_1 = (1 : 1 : \omega : 1)$, $p_2 = (1 : 1 : \omega^2 : 1)$ and denote by X_0, X_1 and X_2 the respective closure of the torus orbit under the action (5) of p_0, p_1 and p_2. In particular, $X_0 = X_A$. Then

$$X_B = X_0 \cup X_1 \cup X_2$$

and X_i is the image of X_0 under the coordinatewise multiplication by p_i, $i = 1, 2$. Note that

$$X_i = \{(\partial_1 : \cdots : \partial_4) : \partial_1\partial_3 - \omega^i \partial_2^2 = \partial_3^2 - \omega^{2i}\partial_2\partial_4 = \partial_2\partial_3 - \omega^i \partial_1\partial_4 = 0\}$$

so that the equations defining X_i are “translations” of the equations for $X_0 = X_A$.

This can be phrased in general as follows: Given $\rho \in G_{\mathcal{B}}$, let X_ρ denote zero scheme of the ideal:

$$I_\rho = \langle \partial^{u_+} - \rho(u)\partial^{u_-} : u = u_+ - u_- \in L\rangle.$$

Then the ideals I_ρ are prime, their intersection gives $I_{\mathcal{B}}$ and $X_{\mathcal{B}} = \cup_{\rho\in G_{\mathcal{B}}} X_\rho$. We refer to [7] for a proof of these facts.

Consider now the case $m = 2$ and recall that the lattice basis ideal associated to $\mathcal{B}$ is the ideal

$$I = \langle \partial^{u_+} - \partial^{u_-} : u \text{ is a column of } \mathcal{B}\rangle.$$

Its zero set consists of the union of $X_{\mathcal{B}}$ with components that lie inside coordinate hyperplanes. The following proposition, whose proof can be found in [5], gives the precise primary decomposition of the ideal I. Denote $b_1, \ldots, b_n \in \mathbb{Z}^2$ the row vectors of $\mathcal{B}$. Let v_{ij} be the index associated to b_i and b_j as in Definition 2.4.

Proposition 4.2. *The ideal I has the following primary decomposition*:

$$I = \left(\cap_{\rho\in G_{\mathcal{B}}} I_\rho\right) \cap \left(\cap_{v_{ij}>0} I_{ij}\right),$$

where $\sqrt{I_{ij}} = \langle \partial_i, \partial_j\rangle$, *and the multiplicity of each* I_{ij} is v_{ij}, *in the sense that*

$$\dim_K(\mathbb{C}[\partial_1, \ldots, \partial_n]/I_{ij})_{\langle \partial_1,\ldots,\hat{\partial}_i,\ldots,\hat{\partial}_j,\ldots,\partial_n\rangle} = v_{ij},$$

where $K = \mathbb{C}(\partial_1, \ldots, \hat{\partial}_i, \ldots, \hat{\partial}_j, \ldots, \partial_n)$.

We then have

Corollary 4.3. *For d_1, d_2 the degrees of the generators of I,*

$$d_1 \cdot d_2 - \sum_{b_i, b_j \text{ dependent}} v_{ij} = g \cdot \mathrm{vol}\,(A) + \sum_{b_i, b_j \text{ independent}} v_{ij}\ , \tag{6}$$

where the first summation runs over linearly dependent pairs b_i, b_j of rows of $\mathcal{B}$ that lie in opposite open quadrants of $\mathbb{Z}^2$, and the second summation runs over linearly independent such pairs.

Proof. The degree of the complete intersection I is $d_1 d_2$. By Proposition 4.2, this number equals

$$g \cdot \deg(I_A) + \sum v_{ij},$$

where the sum runs over all pairs of rows of $\mathcal{B}$ in opposite open quadrants of $\mathbb{Z}^2$. Now the result follows from the fact that the degree of I_A is exactly the normalized volume vol (A) of the polytope obtained by taking the convex hull of the columns of A [27, Theorem 4.16]. $\square$

The following is another result related to the primary decomposition of I.

Proposition 4.4. *Let $\mathcal{B} \in \mathbb{Z}^{n\times 2}$ of rank 2, with rows $b_1, \ldots, b_n$, that add up to zero, and $I_{\mathcal{B}}$, I, the lattice and lattice basis ideals associated to $\mathcal{B}$. For each $1 \leqslant i, j \leqslant n$, v_{ij} is as in Definition 2.4. Set*

$$\alpha_i = \begin{cases} \max_j \ v_{ij} & \text{if } b_{i1} > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\partial^{\alpha} I_{\mathcal{B}} \subseteq I.$$

Proof. By Proposition 4.2, it is enough to prove that $\partial^{\alpha} \in \cap_{v_{ij}>0} I_{ij}$. Assume that $v_{ij} > 0$. Then b_i and b_j lie in the interior of opposite quadrants, so that either b_{i1} or b_{j1} is positive, say $b_{i1} > 0$, so that $\alpha_i \geqslant v_{ij}$. We will be done if we show that $\partial_i^{v_{ij}} \in I_{ij}$. To do this, let $\tilde{I}_{ij}$ be the localization of I_{ij} at $\langle \partial_1, \ldots, \hat{\partial}_i, \ldots, \hat{\partial}_j, \ldots, \partial_n \rangle$ so that $\tilde{I}_{ij}$ is an artinian ideal of multiplicity v_{ij} in $K[\partial_i, \partial_j]$, where $K = \mathbb{C}(\partial_1, \ldots, \hat{\partial}_i, \ldots, \hat{\partial}_j, \ldots, \partial_n)$.

Note that since $\#\{1, \partial_i, \ldots, \partial_i^{v_{ij}}\} = v_{ij}+1$, these monomials must be linearly dependent modulo $\tilde{I}_{ij}$, so we can find $g_0, \ldots, g_{v_{ij}} \in K$ such that

$$g_0 + g_1\partial_i + \cdots + g_{v_{ij}}\partial_i^{v_{ij}} \in \tilde{I}_{ij}.$$

But the radical of $\tilde{I}_{ij}$ is $\langle \partial_i, \partial_j \rangle$, so that $g_0 = 0$. Let $l = \min_{1 \leqslant k \leqslant v_{ij}}\{g_k \neq 0\}$. Then, clearing denominators, we can find polynomials $f_l, \ldots, f_{v_{ij}}$ *not* involving the variables ∂_i, ∂_j, $f_l \neq 0$, such that

$$\partial_i^l (f_l + \cdots + f_{v_{ij}}\partial_i^{v_{ij}-l}) \in I_{ij}.$$

Now, since I_{ij} is primary to $\langle \partial_i, \partial_j \rangle$, and no power of $f_l + \cdots + f_{v_{ij}}\partial_i^{v_{ij}-l}$ belongs to $\langle \partial_i, \partial_j \rangle$, then ∂_i^l must belong to I_{ij}. Since $l \leqslant v_{ij}$, we are done. $\square$

It is an interesting fact that the multiplicities of some of the components of I do not go down under Gröbner deformation. Given $w \in \mathbb{Z}^n$, and $f = \sum f_\alpha x^\alpha$ a homogeneous polynomial in $\mathbb{C}[\partial_1, \ldots, \partial_n]$, let

$$\mathrm{in}_w(f) = \sum_{w\cdot\alpha \text{ maximal over } f_\alpha \neq 0} f_\alpha x^\alpha$$

and define

$$\mathrm{in}_w(I) = \langle \mathrm{in}_w(f) : f \in I \setminus \{0\} \rangle.$$

The ideal $\mathrm{in}_w(I)$ is called the *initial ideal of I with respect to the weight vector w*. It is a monomial ideal if w is generic (see [4] and [6, Chapter 15] for more on initial ideals, especially how to compute them).

Lemma 4.5. *Let b_k and b_l be two linearly dependent rows of $\mathcal{B}$ lying in opposite open quadrants of $\mathbb{Z}^2$. If w is a generic weight vector, then the multiplicity of the ideal $\langle \partial_k, \partial_l \rangle$ as an associated prime of $\mathrm{in}_w(I)$ is the index v_{kl}.*

This proof was suggested to us by Ezra Miller, to whom we are very grateful.

Proof. Recall that the initial variety of $\mathcal{V}(I)$ is the flat limit of a family that is obtained by a one parameter subgroup of the torus acting on the zero set $\mathcal{V}(I)$. The monomial components of $\mathcal{V}(I)$ are invariant under this action, so in the limit, the only way that the multiplicity of $\langle \partial_l, \partial_k \rangle$ could go up is if this prime is associated to $\mathrm{in}_w(I_\mathcal{B})$. Now, if b_k and b_l are linearly dependent, $\langle \partial_k, \partial_l \rangle$ is not associated to $\mathrm{in}_w(I_\mathcal{B})$, this follows from the same arguments that proved [20, Lemma 2.3]. $\square$

5. A-hypergeometric solutions of the Horn system

In this section we study the solutions of the Horn system Horn$(\mathcal{B}, c)$ that arise from the A-hypergeometric system $H_A(A \cdot c)$. Here, we do not use the assumption that $m = 2$. Recall that $\mathcal{B} = (b_{ji})$ is an rank m integer $n \times m$ matrix whose rows add up to zero, and whose columns are denoted $b^{(1)}, \ldots, b^{(m)}$ and let $A = (a_{ij})$ be any rank $(n-m)$ integer $(n-m) \times n$ matrix such that $A \cdot \mathcal{B} = 0$. Here we assume that $n > m$.

Consider the surjective map

$$\begin{aligned} x^{\mathcal{B}} : (\mathbb{C}^*)^n &\to (\mathbb{C}^*)^m, \\ x &\mapsto \left(\prod_{j=1}^n x_j^{b_{j1}}, \ldots, \prod_{j=1}^n x_j^{b_{jm}} \right) = (x^{b^{(1)}}, \ldots, x^{b^{(m)}}). \end{aligned}$$

This map is open in the sense that it takes open sets to open sets. We use it to relate the operators T_i in n variables and the operators H_i in m variables, defined in Section 2.

Lemma 5.1. *Let $U \subseteq (\mathbb{C}^*)^n$ be a simply connected open set and let $V = x^{\mathcal{B}}(U)$. We choose U small enough so that V is also simply connected. Given a holomorphic function $\psi \in \mathcal{O}(V)$, call $\varphi = x^c \psi(x^{\mathcal{B}})$. Then*

(i) $\left(\sum_{j=1}^n a_{kj} x_j \partial_{x_j}\right)(\varphi) = (A \cdot c)_k \varphi$, *for* $k = 1, \ldots, n-m$.
(ii) $T_i(\varphi) = 0$ for $i = 1, \ldots, m$ *if and only if* $H_i(\psi) = 0$ for $i = 1, \ldots, m$.
(iii) *Moreover, for any $u = \mathcal{B} \cdot z \in L_{\mathcal{B}}$, and*

$$H_u = \prod_{u_j > 0} \prod_{l=0}^{u_j - 1} (b_j \cdot \theta_y + c_j - l) - y^z \prod_{u_j < 0} \prod_{l=0}^{|u_j| - 1} (b_j \cdot \theta_y + c_j - l),$$

we have $T_u(\varphi) = 0$ if and only if $H_u(\psi) = 0$.

Proof. The verifications of the three assertions are very similar. The main ingredients are the following identities:

$$\theta_{x_i} x^c = x^c (\theta_{x_i} + c_i) \quad (\text{in } D_n), \tag{7}$$

$$\theta_{x_i}(\psi(x^{\mathcal{B}}))(x) = \left[(b_i \cdot \theta_y)\psi\right](x^{\mathcal{B}}), \tag{8}$$

which are easily checked. Let us prove (ii). Call $\tilde{T}_i = \prod_{b_{ji} > 0} x_j^{b_{ji}} T_i$. We have

$$\tilde{T}_i = \prod_{b_{ji} > 0} x_j^{b_{ji}} \prod_{b_{ji} > 0} \partial_{x_j}^{b_{ji}} - (x^{\mathcal{B}})_i \prod_{b_{ji} < 0} x_j^{-b_{ji}} \prod_{b_{ji} < 0} \partial_{x_j}^{-b_{ji}}. \tag{9}$$

Recall that $(x^{\mathcal{B}})_i = \prod_{j=1}^{n} x_j^{b_{ji}}$. Using the identity

$$x^\alpha \partial_x^\alpha = \prod_{j=1}^{n} \prod_{l=0}^{\alpha_j - 1} (\theta_{x_j} - l),$$

Eq. (9) is transformed into

$$\tilde{T}_i = \prod_{b_{ji}>0} \prod_{l=0}^{b_{ji}-1} (\theta_{x_j} - l) - (x^{\mathcal{B}})_i \prod_{b_{ji}<0} \prod_{l=0}^{-b_{ji}-1} (\theta_{x_j} - l).$$

Using (7),

$$\begin{aligned} \tilde{T}_i(\varphi) &= \tilde{T}_i(x^c \psi(x^{\mathcal{B}})) \\ &= x^c \left(\prod_{l=0}^{b_{ji}-1} (\theta_{x_j} + c_j - l) - (x^{\mathcal{B}})_i \prod_{b_{ji}<0} \prod_{l=0}^{-b_{ji}-1} (\theta_{x_j} + c_j - l) \right) (\psi(x^{\mathcal{B}})). \end{aligned}$$

Now (8) implies that

$$\begin{aligned} \tilde{T}_i(\varphi) &= x^c \left(\prod_{l=0}^{b_{ji}-1} (b_j \cdot \theta_y(\psi) + c_j - l) \right. \\ &\quad \left. -(x^{\mathcal{B}})_i \prod_{b_{ji}<0} \prod_{l=0}^{-b_{ji}-1} (b_j \cdot \theta_y(\psi) + c_j - l) \right) (x^{\mathcal{B}}) \\ &= x^c H_i(\psi)((x^{\mathcal{B}})\ . \end{aligned}$$

This shows that $\tilde{T}_i(\varphi)$ is identically zero if and only if $H_i(\psi)(x^{\mathcal{B}}) = 0$ for all $x \in U$. This is equivalent to $H_i(\psi)$ vanishing identically on V. Since $T_i\varphi = 0$ if and only if $\tilde{T}_i\varphi = 0$, we obtain the desired result. $\square$

Parts (i) and (ii) of Lemma 5.1 have the following consequence.

Corollary 5.2. *The map*

$$\begin{Bmatrix} \textit{Holomorphic solutions of} \\ \mathrm{Horn}\,(\mathcal{B}, c) \textit{ on } V \end{Bmatrix} \longrightarrow \begin{Bmatrix} \textit{Holomorphic solutions of} \\ H_{\mathcal{B}}(c) \textit{ on } U \end{Bmatrix}$$
$$\psi \longmapsto x^c \psi(x^{\mathcal{B}})$$

is a vector space isomorphism, that takes Puiseux polynomials to Puiseux polynomials.

Finally, we can use the solutions of $H_A(A \cdot c)$ to construct solutions of $H_{\mathcal{B}}(c)$ (and thus of $\operatorname{Horn}(\mathcal{B}, c)$). We refer to [25, Section 3] for background on the canonical series solutions of the A-hypergeometric systems introduced by Gel'fand, Kapranov and Zelevinsky. In the case when c is generic, these canonical series solutions are fully supported logarithm-free series.

Theorem 5.3. *Given a generic parameter vector c, and a canonical basis $\{\phi^k : k = 1, \ldots, \operatorname{vol}(A)\}$ for the space of solutions of the A-hypergeometric system $H_A(A \cdot c)$, there exist linearly independent, fully supported solutions with disjoint supports*

$$\{\psi_l^k : k = 1, \ldots, \operatorname{vol}(A), l = 1, \ldots, g\}$$

of $\operatorname{Horn}(\mathcal{B}, c)$ *such that*

$$\phi^k = x^c \sum_{l=1}^{g} \psi_l^k(x^{\mathcal{B}}) \quad for\ all\ \ k = 1, \ldots, \operatorname{vol}(A)\ .$$

Moreover, no (nontrivial) linear combination of the functions ψ_l^k is ever a Puiseux polynomial. This natural decomposition holds as well for canonical series solutions with logarithms.

Proof. By [24, Proposition 5.2], [25, Section 2.5], a canonical series solution ϕ of the A-hypergeometric system $H_A(A \cdot c)$ is of the form

$$\phi = x^\alpha \sum \lambda_{u,v} x^u \log(x^v), \tag{10}$$

with $A \cdot \alpha = A \cdot c$, and $v, u \in L = \ker_{\mathbb{Z}}(A)$. We show that ϕ can be decomposed as a sum of g solutions $\psi_1, \ldots, \psi_g$ of $H_{\mathcal{B}}(c)$ such that, if ψ_j, ψ_l are nonzero, then they have disjoint supports. Observe that, if $u, v \in L$, then

$$\Big((A \cdot \theta)_j - (A \cdot c)_j\Big)\Big(x^{u+\alpha} \log(x^v)\Big) = 0, \quad \text{and} \tag{11}$$

$$\partial_i \Big(x^{u+\alpha} \log(x^v)\Big) = (u + \alpha)_i \ \ x^{u+\alpha-e_i} \log(x^v) \ + \ v_i x^{u+\alpha-e_i}. \tag{12}$$

Consider the lattice $L_{\mathcal{B}} \subseteq \mathbb{Z}^n$ generated by the columns of $\mathcal{B}$, and its saturation $L = \ker_{\mathbb{Z}}(A)$, generated by the columns of a Gale dual B of A (that is, the columns of B form a $\mathbb{Z}$-basis for the integer kernel of A). Let $\{u_l : l = 1, \ldots, g\}$ be a system of representatives for $L/L_{\mathcal{B}}$. Define

$$\psi_l = x^\alpha \sum_{u \equiv u_l \bmod L} \lambda_{u,v} x^u \log(x^v).$$

Clearly, $\phi = \psi_1 + \cdots + \psi_g$, and the summands have pairwise disjoint support. By (11), each ψ_l is a solution of the system of homogeneities $\langle A \cdot \theta - A \cdot c \rangle$. Now we need to check that each ψ_l is a solution of the binomial operators $T_1, \ldots, T_m$ given by the columns of $\mathcal{B}$. Consider $T_j = \partial^{b_+^{(j)}} - \partial^{b_-^{(j)}}$. Certainly $T_j \phi = 0$. We apply the operator T_j to $\phi = \psi_1 + \cdots + \psi_g$, and observe that terms coming from T_j applied to ψ_l cannot cancel with terms coming from $\partial^{b_+^{(j)}}$ nor from $\partial^{b_-^{(j)}}$ applied to $\psi_{l'}$ if $l \neq l'$. This is because the exponents of the monomials appearing in $\left(\partial^{b_+^{(j)}}\right)(\psi_l)$, for instance, are $b_+^{(j)}$-translates of the exponents of the monomials from ψ_l by (12), and $b_+^{(j)} - b_-^{(j)} \in L_{\mathcal{B}}$. The lack of cancellation now follows from the fact that the supports of ψ_l and $\psi_{l'}$ are not congruent modulo $L_{\mathcal{B}}$ by construction.

Now, if we have a canonical basis $\{\phi^k, k = 1, \ldots, \mathrm{vol}(A)\}$ for the space of solutions of $H_A(A \cdot c)$ for generic $c \in \mathbb{C}$, they are of the form

$$\phi^k = x^{\alpha_k} \sum_{u \in L \cap \mathcal{C}_k} \lambda_{u,v} x^u,$$

for different exponents α_k with respect to a generic weight vector, and u ranging over all lattice points in a full-dimensional pointed cone $\mathcal{C}_k$. Note that since c is generic, no pair of the exponents α_k can differ by an integer vector. Decompose each $\phi^k = \phi_1^k + \cdots + \phi_g^k$ as above. Note that all ϕ_l^k are nonzero; in fact, the convex hull of all the supports is full dimensional. Moreover, the collection ϕ_l^k, $k = 1, \ldots, \mathrm{vol}(A)$, $l = 1, \ldots, g$ is linearly independent since the supports are disjoint. By Lemma 5.1, each ϕ_l^k is of the form $x^c \psi_l^k(x^{\mathcal{B}})$, where ψ_l^k is a solution of $\mathrm{Horn}(\mathcal{B}, c)$. Clearly, no (nontrivial) linear combination of the functions ψ_l^k is ever a Puiseux polynomial; in particular, they are linearly independent. $\square$

6. Puiseux polynomial solutions of the Horn system and solutions to hypergeometric recurrences with finite support

Throughout this section we assume that $m = 2$. Denote by $\mathrm{rank}_p(J)$ the dimension of the space of Puiseux polynomial solutions of a D-ideal J.

The first step to compute the dimension of the space of Puiseux polynomial solutions of $\mathrm{Horn}(\mathcal{B}, c)$ is to observe that such a solution gives rise to a solution of a certain system of difference equations. A monomial multiple of a Laurent series $\sum_{u \in \mathbb{Z}^m} a(u) y^u$, say $y^\gamma \sum_{u\in\mathbb{Z}^m} a(u)y^u$, is a solution of $\mathrm{Horn}(\mathcal{B}, c)$ if and only if its coefficients $a(u)$ satisfy the recursions

$$a(u+e_i)Q_i(u+\gamma+e_i) = a(u)P_i(u+\gamma), \quad i = 1,\ldots,m. \tag{13}$$

By the support of a solution $a(u)$ to (13) we mean the set $\{u : a(u) \neq 0\}$. The following proposition is a consequence of Proposition 5 in [21].

Proposition 6.1. *Puiseux polynomial solutions of* $\mathrm{Horn}(\mathcal{B}, c)$ *are in one-to-one correspondence with solutions to* (13) *with finite support.*

Let $\mathcal{B}[i, j]$ be the square submatrix of $\mathcal{B}$ whose rows are b_i and b_j, and let $c[i, j]$ be the vector in $\mathbb{C}^2$ whose coordinates are c_i and c_j. We now reduce the computation of the dimension of the space of Puiseux polynomial solutions to $\mathrm{Horn}(\mathcal{B}, c)$ to the case when $\mathcal{B}$ is a 2×2 matrix.

Lemma 6.2. *For a generic parameter vector c,*

$$\mathrm{rank}_p(\mathrm{Horn}(\mathcal{B}, c)) = \sum_{i<j} \mathrm{rank}_p(\mathrm{Horn}(\mathcal{B}[i, j], c[i, j])).$$

Proof. We call the support S of a solution of $\mathrm{Horn}(\mathcal{B}, c)$ *irreducible* if there exists no other solution whose support is a proper nonempty subset of S. Let $f(y)$ be a series solution to $\mathrm{Horn}(\mathcal{B}, c)$ with irreducible support S and let $s_0 \in S$. It follows by Theorem 1.3 in [23] that if the monomial y^{s_0} is not present in the series $f(y)$ then for no $s \in S$ can y^s be present in $f(y)$. This implies that irreducible supports are disjoint. Indeed, if S_1 and S_2 are irreducible and $s_0 \in S_1 \cap S_2$ then there exist solutions f_1 (respectively f_2) of $\mathrm{Horn}(\mathcal{B}, c)$ supported in S_1 (respectively S_2) such that $f_1 - f_2$ does not contain y^{s_0}. But then, since y^{s_0} does not appear in $f_1 - f_2$, no monomial in S_2 can appear in $f_1 - f_2$, and hence $S_1 \backslash S_2$ supports a solution of $\mathrm{Horn}(\mathcal{B}, c)$. This contradicts the fact that S_1 was irreducible.

Any Puiseux polynomial solution of $\mathrm{Horn}(\mathcal{B}, c)$ can be written as a linear combination of polynomial solutions with irreducible supports. Since Puiseux polynomials with disjoint supports are linearly independent, it is sufficient to count irreducible supports in order to determine $\mathrm{rank}_p(\mathrm{Horn}(\mathcal{B}, c))$.

Remember that the equations of the Horn system translate into recurrence relations (13) for the coefficients of any of its power series solutions. We refer to [23] for a detailed study of these recurrences. They imply that any coefficient in a solution of a Horn system is given by a nonzero multiple of any of its adjacent coefficients, as long as none of the polynomials $\boldsymbol{P}_i$, $\boldsymbol{Q}_i$ vanish at the corresponding exponent. This yields that the support of a solution must be "bounded" by the zeros of these polynomials in the following sense. The exponent of a monomial in a solution must lie in the zero locus of at least one of the polynomials $\boldsymbol{P}_i$, $\boldsymbol{Q}_i$, provided that some of the adjacent exponents are not present in the polynomial solution (see [23, Theorem 1.3]).

Let S be the support of a Puiseux solution of Horn $(\mathcal{B}, c)$. If S is irreducible, then for a generic vector c the set S cannot meet more than two lines of the form $b_j \cdot \theta_y + c_j - l = 0$ corresponding to different parameters c_j. If it only meets one such line then by Theorem 1.3 in [23] the set S cannot be finite (in fact, its convex hull is a half-plane in this case). If S meets two lines of the above form then all the other lines can be removed from the picture without affecting the supports (but not the coefficients) of the Puiseux polynomial solutions which are generated by this specific pair of lines. This implies the desired result. □

Now our goal is to compute $\operatorname{rank}_p(\text{Horn}\,(\mathcal{B}[i,j], c[i,j]))$. The first step is to eliminate the cases when this rank is zero.

Lemma 6.3. *The system* Horn $(\mathcal{B}[i,j], c[i,j])$ *has nonzero Puiseux polynomial solutions only if b_i and b_j are linearly independent in opposite open quadrants of $\mathbb{Z}^2$, or for some special values of c_i, c_j when b_i, b_j are linearly dependent and opposite. The corresponding Puiseux polynomial solutions of $H_{\mathcal{B}[i,j]}(c[i,j])$ are Taylor polynomials, that is, polynomials with natural number exponents.*

Proof. Corollary 5.2 gives a vector space isomorphism between the solution spaces of the hypergeometric systems Horn $(\mathcal{B}[i,j], c[i,j])$ and $H_{\mathcal{B}[i,j]}(c[i,j])$ that takes Puiseux polynomials to Puiseux polynomials. Thus it is enough to investigate the Puiseux polynomial solutions of $H_{\mathcal{B}[i,j]}(c[i,j])$. If b_i and b_j do not lie in the interior of opposite open quadrants, one of the operators in $H_{\mathcal{B}[i,j]}(c[i,j])$ is of the form $\partial^\alpha - 1$ for some $\alpha \in \mathbb{N}^2$. It is clear that such an operator cannot have a Puiseux polynomial solution.

Now assume that b_i and b_j lie in the interior of opposite quadrants. Let us prove the statement about Taylor polynomials. We may without loss of generality assume that $b_{i1} > 0$. If $b_{i2} < 0$, then the change of variables $\tilde{y}_1 = y_1$, $\tilde{y}_2 = 1/y_2$, transforms Horn $(\mathcal{B}[i,j], c[i,j]))$ into a Horn system given by a 2×2 matrix whose first row lies in the first open quadrant of $\mathbb{Z}^2$. Thus we may assume that $b_{i1}, b_{i2} > 0$, and consequently $b_{j1}, b_{j2} < 0$, since b_i and b_j lie in opposite open quadrants.

In this case

$$H_{\mathcal{B}[i,j]}(c[i,j]) = \langle \partial_i^{b_{i1}} - \partial_j^{-b_{j1}}, \partial_i^{b_{i2}} - \partial_j^{-b_{j2}} \rangle,$$

and this is an ideal in the Weyl algebra with generators x_i, x_j, ∂_i, ∂_j.

Let us show that any Puiseux polynomial solution f of $H_{\mathcal{B}[i,j]}(c[i,j])$ with irreducible support is actually a Taylor polynomial. This will imply the statement of the lemma. Choose $(u_0, v_0) \in \operatorname{supp}(f)$ such that $\operatorname{Re} u_0 = \min\{\operatorname{Re} u : (u,v) \in \operatorname{supp}(f) \setminus \mathbb{N}^2\}$. Then $(\partial_i^{b_{i1}} - \partial_j^{-b_{j1}}) f$ contains the monomial $x_i^{u_0 - b_{i1}} x_j^{v_0}$ with a nonzero coefficient unless u_0 is a natural number strictly less than b_{i1}. In this case, $v_0 \notin \mathbb{N}$. Now, since all the elements of $\operatorname{supp}(f)$ differ by integer vectors, and the real part of u_0 is minimal, we have that $u \in \mathbb{N}$ for all $(u, v) \in \operatorname{supp}(f) \backslash \mathbb{N}^2$. Now pick (u_1, v_1) such that the real part of v_1 is minimal, and conclude that, either v_1 is a natural number strictly less than b_{j1} or $x_i^{u_1} x_j^{v_1 - b_{j1}}$ appears with nonzero coefficient in $(\partial_i^{b_{i1}} - \partial_j^{-b_{j1}}) f = 0$. But now $v \in \mathbb{N}$ for all $(u, v) \in \operatorname{supp}(f) \backslash \mathbb{N}^2$. We conclude that $\operatorname{supp}(f) \subset \mathbb{N}^2$.

Finally, let us show that if b_i and b_j are linearly dependent, then the system $\operatorname{Horn}(\mathcal{B}[i, j], c[i, j])$ has only the identically zero solution, as long as c is generic. Using the change of variables $\xi_1 = y_1^{1/b_{i1}}$, $\xi_2 = y_2^{1/b_{i2}}$, we transform the operator $b_i \cdot \theta_y$ to the operator $\theta_{\xi_1} + \theta_{\xi_2}$. By Lemma 11.4 (to be proved in Section 11) there exists a nonzero polynomial in y_1, y_2 which lies in the ideal $\operatorname{Horn}(\mathcal{B}[i, j], c[i, j])$. Thus the only holomorphic solution to the system is the zero function. $\square$

Example 6.4. Let us construct the Puiseux polynomial solutions to the system of equations $\operatorname{Horn}(\mathcal{B}, 0)$, where

$$\mathcal{B} = \begin{pmatrix} 4 & 5 \\ -3 & -5 \end{pmatrix}.$$

The system $H_{\mathcal{B}}(0)$ is defined by the operators

$$\frac{\partial^4}{\partial x_1^4} - \frac{\partial^3}{\partial x_2^3}, \qquad \frac{\partial^5}{\partial x_1^5} - \frac{\partial^5}{\partial x_2^5}. \tag{14}$$

Note that we may use the parameter 0 without loss of generality. The solutions of $H_{\mathcal{B}}(c)$ are exactly the same as those of $H_{\mathcal{B}}(0)$, and in the case of $\operatorname{Horn}(\mathcal{B}, c)$, the only effect is a translation of the supports of the solutions.

The supports of the polynomial solutions to (14) are displayed in Fig. 1. Two exponents are connected if the corresponding monomials are contained in a polynomial solution with irreducible support. Note that in order to obtain these supports, we just connected the (empty) circles inside a certain rectangle to other integer points using the moves given by the columns of $\mathcal{B}$.

The polynomial solutions to (14) are given by

$$1, \quad x_1, \quad x_1^2, \quad x_1^3, \quad x_2, \quad x_1x_2, \quad x_1^2x_2, \quad x_1^3x_2, \quad x_2^2, \quad x_1x_2^2, \quad x_1^2x_2^2, \quad x_1^3x_2^2,$$

$$x_1^4 + 4x_2^3, \quad x_1^4x_2 + x_2^4, \quad 5x_1^4x_2^2 + 2x_1^5 + 2x_2^5 + 40x_1x_2^3.$$

Now let us unravel our isomorphism of solution spaces to obtain the corresponding solutions of $\operatorname{Horn}(\mathcal{B}, 0)$. As in the proof of the previous lemma, if $\psi = \sum \psi_\alpha y^\alpha$ is a Puiseux polynomial solution of $\operatorname{Horn}(\mathcal{B}, 0)$, and $\psi_\alpha \neq 0$, then $\binom{u}{v} = \mathcal{B} \cdot \alpha \in \mathbb{N}^2$.

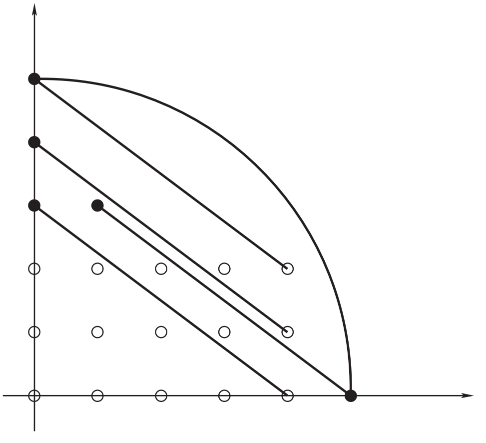

Fig. 1. The supports of the 15 polynomial solutions to (14).

But then

$$\alpha = \mathcal{B}^{-1} \cdot \binom{u}{v} = \begin{pmatrix} 1 & 1 \\ -3/5 & -4/5 \end{pmatrix} \cdot \binom{u}{v}.$$

This implies that α_1 is a natural number, and $\alpha_2 \in (-1/5)\mathbb{N}$. Moreover $\mathcal{B} \cdot \alpha \geqslant 0$. Thus, in order to find the irreducible supports of the Puiseux polynomial solutions of Horn $(\mathcal{B}, 0)$, we need to draw the region $\mathcal{B} \cdot \alpha \geqslant 0$, plot the points $\alpha \in \mathbb{N} \times (-1/5)\mathbb{N}$, and connect those points with horizontal and vertical moves. This is done in Fig. 2. The solid points belong to the supports of Puiseux polynomials, and the empty circles and dotted lines correspond to fully supported solutions. Thus the polynomial solutions to Horn $(\mathcal{B}, 0)$ are as follows:

$$\begin{gathered} 1, \quad y_1 y_2^{-3/5}, \quad y_1^2 y_2^{-6/5}, \quad y_1^3 y_2^{-9/5}, \quad y_1 y_2^{-4/5}, \quad y_1^2 y_2^{-7/5}, \quad y_1^3 y_2^{-2}, \quad y_1^4 y_2^{-13/5}, \\ y_1^2 y_2^{-8/5}, \quad y_1^3 y_2^{-11/5}, \quad y_1^4 y_2^{-14/5}, \quad y_1^5 y_2^{-17/5}, \quad y_1^4 y_2^{-12/5} + 4 y_1^3 y_2^{-12/5}, \\ y_1^5 y_2^{-16/5} + y_1^4 y_2^{-16/5}, \quad 5 y_1^6 y_2^{-4} + 2 y_1^5 y_2^{-3} + 2 y_1^5 y_2^{-4} + 40 y_1^4 y_2^{-3}. \end{gathered}$$

We are now ready to compute $\operatorname{rank}_p(\operatorname{Horn}(\mathcal{B}[i, j], c[i, j]))$.

Lemma 6.5. *The dimension of the space of Puiseux polynomial solutions of the hypergeometric system* Horn $(\mathcal{B}[i, j], c[i, j])$ *equals* v_{ij} *if the vectors* b_i *and* b_j *are linearly independent and lie in opposite open quadrants of* $\mathbb{Z}^2$.

Proof. Suppose that b_i and b_j are linearly independent and lie in opposite open quadrants of $\mathbb{Z}^2$. As in Lemma 6.3, we may assume that b_i lies in the interior of

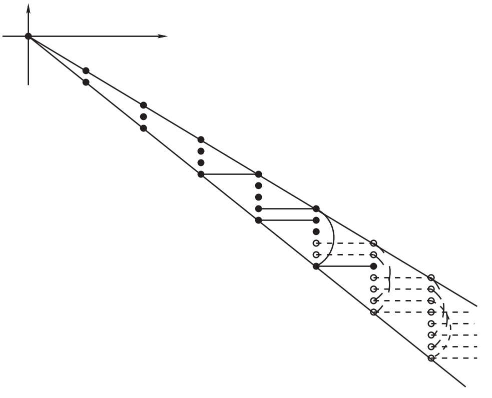

Fig. 2. The supports of the 15 Puiseux polynomial solutions to Horn $(\mathcal{B}, 0)$ in Example 6.4.

the first quadrant (so that b_j lies in the interior of the third). By Corollary 5.2, it is sufficient to compute the number of Puiseux polynomial solutions of $H_{\mathcal{B}[i,j]}(c[i,j])$.

Introduce vectors α, β as follows:

$$\alpha = \begin{cases} (b_{i1}, b_{j1}) & \text{if } |b_{i1}b_{j2}| > |b_{i2}b_{j1}|, \\ (-b_{i1}, -b_{j1}) & \text{if } |b_{i1}b_{j2}| < |b_{i2}b_{j1}|, \end{cases}$$

$$\beta = \begin{cases} (-b_{i2}, -b_{j2}) & \text{if } |b_{i1}b_{j2}| > |b_{i2}b_{j1}|, \\ (b_{i2}, b_{j2}) & \text{if } |b_{i1}b_{j2}| < |b_{i2}b_{j1}|. \end{cases}$$

Furthermore, denote by $\mathcal{R}$ the set of points

$$\mathcal{R} = \begin{cases} \{(u, v) \in \mathbb{N}^2 : u < b_{i2},\ v < -b_{j1}\} & \text{if } |b_{i1}b_{j2}| > |b_{i2}b_{j1}|, \\ \{(u, v) \in \mathbb{N}^2 : u < b_{i1},\ v < -b_{j2}\} & \text{if } |b_{i1}b_{j2}| < |b_{i2}b_{j1}|, \end{cases}$$

and call it the *base rectangle* of $H_{\mathcal{B}[i,j]}(c[i,j])$. By a *path* connecting two points $a, \tilde{a} \in \mathbb{N}^2$ we mean a sequence $a_1, \ldots, a_k \in \mathbb{N}^2$ such that $a_1 = a$, $a_k = \tilde{a}$ and the difference $a_{i+1} - a_i$ is one of the vectors α, $-\alpha$, β, $-\beta$. We say that a path is *increasing* if the differences are always one of α, β, and that the path is *decreasing* if the differences are always one of $-\alpha$, $-\beta$. We say that a point in $\mathbb{N}^2$ is connected with infinity if it can be connected with another point in $\mathbb{N}^2$ which is arbitrarily far removed from the origin.

Since the equations defining $H_{\mathcal{B}[i,j]}(c[i,j])$ can be transformed into recurrence relations for the coefficients of a polynomial solution to this system, it follows that two points can be connected by a path if and only if the monomials whose exponents are these points appear simultaneously in a polynomial solution of $H_{\mathcal{B}[i,j]}(c[i,j])$ that has irreducible support. Note that if a point in $\mathbb{N}^2$ is connected with infinity, then the corresponding monomial cannot be present in any polynomial solution of $H_{\mathcal{B}[i,j]}(c[i,j])$.

Our next observation is that there are no nonconstant increasing paths starting at a point of the base rectangle. This can be verified by direct check of all possible relations between $|b_{i1}b_{j2}|$, $|b_{i2}||b_{j1}|$, b_{i1}, b_{i2}, b_{j1}, b_{j2}: choosing the signs of the differences $|b_{i1}b_{j2}| - |b_{i2}||b_{j1}|$, $b_{i1} - b_{i2}$, $b_{j1} - b_{j2}$, we verify this claim in each of the eight possible situations. It follows from this that no two different points in the base rectangle can be connected by a path, and that no such point is connected with infinity. Thus, any point in $\mathbb{N}^2$ is either connected with a unique point in the base rectangle, or it is connected with infinity. This shows that the number of polynomial solutions of $H_{\mathcal{B}[i,j]}(c[i,j])$ equals the number of lattice points in $\mathcal{R}$, that is, $v_{ij} = \min(|b_{i1}b_{j2}|, |b_{i2}b_{j1}|)$. $\square$

Combining Lemmas 6.2 and 6.5, we obtain a formula for the dimension of the space of Puiseux polynomial solutions of Horn $(\mathcal{B}, c)$.

Theorem 6.6. *For a generic parameter* c,

$$\operatorname{rank}_p(\operatorname{Horn}(\mathcal{B}, c)) = \sum v_{ij} ,$$

where the sum runs over pairs of rows b_i *and* b_j *of* $\mathcal{B}$ *that are linearly independent and lie in opposite open quadrants of* $\mathbb{Z}^2$.

7. Solutions of hypergeometric systems arising from lattices

In this section we consider, for $\beta = A \cdot c$, the lattice hypergeometric system $I_{\mathcal{B}} + \langle A \cdot \theta - \beta \rangle$. This D-ideal is holonomic for all $\beta \in \mathbb{C}^d$, since its fake characteristic ideal, that is, the ideal generated by the principal symbols of the generators of $I_{\mathcal{B}}$ and $\langle A \cdot \theta - \beta \rangle$, has dimension n. In order to compute the holonomic rank of these systems, we need to look at the solutions of the hypergeometric systems arising from the primary components of $I_{\mathcal{B}}$.

Let ρ be a partial character of $L/L_{\mathcal{B}}$, and let I_ρ be as in Section 4. Define $H_\rho(A \cdot c) = I_\rho + \langle A \cdot \theta - A \cdot \beta \rangle$. In particular, since ρ_0 is the trivial character, $H_{\rho_0}(A \cdot c) = H_A(A \cdot c)$.

Lemma 7.1. *For* $\rho, \rho' \in G_{\mathcal{B}}$, *the group of partial characters of* $L/L_{\mathcal{B}}$, *the* D-*modules* $H_\rho(\beta)$ *and* $H_{\rho'}(\beta)$ *are isomorphic.*

Proof. It is enough to consider the case when $\rho' = \rho_0$, so that $I_{\rho'} = I_{\rho_0} = I_A$. Given any partial character $\rho : L \to \mathbb{C}^*$, let p_ρ be any point in X_ρ all of whose coordinates are nonzero. We define the map $\tau_\rho : D \to D$ by setting

$$\tau_\rho \left(\sum x^\alpha \partial^\beta \right) = \sum p_\rho^{\alpha-\beta} x^\alpha \partial^\beta .$$

It is straightforward to check that τ_ρ defines an endomorphism of D, which is clearly an isomorphism. It is also easily checked that $\tau_\rho(I_A) = I_\rho$, and $\tau_\rho(\langle A \cdot \theta - \beta \rangle) = \langle A \cdot \theta - \beta \rangle$, so that $\tau_\rho(H_A(\beta)) = H_\rho(\beta)$ and the D-modules $D/H_A(\beta)$ and $D/H_\rho(\beta)$ are isomorphic. $\square$

Corollary 7.2. *If $\rho \in G_{\mathcal{B}}$, the D-module $D/H_\rho(A \cdot c)$ is regular holonomic for all $c \in \mathbb{C}^n$.*

Proof. Hotta has shown (see [17]) that $D/H_A(A \cdot c)$ is regular holonomic for all parameters $c \in \mathbb{C}^n$, since the condition that the sum of the rows of $\mathcal{B}$ equals zero implies that the vector $(1, 1, \ldots, 1) \in \mathbb{Z}^n$ belongs to the row-span of A. Now apply Lemma 7.1. $\square$

We have shown that the hypergeometric systems arising from the primary components of the lattice ideal $I_{\mathcal{B}}$ are regular holonomic for all parameters. This implies that the solutions of these systems belong to the Nilsson class [2, Chapter 6.4]. We will show that the solutions of the hypergeometric system $I_{\mathcal{B}} + \langle A \cdot \theta - \beta \rangle$ satisfy the same properties.

Recall that $I_{\mathcal{B}} = \cap_{\rho \in G_{\mathcal{B}}} I_\rho$, where $G_{\mathcal{B}}$ is the order g group of partial characters, with corresponding ideals I_ρ. For any $\mathcal{J} \subseteq G_{\mathcal{B}}$, we denote by $I_{\mathcal{J}}$ the intersection $\cap_{\rho \in \mathcal{J}} I_\rho$. We first need the following result.

Proposition 7.3. *Let $w \in \mathbb{N}^n \backslash \{0\}$. For generic β, the map*

$$D/(I_{\mathcal{J}} + \langle A \cdot \theta - \beta - A \cdot w \rangle) \xrightarrow{\ \cdot\, \partial^w\ } D/(I_{\mathcal{J}} + \langle A \cdot \theta - \beta \rangle)\,,$$

given by right multiplication by ∂^w, is an isomorphism of left D-modules.

Proof. It is sufficient to consider the case when $w = e_i$, so that our map is right multiplication by ∂_i. In order to use the exact argument of the proof of [25, Theorem 4.5.10] (the analogous result for A-hypergeometric systems), we need to show that there exists a nonzero parametric b-function (see [25, Section 4.4]), that is, we need to prove that the following elimination ideal in the polynomial ring $\mathbb{C}[s_1, \ldots, s_d] = \mathbb{C}[s]$:

$$\big(D[s]\ I_{\mathcal{J}} + \langle A \cdot \theta - s \rangle + D[s]\ \langle \partial_i \rangle\big) \cap \mathbb{C}[s]$$

is nonzero, where $D[s]$ is the parametric Weyl algebra. In order to do this, we first go through an intermediate step:

$$\begin{aligned}(D[s]\ I_{\mathcal{J}} + \langle A \cdot \theta - s \rangle + D[s]\ \langle \partial_i \rangle) \cap \mathbb{C}[\theta, s] \\ \supseteq \big(D[s](I_{\mathcal{B}} + \langle \partial_i \rangle) + \langle A \cdot \theta - s \rangle\big) \cap \mathbb{C}[\theta, s]\end{aligned}$$

$$
\begin{aligned}
&= \big(D[s](\mathrm{in}_{-e_i}(I_{\mathcal{B}} + \langle \partial_i \rangle)) + \langle A \cdot \theta - s \rangle\big) \cap \mathbb{C}[\theta, s] \\
&= \mathrm{in}_{(-e_i, e_i, 0)}\big(I_{\mathcal{B}} + \langle A \cdot \theta - s \rangle\big) \cap \mathbb{C}[\theta, s] \\
&\supseteq \langle [\theta]_u : \partial^u \in \mathrm{in}_{-e_i}(I_{\mathcal{B}}) \rangle + \langle \theta_i \rangle + \langle A \cdot \theta - s \rangle \\
&\supseteq \langle [\theta]_{g\ u} : \partial^u \in \mathrm{in}_{-e_i}(I_A) \rangle + \langle \theta_i \rangle + \langle A \cdot \theta - s \rangle.
\end{aligned}
$$

Here $[\theta]_u = \prod_{k=1}^n \prod_{l=0}^{u_k-1}(\theta_k - l)$. The first containment holds because $I_{\mathcal{B}} \subseteq I_{\mathcal{J}}$. The next equality is true since

$$
I_{\mathcal{B}} + \langle \partial_i \rangle = \mathrm{in}_{-e_i}(I_{\mathcal{B}}) + \langle \partial_i \rangle.
$$

The equality in the third line holds by the proof of [25, Theorem 3.1.3], which applies here since $I_{\mathcal{B}}$ is homogeneous with respect to the multi-grading given by the columns of A. The next inclusion is easy to check, given that, for a monomial ∂^u, $x^u \partial^u = [\theta]_u$. The last containment follows from the fact that $g\ u \in L_B$ for all $u \in \ker_{\mathbb{Z}}(A)$. Now if we prove that

$$
\big(\langle [\theta]_{g\ u} : \partial^u \in \mathrm{in}_{-e_i}(I_A) \rangle + \langle \theta_i \rangle + \langle A \cdot \theta - s \rangle\big) \cap \mathbb{C}[s]
$$

is nonzero, we will be done. But this is a commutative elimination, so all we need to do is show that the projection of the zero set of $\langle [\theta]_{g\ u} : \partial^u \in \mathrm{in}_{-e_i}(I_A) \rangle + \langle \theta_i \rangle + \langle A \cdot \theta - s \rangle$ onto the s-variables is not surjective.

Observe that the projection of $\langle [\theta]_u : \partial^u \in \mathrm{in}_{-e_i}(I_A) \rangle + \langle \theta_i \rangle + \langle A \cdot \theta - s \rangle$ onto the s-variables is not surjective (by [25, Corollary 4.5.9]). This projection is clearly the union of affine spaces of different dimensions. But then the projection that we want is not surjective, since it is obtained from this one by adding translates of some of the affine spaces appearing in it. This concludes the proof. $\square$

Theorem 7.4. *For generic β, any solution f of $I_{\mathcal{J}} + \langle A \cdot \theta - \beta \rangle$ can be written as a linear combination*

$$
f = \sum_{\rho \in \mathcal{J}} f_\rho,
$$

where f_ρ is a solution of $I_\rho + \langle A \cdot \theta - \beta \rangle$. In particular, the solutions of $I_{\mathcal{B}} + \langle A \cdot \theta - \beta \rangle$ are linear combinations of the solutions of the systems $I_\rho + \langle A \cdot \theta - \beta \rangle$, for $\rho \in G_{\mathcal{B}}$.

Proof. We proceed by induction on the cardinality of $\mathcal{J}$, the base case being trivial. Assume that our conclusion is valid for subsets of $G_{\mathcal{B}}$ of cardinality $r - 1 \geqslant 1$, pick $\mathcal{J} \subseteq G_{\mathcal{B}}$ of cardinality r and fix $\rho \in \mathcal{J}$.

Let P be an element of $I_{\mathcal{J}\setminus\{\rho\}}$ such that $P \notin I_\rho$. Since all of the ideals I_τ, $\tau \in G_{\mathcal{B}}$, are homogeneous with respect to the multi-grading given by A, we may assume that P is homogeneous, and write

$$P = \lambda_1 \partial^{u^{(1)}} + \cdots + \lambda_{p-1}\partial^{u^{(p-1)}} + \partial^w,$$

where $\lambda_1, \ldots, \lambda_{p-1} \in \mathbb{C}$ and $A \cdot u^{(1)} = A \cdot u^{(2)} \cdots = A \cdot u^{(p-1)} = A \cdot w$. Note that the polynomial

$$\bar{P} = \lambda_1 \partial^{u^{(1)}} + \cdots + \lambda_{p-1}\partial^{u^{(p-1)}} - [\lambda_1 \rho(u^{(1)} - w) + \cdots + \lambda_{p-1}\rho(u^{(p-1)} - w)]\partial^w$$

is an element of the ideal I_ρ, since this ideal is generated by all binomials of the form $\partial^\alpha - \rho(\alpha - \gamma)\partial^\gamma$, where $A \cdot \alpha = A \cdot \gamma$. To simplify the notation, set $-\lambda$ to be the coefficient of ∂^w in $\bar{P}$, that is,

$$\lambda = \lambda_1 \rho(u^{(1)} - w) + \cdots + \lambda_{p-1}\rho(u^{(p-1)} - w).$$

Now let f be a solution of $I_{\mathcal{J}} + \langle A \cdot \theta - \beta \rangle$, and consider the function $\bar{P} f$. For any $Q \in I_{\mathcal{J}\setminus\{\rho\}}$, we have $Q\bar{P} \in I_{\mathcal{J}}$. This implies that $Q\bar{P}f = 0$. Furthermore, noting that $\bar{P}$ is A-homogeneous of multi-degree $A \cdot w$, we conclude that $\bar{P}f$ is a solution of $I_{\mathcal{J}\setminus\{\rho\}} + \langle A \cdot \theta - \beta - A \cdot w\rangle$. Since β is generic, so is $\beta + A \cdot w$, and by the inductive hypothesis we can write $\bar{P} f = \sum_{\tau \in \mathcal{J}\setminus\{\rho\}} g_\tau$, where each g_τ is a solution of $I_\tau + \langle A \cdot \theta - \beta - A \cdot w \rangle$.

By Proposition 7.3, ∂^w induces an isomorphism between the solution spaces of $I_\tau + \langle A \cdot \theta - \beta\rangle$ and $I_\tau + \langle A \cdot \theta - \beta - A \cdot w\rangle$, so that we can find a solution $\tilde{g}_\tau$ of $I_\tau + \langle A \cdot \theta - \beta \rangle$ such that $\partial^w \tilde{g}_\tau = g_\tau$. Now

$$\begin{aligned} \bar{P}\tilde{g}_\tau &= \sum_{i=1}^{p-1} \lambda_i \partial^{u(i)} \tilde{g}_\tau - \lambda \partial^w \tilde{g}_\tau \\ &= \left(\sum_{i=1}^{p-1} \lambda_i \tau(u^{(i)} - w) - \lambda \right) g_\tau. \end{aligned}$$

The last equality holds because $\tilde{g}_\tau$ is a solution of I_τ, and therefore $\partial^{u^{(i)}} - \tau(u^{(i)} - w)\partial^w$ annihilates it, yielding $\partial^{u^{(i)}}\tilde{g}_\tau = \tau(u(i) - w)\partial^w \tilde{g}_\tau = \tau(u^{(i)} - w) g_\tau$.

Note that the coefficient $\sum_{i=1}^{p-1} \lambda_i \tau(u^{(i)} - w) - \lambda$ is nonzero, for otherwise we could rewrite $\bar{P}$ using the sum instead of λ, and conclude that $\bar{P} \in I_\tau$. But we know $P \in I_\tau$, so $\bar{P} - P \in I_\tau$, a contradiction since this is a nonzero multiple of ∂^w, and the ideal I_τ contains no monomials. (The fact that $\bar{P} - P \neq 0$ follows from $\bar{P} \in I_\rho$ and $P \notin I_\rho$.)

Finally define $f_\tau = \left(\sum_{i=1}^{p-1} \lambda_i \tau(u^{(i)} - w) - \lambda\right)^{-1} \tilde{g}_\tau$, so that f_τ is a solution of $I_\tau + \langle A \cdot \theta - \beta \rangle$ and

$$\bar{P} \sum_{\tau \in \mathcal{J} \setminus \{\rho\}} f_\tau = \sum_{\tau \in \mathcal{J} \setminus \{\rho\}} g_\tau = \bar{P} f.$$

If $h = f - \sum_{\tau \in \mathcal{J} \setminus \{\rho\}} f_\tau$, then h is a solution of $I_{\mathcal{J}} + \langle A \cdot \theta - \beta \rangle$ that satisfies $\bar{P}h = 0$. Now consider Ph. Since $P \in I_{\mathcal{J} \setminus \{\rho\}}$, Ph is a solution of $I_\rho + \langle A \cdot \theta - \beta - A \cdot w \rangle$, and a similar argument as before yields a solution f_ρ of $I_\rho + \langle A \cdot \theta - \beta \rangle$ such that $Ph = Pf_\rho$. Let $\tilde{h} = h - f_\rho$, so that $f = \sum f_\tau + f_\rho + \tilde{h}$ and $P\tilde{h} = 0$. But $\bar{P}\tilde{h} = \bar{P}h - \bar{P} f_\rho = 0$ since $\bar{P} \in I_\rho$.

Now $P\tilde{h} = \bar{P}\tilde{h} = 0$ implies $(P - \bar{P})\tilde{h} = 0$, so that $\partial^w \tilde{h} = 0$, because $P - \bar{P}$ is a nonzero multiple of ∂^w. But then $\tilde{h}$ is a solution of $I_{\mathcal{J}} + \langle A \cdot \theta - \beta \rangle$ that is mapped under ∂^w to the zero element in the solution space of $I_{\mathcal{J}} + \langle A \cdot \theta - \beta - A \cdot w \rangle$, which, using the genericity of β and Proposition 7.3, implies that $\tilde{h} = 0$. Thus we have obtained an expression for f as a linear combination of solutions of the systems $I_\tau + \langle A \cdot \theta - \beta \rangle$, $\tau \in \mathcal{J}$, and the proof of the inductive step is finished. $\square$

Considering $\mathcal{J} = G_{\mathcal{B}}$, we deduce that all solutions of $D/(I_{\mathcal{B}} + \langle A \cdot \theta - \beta \rangle)$ split as a sum of solutions for each I_ρ, yielding a kind of converse to Theorem 5.3. We remark that this result is not true without the genericity assumption on β, since for certain parameters (for instance for $\beta = 0$, where the constant function 1 is a solution), the solutions to the different ideals $H_\rho(\beta)$ are not linearly independent.

Corollary 7.5. *Suppose that $\mathcal{B}$ has zero column sums, and $\beta \in \mathbb{C}^d$ is generic. Then*

$$\operatorname{rank}\,(I_{\mathcal{B}} + \langle A \cdot \theta - \beta \rangle) \leqslant g \cdot \operatorname{vol}(A).$$

Proof. Under these hypotheses, the solutions of $I_{\mathcal{B}}$ are linear combinations of solutions of the g systems $I_\rho + \langle A \cdot \theta - \beta \rangle$, by the previous theorem. Each of these systems has rank $\operatorname{vol}(A)$. $\square$

8. Holonomicity and solutions of the Horn system $H_{\mathcal{B}}(c)$

In this section we assume that $m = 2$. Our goal is to investigate both the holonomicity of $H_{\mathcal{B}}(c)$ and to find out the form of its solutions. First let us show that $H_{\mathcal{B}}(c)$ is holonomic for generic c.

Theorem 8.1. *Let $m = 2$ and c generic parameter vector. Then $H_{\mathcal{B}}(c)$ is holonomic.*

Proof. Write $I = \langle \partial^{u_+} - \partial^{u_-}, \partial^{v_+} - \partial^{v_-} \rangle$, where u and v are the columns of $\mathcal{B}$. Consider first the case when $\mathcal{B}$ has no linearly dependent rows in opposite open quadrants of

$\mathbb{Z}^2$. Then the ring

$$\frac{\mathbb{C}[x_1, \ldots, x_n, z_1, \ldots, z_n]}{\langle z^{u_+} - z^{u_-}, z^{v_+} - z^{v_-} \rangle + \langle \sum_{j=1}^n a_{ij} x_j z_j : i = 1, \ldots, n-m \rangle}$$

has dimension n (see Lemma 12.1). Since the polynomial ring modulo the characteristic ideal of $H_{\mathcal{B}}(c)$ is a subring of this one, we conclude that $H_{\mathcal{B}}(c)$ is holonomic *for all* $c \in \mathbb{C}^m$.

Now assume that $\mathcal{B}$ has linearly dependent rows b_i, b_j in opposite open quadrants of $\mathbb{Z}^2$. In this case, the ideal $\langle z^{u_+} - z^{u_-}, z^{v_+} - z^{v_-} \rangle + \langle \sum a_{ij} x_j z_j : j = 1, \ldots, n-m \rangle$ will have a lower-dimensional component corresponding to the vanishing of z_i and z_j, by the results in Section 4 about primary decomposition of codimension 2 lattice basis ideals.

To ensure holonomicity of $H_{\mathcal{B}}(c)$, we will construct, for each pair b_i, b_j of linearly dependent rows of $\mathcal{B}$ in opposite open quadrants of $\mathbb{Z}^2$, an element of the ideal $H_{\mathcal{B}}(c)$ that contains no x_i, x_j, ∂_i, ∂_j, and that, for generic c, is nonzero. The principal symbol of this element will therefore not depend on z_i or z_j.

To simplify the notation, assume b_1 and b_2 are linearly dependent in opposite open quadrants of $\mathbb{Z}^2$. Then the complementary square submatrix of A has determinant zero, so that, by performing row and column operations, we can find $p, q \in \mathbb{Q}$, $r \in \mathbb{C}$, such that $p\,\theta_1 + q\,\theta_2 - r$ lies in $H_{\mathcal{B}}(c)$. The numbers p and q are rational combinations of some of the elements a_{ij} of the matrix A, the number r is a linear combination of the coordinates of the vector c.

Also, since b_1 and b_2 are linearly dependent, we can find a nonzero element $w \in L_{\mathcal{B}}$ such that $w_1 = w_2 = 0$. Then we can find two monomials m_1, m_2 in $\mathbb{C}[\partial]$ with disjoint supports, that are not divisible by either ∂_1 or ∂_2 such that $\partial_1^k m_1(\partial^{w_+} - \partial^{w_-}) \in I$ for some $k > 0$ and $\partial_2^l m_2(\partial^{w_+} - \partial^{w_-}) \in I$ for some $l > 0$. This follows from the arguments that proved Proposition 4.4. Call $\mu = m_1(\partial^{w_+} - \partial^{w_-})$ and $\lambda = m_2(\partial^{w_+} - \partial^{w_-})$. Note that μ, λ do not depend on ∂_1, ∂_2. Then, using $x_1^k \partial_1^k = \theta_1(\theta_1 - 1) \cdots (\theta_1 - k + 1) = [\theta_1]_k$ we see that $[\theta_1]_k \mu \in H_{\mathcal{B}}(c)$. Similarly, $[\theta_2]_l \lambda \in H_{\mathcal{B}}(c)$.

Consider the left ideal in the Weyl algebra generated by

$$p\,\theta_1 + q\,\theta_2 - r,\, [\theta_1]_k \mu,\, [\theta_2]_l \lambda.$$

This ideal is contained in $H_{\mathcal{B}}(c)$. Now note that θ_1, θ_2, λ and μ are pairwise commuting elements of D_n. This means that we can think of $\langle p\,\theta_1 + q\,\theta_2 - r, [\theta_1]_k \mu, [\theta_2]_l \lambda \rangle$ as an ideal in $\mathbb{C}[\theta_1, \theta_2, \partial_3, \ldots, \partial_n]$, which is a commutative subring of D_n. We will go one step further and think of r also as an indeterminate, which commutes with θ_1, θ_2, $\partial_3, \ldots, \partial_n$.

Finding the element of $H_{\mathcal{B}}(c)$ that we want has now been reduced to eliminating θ_1 and θ_2 from

$$\langle p\,\theta_1 + q\,\theta_2 - r, [\theta_1]_k \mu, [\theta_2]_l \lambda \rangle \subset \mathbb{C}[\theta_1, \theta_2, \partial_3, \ldots, \partial_n, r]. \tag{15}$$

Since the geometric counterpart of elimination is projection, in order to check that the elimination ideal

$$\langle p\,\theta_1 + q\,\theta_2 - r, [\theta_1]_k\mu, [\theta_2]_l\lambda\rangle \cap \mathbb{C}[\lambda, \mu, r]$$

is nonzero, we need to show that there exist complex numbers $\partial_3, \ldots, \partial_n$ and r such that, for all values of $\theta_1, \theta_2 \in \mathbb{C}$, the tuple $(\theta_1, \theta_2, \partial_3, \ldots, \partial_n, r)$ is not a solution of (15). If $(\partial_3, \ldots, \partial_n)$ is generic, the polynomials μ and λ evaluated at that point will be nonzero. Thus, in order for $[\theta_1]_k\mu$ to vanish, θ_1 must be an integer between 0 and k. Analogously, θ_2 must be an integer between 0 and l. But then, for most values of r, $p\,\theta_1 + q\,\theta_2 - r$ is nonzero. Thus, the projection of the zero set of (15) onto the $\partial_3, \ldots, \partial_n, r$ coordinates is not surjective. This implies that (15) contains an element P that does not depend on θ_1 or θ_2. Note that P does depend (polynomially) on r, which is itself a linear combination of the coordinates of c. Thus, for generic c, P will be nonzero. Now P is also an element of the ideal $H_{\mathcal{B}}(c)$, that does not depend on x_1, x_2, ∂_1, ∂_2, and is nonzero for generic c. □

Example 8.2. Consider the matrix

$$\mathcal{B} = \begin{pmatrix} 1 & 2 \\ -2 & -4 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

To prove that $H_{\mathcal{B}}(c)$ is holonomic for generic c, we need to find an element of $H_{\mathcal{B}}(c)$ whose principal symbol does not vanish if we set $z_1 = z_2 = 0$. To find this element, we follow the procedure outlined in the proof of the previous theorem. The first thing we need is an element of $L_{\mathcal{B}}$ with its first two coordinates equal to zero. The vector $(0, 0, -1, 1)$ works. It is easy to check that $\partial_1^2\partial_3^2(\partial_3 - \partial_4)$ and $\partial_2^4(\partial_3 - \partial_4)$ are both elements of the lattice basis ideal I. We can also assume that $(2, 1, 0, 0)$ is a row of the matrix A. Now what remains is to eliminate θ_1 and θ_2 from the $\mathbb{C}[\theta_1, \theta_2, \partial_3, \partial_4, r]$-ideal:

$$\langle \theta_1(\theta_1 - 1)\partial_3^2(\partial_3 - \partial_4), \theta_2(\theta_2 - 1)(\theta_2 - 2)(\theta_2 - 3)(\partial_3 - \partial_4), 2\theta_1 + \theta_2 - r\rangle,$$

where $r = 2c_1 + c_2$. We perform the elimination on a computer algebra system to obtain the following element of $H_{\mathcal{B}}(c)$:

$$\left(\prod_{i=0}^{5}(2c_1 + c_2 - i)\right)\partial_3^2(\partial_3 - \partial_4),$$

whose principal symbol

$$\left(\prod_{i=0}^{5}(2c_1 + c_2 - i)\right) z_3^2(z_3 - z_4)$$

does not vanish along $z_1 = z_2 = 0$ for generic c.

Our goal now is to characterize all the solutions of the Horn system $H_{\mathcal{B}}(c)$ for generic c. The first step is the following result.

Lemma 8.3. *Let α be as in Proposition 4.4. For generic c, the sequence*

$$0 \to \frac{D}{(I_{\mathcal{B}}+\langle A\cdot\theta - A\cdot(c+\alpha)\rangle)} \xrightarrow{\cdot\,\partial^{\alpha}} \frac{D}{H_{\mathcal{B}}(c)} \xrightarrow{\pi} \frac{D}{(I+\langle\partial^{\alpha}\rangle+\langle A\cdot\theta - A\cdot c\rangle)} \to 0, \quad (16)$$

where π is the natural projection, is exact.

Proof. The only part of exactness that is not clear is that right multiplication by ∂^{α} is injective (it is well defined since $\partial^{\alpha} I_{\mathcal{B}} \subseteq I$). To see this, consider the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & D/(I_{\mathcal{B}} + \langle A\cdot\theta - A\cdot(c+\alpha)\rangle) \arrow[r, "\cdot\,\partial^{\alpha}"] \arrow[dr, "\cdot\,\partial^{\alpha}"'] & D/(I_{\mathcal{B}} + \langle A\cdot\theta - A\cdot c\rangle) \arrow[r] \arrow[d] & 0 \\ & & D/H_{\mathcal{B}}(c) & \end{tikzcd} .$$

where the vertical arrow is the natural inclusion. The upper row of the diagram is exact by Theorem 7.3, since c is generic. But then the commutativity implies that the diagonal arrow is injective. $\square$

Lemma 8.4. *Let $u, v \in \mathbb{N}^n$ such that $\langle\partial^u, \partial^v\rangle$ is a complete intersection. If c is generic, then*

$$\langle\partial^u, \partial^v\rangle + \langle A\cdot\theta - A\cdot c\rangle$$

is a holonomic system of differential equations, whose solution space has a basis of Puiseux monomials.

Proof. It is enough to show that the system

$$\langle x^u \partial^u, x^v \partial^v \rangle + \langle A \cdot \theta - A \cdot c \rangle$$

satisfies the desired properties since x^u and x^v are units in $\mathbb{C}(x)$.

Now

$$\langle x^u \partial^u, x^v \partial^v \rangle + \langle A \cdot \theta - A \cdot c \rangle = \langle [\theta]_u, [\theta]_v \rangle + \langle A \cdot \theta - A \cdot c \rangle = D \cdot F,$$

where

$$[\theta]_u = \prod_{k=1}^{n} \prod_{l=0}^{u_k-1} (\theta_k - l),$$

and

$$F = \langle [\theta]_u, [\theta]_v \rangle + \langle A \cdot \theta - A \cdot c \rangle \subseteq \mathbb{C}[\theta].$$

This means that $D \cdot F$ is a Frobenius ideal (see [25, Section 2.3]). By [25, Proposition 2.3.6, Theorem 2.3.11], if we can show that F is artinian and radical, it will follow that $D \cdot F$ is holonomic, with solution space spanned by $\{x^p : p \in \mathcal{V}(F)\}$, where $\mathcal{V}(F)$ is the zero set of the ideal $F \subseteq \mathbb{C}[\theta]$, and we will be done.

To show that F is artinian and radical, we proceed as in [25, Theorem 3.2.10]. Let $p \in \mathcal{V}(F)$. Then there exist $1 \leqslant i < j \leqslant n$ such that p_i and p_j are nonnegative integers between zero and $\max\{u_i, v_i\}$, $\max\{u_j, v_j\}$, respectively. This follows from $[\theta]_u(p) = [\theta]_v(p) = 0$ and the fact that u and v have disjoint supports, because $\langle \partial^u, \partial^v \rangle$ is a complete intersection. Since c is generic, the minor of A complementary to $\{i, j\}$ must be nonzero (otherwise the equations $\theta_i = p_i$, $\theta_j = p_j$ and $A \cdot \theta = A \cdot c$ would be incompatible). Hence its ith and jth coordinates determine p uniquely in $\mathcal{V}(F)$. $\square$

Remark 8.5. If all maximal minors of A are nonzero, the above lemma holds without restriction on c.

Theorem 8.6. *Write $I = \langle \partial^{u+} - \partial^{u-}, \partial^{v+} - \partial^{v-} \rangle$, where u and v are the columns of $\mathcal{B}$. Let ∂^α be a monomial satisfying*:

$$\alpha_i > 0 \Longrightarrow u_i > 0. \tag{17}$$

Then, for generic c, the D-ideal $I + \langle \partial^\alpha \rangle + \langle A \cdot \theta - A \cdot c \rangle$ has only Puiseux polynomial solutions.

Proof. We proceed by induction on $|\alpha| = \alpha_1 + \cdots + \alpha_n$, the length of α. If $|\alpha| \leqslant \min\{u_i : u_i > 0\}$, in particular, if $|\alpha| = 1$ (recall that $|u| = 0$), then ∂^α divides ∂^{u_+}, so that all solutions of $I + \langle \partial^\alpha \rangle + \langle A \cdot \theta - A \cdot c \rangle$ are solutions of $\langle \partial^\alpha, \partial^{u_-} \rangle + \langle A \cdot \theta - A \cdot c \rangle$. But the latter ideal has only Puiseux polynomial solutions by Lemma 8.4, since c is generic.

Assume now that our result is true for length s and let α be of length $s + 1$ satisfying (17). Choose i such that $\alpha_i > 0$ (and so $u_i > 0$), and let φ be a solution of $I + \langle \partial^\alpha \rangle + \langle A \cdot \theta - A \cdot c \rangle$. The function $\partial_i \varphi$ is a solution of $I + \langle \partial^{\alpha - e_i} \rangle + \langle A \cdot \theta - A \cdot c - A \cdot e_i \rangle$. But $|\alpha - e_i| = s$ and $c + e_i$ is still generic, so the inductive hypothesis implies that $\partial_i \varphi$ is a Puiseux polynomial. Write

$$\partial_i \varphi = \sum_{l=0}^{N_0} g_l^{(0)} x_i^l + \sum_{l=0}^{N_1} g_l^{(1)} x_i^{\mu_1 + l} + \cdots + \sum_{l=0}^{N_t} g_l^{(t)} x_i^{\mu_t + l},$$

where the $g_l^{(k)}$ are Puiseux polynomials, constant with respect to x_i, t is a natural number, and $\mu_1, \ldots, \mu_t \in \mathbb{C}$ are nonintegers with noninteger pairwise differences. Then

$$\begin{aligned} \varphi = {} & \sum_{l=0}^{N_0} g_l^{(0)} \frac{x_i^{l+1}}{l+1} + \sum_{l=0}^{N_1} g_l^{(1)} \frac{x_i^{\mu_1 + l + 1}}{\mu_1 + l + 1} + \cdots + \sum_{l=0}^{N_t} g_l^{(t)} \frac{x_i^{\mu_t + l + 1}}{\mu_t + l + 1} \\ & + G(x_1, \ldots, \hat{x}_i, \ldots, x_n). \end{aligned} \tag{18}$$

If we prove that G is a Puiseux polynomial, it will follow that so is φ, and the proof will be finished. We know that φ is a solution of $\langle A \cdot \theta - A \cdot c \rangle$. By construction, so is $\varphi - G$. Then G is a solution of $\langle A \cdot \theta - A \cdot c \rangle$. Recall that $\partial_i G = 0$.

We also know that $\partial^{u_+} \varphi = \partial^{u_-} \varphi$. We want to compare the coefficients of the integer powers of x_i in the expressions we obtain by applying ∂^{u_+} and ∂^{u_-} to (18). Since we are only looking at the integer powers of x_i, we need only look at $\sum_{l=0}^{N_0} g_l^{(0)} (x_i^{l+1}/(l+1)) + G$.

$$\partial^{u_+} \left(\sum_{l=0}^{N_0} g_l^{(0)} \frac{x_i^{l+1}}{l+1} + G \right) = \sum_{l=0}^{N_0} l(l-1) \cdots (l + 2 - u_i) (\partial^{u_+ - u_i e_i} g_l^{(0)}) x_i^{l+1-u_i}. \tag{19}$$

Note that there is no G in the above expression, since $\partial_i G = 0$ and $u_i > 0$. Also, the highest power of x_i appearing in (19) is $x_i^{N_0 + 1 - u_i}$.

$$\partial^{u_-} \left(\sum_{l=0}^{N_0} g_l^{(0)} \frac{x_i^{l+1}}{l+1} + G \right) = \sum_{l=0}^{N_0} (\partial^{u_-} g_l^{(0)}) \frac{x_i^{l+1}}{l+1} + \partial^{u_-} G. \tag{20}$$

We equate the coefficients of x_i^{l+1} in (19) and (20) to obtain

$$\frac{\partial^{u_-} g_l^{(0)}}{l+1} = (l+u_i)\cdots(l+2)\partial^{u_+ - u_i e_i} g_{l+u_i}^{(0)} \quad \text{for } l = 0, \ldots, N_0 - u_i. \tag{21}$$

If $l = N_0 + 1 - u_i, \ldots, N_0$, then $\partial^{u_-} g_l^{(0)} = 0$. Also,

$$\partial^{u_-} G = (u_i - 1)(u_i - 2)\cdots 2 \cdot 1 \cdot \partial^{u_+ - u_i e_i} g_{u_i - 1}^{(0)}.$$

Applying ∂^{u_-} to (21), we see that, for $l = N_0 + 1 - 2u_i, \ldots, N_0 - u_i$:

$$\partial^{2u_-} g_l^{(0)} = (l+u_1)\cdots(l+2)(l+1)\partial^{u_+ - u_i e_i}\partial^{u_-} g_{l+u_i}^{(0)} = 0.$$

Applying ∂^{u_-} enough times, we conclude that, if $ku_i > N_0 + 1$, then $\partial^{ku_-} G = 0$. But now, G is a solution of $\langle \partial_i, \partial^{ku_-}\rangle + \langle A\cdot\theta - A\cdot c\rangle$, and c is generic. By Lemma 8.4, G is a Puiseux polynomial. $\square$

Proposition 8.7. *Let α be as in Proposition 4.4 (in particular, α satisfies (17)), let c be generic, and let f be a solution of $H_{\mathcal{B}}(c)$. Then $f = g + h$, where g is a solution of the lattice hypergeometric system $I_{\mathcal{B}} + \langle A\cdot\theta - A\cdot c\rangle$ and h is a solution of $I + \langle \partial^{\alpha}\rangle + \langle A\cdot\theta - A\cdot c\rangle$.*

Proof. Let $\psi = \partial^{\alpha} f$. Then ψ is a solution of $I_B + \langle A\cdot\theta - A\cdot(c+\alpha)\rangle$. This is because the D-module map

$$\frac{D}{I_{\mathcal{B}} + \langle A\cdot(c+\alpha)\rangle} \xrightarrow{\cdot\,\partial^{\alpha}} \frac{D}{H_{\mathcal{B}}(c)}$$

induces a vector space map between the solution spaces of $H_{\mathcal{B}}(c)$ and $I_{\mathcal{B}} + \langle A\cdot\theta - A\cdot(c+\alpha)\rangle$.

Now by Lemma 7.1, right multiplication by ∂^{α} is an D-module isomorphism between $D/(I_{\mathcal{B}} + \langle A\cdot\theta - A\cdot(c+\alpha)\rangle)$ and $D/(I_{\mathcal{B}} + \langle A\cdot\theta - A\cdot c\rangle)$, so there exists $Q \in D$ and $P \in I_{\mathcal{B}} + \langle A\cdot\theta - A\cdot(c+\alpha)\rangle$ such that $\partial^{\alpha} Q = 1 + P$. Let $g = Q\psi$. Then g is a solution of $I_{\mathcal{B}} + \langle A\cdot\theta - A\cdot c\rangle$, and

$$\partial^{\alpha} g = \partial^{\alpha} Q\psi = (1+P)\psi = \psi = \partial^{\alpha} f, \tag{22}$$

where the next to last equality holds because $P \in I_{\mathcal{B}} + \langle A\cdot\theta - A\cdot(c+\alpha)\rangle$. Now let $h = f - g$. All we need to finish this proof is to show that h is a solution of

$I + \langle \partial^{\alpha} \rangle + \langle A \cdot \theta - A \cdot c \rangle$. But, since $I \subset I_{\mathcal{B}}$, g is also a solution of $H_{\mathcal{B}}(c)$, and thus so is h. Moreover $\partial^{\alpha} h = 0$ by (22). $\square$

Corollary 8.8. *For generic c, we have*

$$\operatorname{rank}(H_{\mathcal{B}}(c)) \leqslant g \cdot \operatorname{vol}(A) + \sum v_{ij},$$

where the sum runs over pairs of linearly independent rows of $\mathcal{B}$ in opposite open quadrants of $\mathbb{Z}^2$.

Proof. By Proposition 8.7, the solution space of $H_{\mathcal{B}}(c)$ is contained in the sum of the solution spaces of $I_{\mathcal{B}} + \langle A \cdot \theta - A \cdot c \rangle$ and $I + \langle \partial^{\alpha} \rangle + \langle A \cdot \theta - A \cdot c \rangle$. The first solution space has rank at most $g \cdot \operatorname{vol}(A)$ by Corollary 7.5. The second solution space contains only Puiseux polynomials and therefore has rank at most $\operatorname{rank}_p(H_{\mathcal{B}}) = \sum v_{ij}$ by Theorem 6.6. $\square$

9. Initial ideals, indicial ideals and holonomic ranks

In this section we finish the proofs of our rank formulas for generic parameters, by showing the reverse inequalities in Corollaries 7.5 and 8.8. We will assume $m = 2$ when dealing with Horn systems, although the arguments will work for general m as long as I is a complete intersection and $H_{\mathcal{B}}(c)$ is holonomic for generic c.

Our main tool will be the fact that holonomic rank is lower semicontinuous when we pass to initial ideals with respect to weight vectors of the form $(-w, w)$; this is [25, Theorem 2.2.1]. For an introduction to initial ideals in the Weyl algebra, including algorithms, see [25, Chapters 1 and 2].

Theorem 9.1 (*Saito et al.* [25, Theorem 2.2.1]). *If J is a holonomic D_n-ideal, and w is a generic weight vector, then the initial D_n-ideal* $\operatorname{in}_{(-w,w)}(J)$ *is also holonomic, and*

$$\operatorname{rank}(\operatorname{in}_{(-w,w)}(J)) \leqslant \operatorname{rank}(J).$$

Remark 9.2. If we assume that J is *regular holonomic*, then equality will hold in the above theorem.

Our goal is now to compute the holonomic ranks of $\operatorname{in}_{(-w,w)}(H_{\mathcal{B}}(c))$ and $\operatorname{in}_{(-w,w)}(I_{\mathcal{B}}) + \langle A \cdot \theta - A \cdot c \rangle$ for generic c. In order to do this, we introduce indicial ideals, which are modifications of initial ideals, and have the advantage of belonging to the (commutative) polynomial ring $\mathbb{C}[\theta]$.

Definition 9.3. If J is a holonomic left D_n-ideal, and w is a generic weight vector, the *indicial ideal* of J is

$$\operatorname{ind}_w(J) = R \cdot \operatorname{in}_{(-w,w)}(J) \cap \mathbb{C}[\theta_1, \ldots, \theta_n],$$

where R is the ring of linear partial differential equations with rational function coefficients.

A D_n-ideal whose generators belong to $\mathbb{C}[\theta] = \mathbb{C}[\theta_1, \ldots, \theta_n]$ is called a *Frobenius ideal*. The commutative ideal in $\mathbb{C}[\theta]$ given by the generators of a Frobenius ideal is called the *underlying commutative ideal*. The following theorem justifies our interest in indicial ideals.

Theorem 9.4 (*Saito et al. [25, Theorem 2.3.9]*). *Let J be a holonomic D_n-ideal and w a generic weight vector. Then $D_n \cdot \operatorname{ind}_w(J)$ is a holonomic Frobenius ideal whose rank equals* $\operatorname{rank}(\operatorname{in}_{(-w,w)}(J))$.

Finally, computing the rank of a holonomic Frobenius ideal (such as $\operatorname{ind}_w(J)$ for holonomic J) is a commutative operation.

Proposition 9.5 (*Saito et al. [25, Proposition 2.3.6]*). *Let $D_n F$ be a Frobenius ideal, where $F \subset \mathbb{C}[\theta]$ is the underlying commutative ideal. Then $D_n F$ is holonomic if and only if F is zero dimensional, in which case*

$$\operatorname{rank}(D_n F) = \deg(F).$$

Although indicial ideals are extremely useful, they are hard to get a hold of in general. However, for generic parameters, we know explicitly what the indicial ideal of an A-hypergeometric system is [25, Corollary 3.1.6], and the same ideas work for the case of Horn systems and hypergeometric systems arising from lattices.

Theorem 9.6. *For generic parameters c, we have*

$$\operatorname{ind}_w(H_{\mathcal{B}}(c)) = \big((R \cdot \operatorname{in}_w(I)) \cap \mathbb{C}[\theta]\big) + \langle A \cdot \theta - A \cdot c \rangle,$$

and

$$\operatorname{ind}_w(I_{\mathcal{B}} + \langle A \cdot \theta - A \cdot c \rangle) = \big((R \cdot \operatorname{in}_w(I_{\mathcal{B}})) \cap \mathbb{C}[\theta]\big) + \langle A \cdot \theta - A \cdot c \rangle.$$

Proof. The proof of the analogous fact for A-hypergeometric systems follows from [25, Theorem 3.1.3 and Proposition 3.1.5]. But [25, Proposition 3.1.5] carries over to the cases that interest us without any modification in its proof. Moreover the proof of [25, Theorem 3.1.3] only uses the fact that I_A is homogeneous with respect to the multi-grading given by the columns of A, a property that both I and $I_{\mathcal{B}}$ satisfy. $\square$

Our next goal is to compute the primary decomposition of the indicial ideals of $H_{\mathcal{B}}(c)$ and $I_{\mathcal{B}}+\langle A\cdot\theta - A\cdot c\rangle$ when c is generic. The first step is to recall the definition of certain combinatorial objects that correspond to the irreducible components of a monomial ideal in a polynomial ring.

Definition 9.7. Let M be a monomial ideal in $\mathbb{C}[\partial_1,\ldots,\partial_n]$. A *standard pair* of M is a pair (∂^η,σ), where σ is a possibly empty subset of $\{1,\ldots,n\}$, that satisfies

(i) $\eta_i = 0$ for all $i\in\sigma$;
(ii) for any choice of integers $\mu_j \geqslant 0$, $j\in\sigma$, the monomial $\partial^\eta \prod_{j\in\sigma}\partial_j^{\mu_j}$ is not in M;
(iii) for all $l\notin\sigma$, there exist integers $\mu_l\geqslant 0$ and $\mu_j\geqslant 0$, $j\in\sigma$, such that $\partial^\eta\partial_l^{\mu_l}\prod_{j\in\sigma}\partial_j^{\mu_j}$ lies in M.

We denote the set of standard pairs of a monomial ideal M by $S(M)$. By [28, Eq. (3.2)],

$$M = \bigcap_{(\partial^\eta,\sigma)\in S(M)} \langle \partial_i^{\eta_i+1} : i\notin\sigma\rangle.$$

The prime ideal $\langle\partial_i : i\notin\sigma\rangle$ is associated to M if and only if there exists a standard pair of the form $(\cdot,\sigma)$ in $S(M)$. A standard pair (∂^η,σ) is called *top dimensional* if $\langle\partial_i : i\notin\sigma\rangle$ is a minimal associated prime of M, it is called *embedded* otherwise. It is clear from the above formula that the degree of M is equal to the cardinality of the set of top dimensional standard pairs of M.

Now, since the ideals I and $I_{\mathcal{B}}$ are unmixed (I is a complete intersection, and the associated primes of $I_{\mathcal{B}}$ are all isomorphic to I_A), all of the minimal primes of all the initial ideals of I have the same dimension, d (see [18, Corollary 1]), and the same holds for $I_{\mathcal{B}}$. This means that a standard pair (∂^η,σ) of either $\mathrm{in}_w(I)$ or $\mathrm{in}_w(I_{\mathcal{B}})$ is top dimensional if and only if $\#\sigma = d$.

Let $T(\mathrm{in}_w(I))$ be the set of top dimensional standard pairs (∂^η,σ) of $\mathrm{in}_w(I)$ such that the rows of $\mathcal{B}$ indexed by $i\notin\sigma$ are linearly independent.

Note that if (∂^η,σ) is a top-dimensional standard pair of $\mathrm{in}_w(I_{\mathcal{B}})$, then the rows of $\mathcal{B}$ indexed by $i\notin\sigma$ are linearly independent (the proof of [20, Lemma 2.3] works for lattice ideals too). Then $T(\mathrm{in}_w(I_{\mathcal{B}}))$ equals the set of top-dimensional standard pairs of $\mathrm{in}_w(I_{\mathcal{B}})$.

Given a standard pair in either $T(\mathrm{in}_w(I))$ or $T(\mathrm{in}_w(I_{\mathcal{B}}))$, and an arbitrary parameter vector c, there exists a unique vector v such that $A\cdot v = A\cdot c$, and $v_k=\eta_k$, $v_l=\eta_l$.

Suppose that (∂^η,σ) is a standard pair of $\mathrm{in}_w(I)$ that does not belong to the set $T(\mathrm{in}_w(I))$. Then either $\#\sigma < m$ or $\#\sigma = n-2$ and the columns of $\mathcal{B}$ corresponding to the indices not in σ are linearly dependent. In both of these cases, for a generic choice of c, the system $A\cdot v = A\cdot c$, $v_i = \eta_i$ for $i\notin\sigma$, has no solutions. The same holds for standard pairs not in $T(\mathrm{in}_w(I_{\mathcal{B}}))$.

We can now describe the primary decomposition of the indicial ideals of $H_{\mathcal{B}}(c)$ and $I_{\mathcal{B}} + \langle A \cdot \theta - A \cdot c \rangle$ with respect to w, in analogy to [25, Theorem 3.2.10].

Proposition 9.8. *For a generic parameter c, the indicial ideal of $H_{\mathcal{B}}(c)$ with respect to w equals the following intersection of maximal ideals*:

$$\bigcap_{(\partial^\eta, \sigma) \in T(\mathrm{in}_w(I))} \big(\langle \theta_i - \eta_i : i \notin \sigma \rangle + \langle A \cdot \theta - A \cdot c \rangle\big), \tag{23}$$

and the indicial ideal of $I_{\mathcal{B}} + \langle A \cdot \theta - A \cdot c \rangle$ equals:

$$\bigcap_{(\partial^\eta, \sigma) \in T(\mathrm{in}_w(I_{\mathcal{B}}))} \big(\langle \theta_i - \eta_i : i \notin \sigma \rangle + \langle A \cdot \theta - A \cdot c \rangle\big). \tag{24}$$

Proof. We prove the statement for the indicial ideal of $H_{\mathcal{B}}(c)$. The other indicial ideal is computed in exactly in the same manner.

By [25, Corollary 3.2.3], the indicial ideal is

$$J = \langle A \cdot \theta - A \cdot c \rangle + \bigcap_{(\partial^\eta, \sigma) \in S(\mathrm{in}_w(I))} \langle \theta_i - \eta_i : i \notin \sigma \rangle.$$

It is clear that the ideal (23) is radical. If we show that it has the same zero set as J, and that J has no multiple roots, we will be done.

Let v be a zero of J. Then $A \cdot v = A \cdot c$, and for some $(\partial^\eta, \sigma) \in S(\mathrm{in}_w(I))$, we have that $v_i = \eta_i$ for all $i \notin \sigma$. Since our parameter c is generic, we must have that (∂^η, σ) belongs to $T(\mathrm{in}_w(I))$. These are exactly the roots of the ideal (23). It also follows from the genericity of c that all the zeros of J are distinct, and the proof is finished. □

Note that the degree of $\mathrm{in}_w(I)$ is $d_1 \cdot d_2$, since it coincides with the degree of the complete intersection I. Then the cardinality of the set of top-dimensional standard pairs is exactly $d_1 \cdot d_2$. This and the previous proposition imply the following result.

Corollary 9.9. *Let ν be the sum of the multiplicities of the minimal primes of $\mathrm{in}_w(I)$ corresponding to linearly dependent sets of two rows of $\mathcal{B}$. For a generic parameter vector c, the degree of the fake indicial ideal is exactly $d_1 \cdot d_2 - \nu$. Therefore,*

$$\mathrm{rank}\,(H_{\mathcal{B}}(c)) = \mathrm{rank}\,(\mathrm{Horn}\,(\mathcal{B}, c)) = d_1 \cdot d_2 - \nu = \#T(\mathrm{in}_w(I)).$$

Our desired formula for the generic rank of a bivariate Horn system now follows from Proposition 4.2.

Theorem 9.10. *For generic c and $m = 2$,*

$$\operatorname{rank}(H_{\mathcal{B}}(c)) = \operatorname{rank}(\operatorname{Horn}(\mathcal{B}, c)) = d_1 \cdot d_2 - \sum v_{ij},$$

where the sum runs over linearly dependent rows of $\mathcal{B}$ that lie in opposite open quadrants of $\mathbb{Z}^2$.

Proof. By Proposition 4.2, the sum of the multiplicities of the minimal primes of I corresponding to linearly dependent rows of $\mathcal{B}$ is the sum of the corresponding indices $\sum v_{ij}$. This implies that

$$\deg(\operatorname{ind}_w(H_{\mathcal{B}}(c)) = d_1 \cdot d_2 - \sum v_{ij},$$

where the sum runs over linearly independent rows of $\mathcal{B}$ lying in opposite open quadrants of $\mathbb{Z}^2$. But then, since

$$\deg(\operatorname{ind}_w(H_{\mathcal{B}}(c))) = \operatorname{rank}(\operatorname{in}_{(-w,w)}(H_{\mathcal{B}}(c)) \leqslant \operatorname{rank}(H_{\mathcal{B}}(c))$$

we conclude that

$$\operatorname{rank}(H_{\mathcal{B}}(c)) = \operatorname{rank}(\operatorname{Horn}(\mathcal{B}, c)) \geqslant d_1 \cdot d_2 - \sum v_{ij}.$$

The reverse inequality follows from Corollary 8.8. $\square$

The same method that exactly proved Theorem 9.10 will compute the rank of the hypergeometric system arising from a lattice (actually, this proof is easier, since $\#T(\operatorname{in}_w(I_{\mathcal{B}})) = \deg(I_{\mathcal{B}}) = g \cdot \operatorname{vol}(A)$ is easier to compute than $\#T(\operatorname{in}_w(I)))$. Note that here we do not need to require that $m = 2$, since we know what the solutions of these systems look like without restriction on the codimension of $I_{\mathcal{B}}$.

Theorem 9.11. *For generic c,*

$$\operatorname{rank}(I_{\mathcal{B}} + \langle A \cdot \theta - A \cdot c \rangle) = \#T(\operatorname{in}_w(I_{\mathcal{B}})) = \deg(I_{\mathcal{B}}) = g \cdot \operatorname{vol}(A).$$

10. Explicit construction of fully supported hypergeometric functions

We already know how to explicitly write down Puiseux polynomial solutions of a bivariate Horn system with generic parameters. This is done by taking pairs of rows of the matrix $\mathcal{B}$ that are linearly independent and lie in opposite open quadrants of $\mathbb{Z}^2$, obtaining a cone from these vectors, and joining together lattice points in the cone using horizontal and vertical moves to obtain the finite supports of Puiseux polynomial solutions. We have not described the coefficients appearing in these Puiseux polynomials, although they are easily computed on a case by case basis.

The goal of this section is to be even more explicitly describe the fully supported solutions of $H_{\mathcal{B}}(c)$, and thus of Horn $(\mathcal{B}, c)$. In particular, we will show that the fully supported solutions of Horn $(\mathcal{B}, c)$ are hypergeometric in the following classical sense.

Definition 10.1. A formal power series $\sum_{(s,t)\in\mathbb{Z}^2} \lambda(s,t) y_1^s y_2^t$ is *hypergeometric* if there exist rational functions R_1 and R_2 such that

$$\lambda(s+1,t) = R_1(s,t)\lambda(s,t) \quad \text{and} \quad \lambda(s,t+1) = R_2(s,t)\lambda(s,t).$$

In this paper we restrict our attention to the case when the numerator and the denominator of the rational functions R_1, R_2 are products of affine linear functions with integer coefficients by s, t and arbitrary constant terms.

A formal power series such as in Definition 10.1 satisfies a Horn system of differential equations. We will now show that the other fully supported solutions of this system are spanned by monomial multiples of series of this form. We know that the fully supported solutions of $H_{\mathcal{B}}(c)$ are simply the solutions of the lattice hypergeometric system $I_{\mathcal{B}} + \langle A\cdot\theta - A\cdot c\rangle$. The following result is proved using the methods from [25, Section 3.4]. We start by setting up some notation. Recall that $L_{\mathcal{B}}$ is the lattice in $\mathbb{Z}^n$ spanned by the columns of $\mathcal{B}$.

Given $v \in \mathbb{C}^n$ we let

$$N_v = \{u \in L_{\mathcal{B}} : v_i \in \mathbb{Z}_{<0} \Leftrightarrow (u+v)_i \in \mathbb{Z}_{<0} \text{ and } v_i \in \mathbb{Z}_{\geqslant 0} \Leftrightarrow (u+v)_i \in \mathbb{Z}_{\geqslant 0}\},$$

and define a formal power series

$$\phi_v := x^v \sum_{u\in N_v} \frac{[v]_{u_-}}{[v+u]_{u_+}} x^u, \tag{25}$$

where

$$[v]_{u_-} = \prod_{i:u_i<0} \prod_{j=1}^{-u_i} (v_i - j + 1) \quad \text{and} \quad [v+u]_{-} = \prod_{i:u_i>0} \prod_{j=1}^{u_i} (v_i + j).$$

Theorem 10.2. *Let c be generic and w a generic weight vector. Denote by $v^{(1)}, \ldots, v^{(g\cdot \mathrm{vol}\,(A))}$ be the zeros of the indicial ideal* $\mathrm{ind}_w(I_{\mathcal{B}} + \langle A\cdot\theta - A\cdot c\rangle)$. *Then the formal power series $\{\phi_{v^{(i)}} : i = 1, \ldots, g\cdot \mathrm{vol}\,(A)\}$ are linearly independent holomorphic solutions of $I_{\mathcal{B}} + \langle A\cdot\theta - A\cdot c\rangle$.*

Proof. For sufficiently generic c, the vectors $v^{(i)}$ have no negative integer coordinates. Now use the arguments from [25, Theorem 3.4.2]. In particular, the support of each of these series is contained in a strongly convex cone. □

We now have an explicit description of a basis of the solution space of the system Horn $(\mathcal{B}, c)$ (and $H_{\mathcal{B}}(c)$).

Theorem 10.3. *If c is generic, the fully supported series obtained by applying the isomorphism from Corollary 5.2 to the fully supported series constructed in Theorem 10.2 and the Puiseux polynomials constructed in Theorem 6.6 form a basis for the solution space of* Horn $(\mathcal{B}, c)$.

Proof. Theorem 10.2 and Corollary 5.2 give us $g \cdot \mathrm{vol}\,(A) + \sum v_{ij}$ linearly independent solutions of $H_{\mathcal{B}}(c)$ (here the sum runs over linearly independent rows of $\mathcal{B}$). By Theorem 9.10, these must span the solution space of $H_{\mathcal{B}}(c)$. □

Note that applying the change of variables from Corollary 5.2 to the functions $\phi_{v^{(i)}}$ from Theorem 10.2 is particularly easy.

Corollary 10.4. *For c generic and $v^{(i)}$ as in Theorem 10.2, let $\alpha^{(i)}$ be the unique vector that satisfies $v^{(i)} - c = \mathcal{B} \cdot \alpha^{(i)}$. Then the space of fully supported solutions of* Horn $(\mathcal{B}, c)$ *is spanned by the functions*

$$y_1^{\alpha_1^{(i)}} y_2^{\alpha_2^{(i)}} \sum_{\mathcal{B}\cdot z \in N_{v^{(i)}}} \frac{[v^{(i)}]_{(\mathcal{B}\cdot z)_-}}{[v^{(i)} + \mathcal{B} \cdot z]_{(\mathcal{B}\cdot z)_+}} y_1^{z_1} y_2^{z_2}.$$

In particular, all the fully supported solutions of Horn $(\mathcal{B}, c)$ *are spanned by monomial multiples of hypergeometric series in the sense of Definition 10.1.*

11. Holonomicity of Horn $(\mathcal{B}, c)$

Throughout this section we assume that $m = 2$. Since we do not have a D-module isomorphism between $H_{\mathcal{B}}(c)$ and Horn $(\mathcal{B}, c)$, the holonomicity of $H_{\mathcal{B}}(c)$ does not directly prove that Horn $(\mathcal{B}, c)$ is holonomic. In this section we prove that the bivariate hypergeometric system Horn $(\mathcal{B}, c)$ is holonomic.

Recall that a system of differential equations is said to be *holonomic* if the dimension of its characteristic variety is the same as the dimension of the variable space.

We recall that we are dealing with the system of equations defined by the hypergeometric operators

$$\begin{aligned} H_1 &= \boldsymbol{Q}_1(\theta) - y_1 \boldsymbol{P}_1(\theta), \\ H_2 &= \boldsymbol{Q}_2(\theta) - y_2 \boldsymbol{P}_2(\theta). \end{aligned} \tag{26}$$

By the definition of the Horn system (see Section 2) the bivariate polynomials $\boldsymbol{P}_i, \boldsymbol{Q}_i$ satisfy the compatibility condition

$$R_1(s+e_2)R_2(s) = R_2(s+e_1)R_1(s), \tag{27}$$

where $R_i(s) = \boldsymbol{P}_i(s)/\boldsymbol{Q}_i(s+e_i)$ and $\{e_1, e_2\}$ is the standard basis of $\mathbb{Z}^2$.

Theorem 11.1. *A bivariate Horn system with generic parameters is holonomic.*

To prove this theorem we need some intermediate results and notation. Denote by $(H_1, H_2) \subset D_2$ the ideal generated by the hypergeometric operators defining the Horn system. By $\sigma(P)$ we denote the principal symbol of a differential operator P. This is an element of the polynomial ring $\mathbb{C}[y_1, y_2, z_1, z_2]$. The only case when a bivariate Horn system is not holonomic is when the principal symbols of all the operators in (H_1, H_2) have a nontrivial greatest common divisor (for otherwise we have two independent algebraic equations and hence the dimension of the characteristic variety of the Horn system is 2). Thus to prove holonomicity of (26) it suffices to construct a family of operators in (H_1, H_2) such that the greatest common divisor of their principal symbols is 1.

By the construction of the operators in the Horn system (see Section 2) the greatest common divisor of the principal symbols of H_1 and H_2 is given by a product of powers of linear forms $ay_1z_1 + by_2z_2$, where $a, b \in \mathbb{Z}$. Thus to prove Theorem 11.1 it suffices to show that for any $a, b \in \mathbb{Z}$ such that $ay_1z_1 + by_2z_2$ divides $\gcd(\sigma(H_1), \sigma(H_2))$ there exists an operator $T_{a,b} \in (H_1, H_2)$ whose principal symbol is not divisible by $ay_1z_1 + by_2z_2$.

Remark 11.2. For generic parameters the compatibility condition (27) is equivalent to the relations

$$[y_1\boldsymbol{P}_1(\theta), y_2\boldsymbol{P}_2(\theta)] = 0, \qquad (E_2\boldsymbol{Q}_2)(\theta)(E_1E_2\boldsymbol{Q}_1)(\theta) = (E_1\boldsymbol{Q}_1)(\theta)(E_1E_2\boldsymbol{Q}_2)(\theta), \tag{28}$$

where $[\ ,\ ]$ denotes the commutator of two operators, $(E_i^{\lambda}P)(s) = P(s+\lambda e_i)$ and $E_i = E_i^1$. Indeed, equalities (28) mean that the numerators (respectively the denominators) of the rational functions in (27) are equal. The generic parameters assumption implies that no cancellations can occur and hence this is indeed the case.

Lemma 11.3. *For any $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ and $P_1(\theta), P_2(\theta), Q_1(\theta), Q_2(\theta)$ satisfying the relations*

$$[y_1P_1(\theta), y_2P_2(\theta)] = 0, \quad (E_2Q_2)(\theta)(E_1E_2Q_1)(\theta) = (E_1Q_1)(\theta)(E_1E_2Q_2)(\theta), \tag{29}$$

it holds that

$$(\alpha(E_2^{-1}Q_1)(\theta) - \beta y_1 P_1(\theta))(\gamma Q_2(\theta) - \delta y_2 P_2(\theta))$$
$$-(\alpha(E_1^{-1}Q_2)(\theta) - \beta y_2 P_2(\theta))(\gamma Q_1(\theta) - \delta y_1 P_1(\theta)) = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \Psi, \tag{30}$$

where $\Psi = y_1 Q_2(\theta) P_1(\theta) - y_2 Q_1(\theta) P_2(\theta)$.

The proof of Lemma 11.3 is a direct computation which uses the compatibility conditions (29) and the Weyl algebra identity $(E_i^{-1}Q_j)(\theta)y_i = y_i Q_j(\theta)$.

Let us now consider a special case to which we will later reduce the case of an arbitrary bivariate Horn system with generic parameters. Namely, let us find a holonomicity condition for the system defined by the operators

$$\begin{aligned} U_1 &= f(t)Q_1(\theta) - y_1 g(t) P_1(\theta), \\ U_2 &= f(t)Q_2(\theta) - y_2 g(t) P_2(\theta), \end{aligned} \tag{31}$$

where f, g are arbitrary nonzero univariate polynomials, $t = \theta_1 + \theta_2$ and P_i, Q_i are arbitrary bivariate polynomials such that $\deg f + \deg Q_i = \deg g + \deg P_i$ and that P_i, Q_i satisfy (29). Note that these relations are satisfied if $g(t)P_i$, $g(t)Q_i$ satisfy the equivalent relations. We assume also that t is not present in $P_i(\theta), Q_i(\theta)$, i.e., that none of the principal symbols of these operators vanish along the hypersurface $y_1 z_1 + y_2 z_2 = 0$.

Our goal is to "eliminate t" from (31), i.e., to construct an operator in the ideal (U_1, U_2) whose principal symbol is not divisible by $\sigma(t) = y_1 z_1 + y_2 z_2$. We do it as follows.

Lemma 11.4. *Let* Ψ *be as in Lemma* 11.3. *Then* $R(f(t), g(t))\Psi \in (U_1, U_2)$, *where* $R(f(t), g(t))$ *is the resultant of* f, g.

Proof. Let us write the polynomials f, g in the form $f(t) = \sum_{i=0}^d f_i t^i$, $g(t) = \sum_{i=0}^d g_i t^i$. Note that f, g do not have to be of the same degree since some of f_i, d_i may be zero. Using (30), and the fact that the subring of the Weyl algebra generated by θ_1 and θ_2 is commutative, we conclude that for any $j = 0, \ldots, d$

$$\begin{aligned} &\sum_{i=0}^d \begin{vmatrix} f_j & g_j \\ f_i & g_i \end{vmatrix} \Psi t^i \\ &= \sum_{i=0}^d \Theta_{1j}(f_i t^i Q_2(\theta) - y_2 g_i t^i P_2(\theta)) - \sum_{i=0}^d \Theta_{2j}(f_i t^i Q_1(\theta) - y_1 g_i t^i P_1(\theta)) \\ &= \Theta_{1j} U_2 + \Theta_{2j} U_1 \in (U_1, U_2), \end{aligned}$$

where $\Theta_{1j} = f_j(E_2^{-1}Q_1)(\theta) - g_j y_1 P_1(\theta)$ and $\Theta_{2j} = f_j(E_1^{-1}Q_2)(\theta) - g_j y_2 P_2(\theta)$.

Now clearly,

$$\sum_{i=0}^{d} \begin{vmatrix} f_j & g_j \\ f_i & g_i \end{vmatrix} \Psi t^i = \Psi \begin{vmatrix} f_j & g_j \\ f(t) & g(t) \end{vmatrix},$$

so that

$$\Psi \begin{vmatrix} f_j & g_j \\ f(t) & g(t) \end{vmatrix} \in (U_1, U_2).$$

In the trivial case when the polynomials f and g are proportional we have that $R(f(t), g(t)) = 0$ and the conclusion of the lemma is obviously true. If f is not proportional to g then the rank of the $2 \times (m+1)$-matrix $\left(\begin{smallmatrix} f_0 \cdots f_m \\ g_0 \cdots g_m \end{smallmatrix}\right)$ equals 2 and hence $\Psi f(t), \Psi g(t) \in (U_1, U_2)$. Since $(t-1)\Psi = \Psi t$, it follows that $\Psi h(t) \in (U_1, U_2)$, for any $h(t) \in (f(t), g(t))$, where $(f(t), g(t))$ denotes the ideal in the ring of (commuting) univariate polynomials generated by f, g. It is known that the resultant of two polynomials lies in the ideal generated by these polynomials and hence $R(f(t), g(t))\Psi \in (U_1, U_2)$. The proof is complete. $\square$

Corollary 11.5. *Suppose that* $\gcd(\sigma(U_1), \sigma(U_2))$ *is a power of* $x_1 z_1 + x_2 z_2$. *Then the hypergeometric system* (31) *is holonomic if and only if* $R(f(t), g(t))$ *is nonzero.*

Proof. Suppose that $R(f(t), g(t)) = 0$ and let $\zeta \in \mathbb{C}$ be a common root of the polynomials f, g. Since for any smooth univariate function h the product $y_2^{\zeta} h(y_1/y_2)$ is annihilated by the operator $t - \zeta = \theta_1 + \theta_2 - \zeta$, it follows that the space of analytic solutions to (31) has infinite dimension. It is known that a holonomic system can only have finitely many linearly independent solutions and hence (31) is not holonomic in this case.

On the other hand, if $R(f(t), g(t)) \neq 0$, then by Lemma 11.4 the operator Ψ is an element of the ideal (U_1, U_2). By the assumption of the corollary the principal symbols of U_1, U_2 and Ψ are relatively prime and hence system (31) is holonomic. $\square$

Example 11.6. Consider the system quoted in the introduction, given by the two hypergeometric operators

$$\begin{aligned} H_1 &= x(\theta_x + \theta_y + a)(\theta_x + b) - \theta_x(\theta_x + \theta_y + c - 1), \\ H_2 &= y(\theta_x + \theta_y + a)(\theta_y + b') - \theta_y(\theta_x + \theta_y + c - 1), \end{aligned}$$

for Appell's function F_1. The operator Ψ in Lemma 11.3 equals in this case

$$\Psi = (x \quad y) \ \Psi' \ , \quad \text{where} \quad \Psi' = (x-y)\partial_x\partial_y - b'\partial_x + b\partial_y.$$

When $a-c+1 \neq 0$, we deduce from Lemma 11.4 that $(x \quad y) \ \Psi'$ lies in the D-ideal $\langle H_1, H_2\rangle$. In particular, all holomorphic solutions φ of the Appell system will also satisfy $\Psi'(\varphi) = 0$. We point out that some authors add this third equation to the system (cf. for instance [25, p. 48]). In fact, having this operator, the holonomicity of the system follows immediately.

We are now in a position to complete the proof of Theorem 11.1.

Proof of Theorem 11.1. Suppose that $\gcd(\sigma(H_1), \sigma(H_2))$ vanishes along the hypersurface $ay_1z_1+by_2z_2{=}0$. We aim to construct an operator in the ideal (H_1, H_2) whose principal symbol is not divisible by $ay_1z_1+by_2z_2$. The change of variables $\xi_1 = y_1^{1/a}$, $\xi_2 = y_2^{1/b}$ transforms the operator $a\theta_{y_1}+b\theta_{y_2}$ into the operator $\theta_{\xi_1}+\theta_{\xi_2}$ and system (26) into the system generated by the operators

$$\begin{aligned}&\hat{Q}_1(\theta_{\xi_1}, \theta_{\xi_2}) - \xi_1^a \, \hat{P}_1(\theta_{\xi_1}, \theta_{\xi_2}),\\ &\hat{Q}_2(\theta_{\xi_1}, \theta_{\xi_2}) - \xi_2^b \, \hat{P}_2(\theta_{\xi_1}, \theta_{\xi_2}),\end{aligned} \tag{32}$$

where $\hat{P}_i(u, v) = P_i(u/a, v/b)$, $\hat{Q}_i(u, v) = Q_i(u/a, v/b)$.

Let us introduce operators $\lambda_{ia}^k, \mu_{ia}^k$ acting on a bivariate polynomial P as follows:

$$\lambda_{ia}^k(P) = \prod_{j=1}^{k}(E_i^{-ja}P), \qquad \mu_{ia}^k(P) = \prod_{j=0}^{k-1}(E_i^{ja}P). \tag{33}$$

(Note that the upper index here is **not** a power.) The next Weyl algebra identities follow directly from the definition of $\lambda_{ia}^k, \mu_{ia}^k$ (the arguments of all of the involved polynomials being $\theta_{\xi_1}, \theta_{\xi_2}$):

$$\begin{aligned}\lambda_{1a}^k\,(\hat{Q}_1)\xi_1^a \;&=\; \xi_1^a\,\hat{Q}_1\,\lambda_{1a}^{k-1}\,(\hat{Q}_1),\\ \lambda_{2b}^k\,(\hat{Q}_2)\xi_2^b \;&=\; \xi_2^b\,\hat{Q}_2\,\lambda_{2b}^{k-1}\,(\hat{Q}_2),\\ \mu_{1a}^k\,(\hat{P}_1)\xi_1^a\,\hat{P}_1 \;&=\; \xi_1^a\,\mu_{1a}^{k+1}\,(\hat{P}_1),\\ \mu_{2b}^k\,(\hat{P}_2)\xi_2^b\,\hat{P}_2 \;&=\; \xi_2^b\,\mu_{2b}^{k+1}\,(\hat{P}_2).\end{aligned} \tag{34}$$

Using (34) we arrive at the equalities

$$\begin{aligned}&\left(\sum_{v=0}^{b-1}\xi_1^{va}\lambda_{1a}^{b-1-v}(\hat{Q}_1)(\theta_\xi)\mu_{1a}^{v}(\hat{P}_1)(\theta_\xi)\right)(\hat{Q}_1(\theta_\xi) - \xi_1^a\hat{P}_1(\theta_\xi))\\ &\quad = \hat{Q}_1(\theta_\xi)\lambda_{1a}^{b-1}(\hat{Q}_1)(\theta_\xi) - \xi_1^{ab}\mu_{1a}^{b}(\hat{P}_1)(\theta_\xi),\end{aligned} \tag{35}$$

$$\left(\sum_{v=0}^{a-1} \xi_2^{vb} \lambda_{2b}^{a-1-v}(\hat{Q}_2)(\theta_\xi)\mu_{2b}^{v}(\hat{P}_2)(\theta_\xi)\right)(\hat{Q}_2(\theta_\xi) - \xi_2^b \hat{P}_2(\theta_\xi))$$
$$= \hat{Q}_2(\theta_\xi)\lambda_{2b}^{a-1}(\hat{Q}_2)(\theta_\xi) - \xi_2^{ab}\mu_{2b}^{a}(\hat{P}_2)(\theta_\xi). \tag{36}$$

The differential operators (35) and (36) are Horn-type hypergeometric operators in the variables $\eta_1 = \xi_1^{ab}$ and $\eta_2 = \xi_2^{ab}$. Let us write these operators in the form

$$\tilde{U}_1 = f(\tau)\tilde{Q}_1(\theta_\eta) - \eta_1 g(\tau)\tilde{P}_1(\theta_\eta),$$
$$\tilde{U}_2 = f(\tau)\tilde{Q}_2(\theta_\eta) - \eta_2 g(\tau)\tilde{P}_2(\theta_\eta),$$

where f, g are univariate polynomials, $\tau = \theta_{\eta_1} + \theta_{\eta_2}$ and none of the principal symbols of the operators $\tilde{P}_i(\theta_\eta), \tilde{Q}_i(\theta_\eta)$ vanish along the hypersurface $\eta_1 z_1 + \eta_2 z_2 = 0$. The existence of such polynomials f, g follows from the compatibility condition which is satisfied by (35), (36).

By Lemma 11.4 the operator $\tilde{\Psi} = \eta_1\tilde{Q}_2(\theta_\eta)\tilde{P}_1(\theta_\eta) - \eta_2\tilde{Q}_1(\theta_\eta)\tilde{P}_2(\theta_\eta)$ lies in the ideal $(\tilde{U}_1, \tilde{U}_2)$ as long as the parameters of the original Horn system (26) are generic. Note that by construction the principal symbol of $\tilde{\Psi}$ does not vanish along the hypersurface $\eta_1 z_1 + \eta_2 z_2 = 0$. Going back to the variables y_1, y_2, we conclude that there exists an operator in (H_1, H_2) whose principal symbol is not divisible by $ay_1z_1 + by_2z_2$. This completes the proof of Theorem 11.1. $\square$

12. The Cohen–Macaulay property as a tool to compute rank, and further research directions

Since the lattice basis ideal I is a complete intersection and therefore Cohen–Macaulay, it is natural to try to apply the methods that proved that the holonomic rank $H_A(A \cdot c)$ is always $\mathrm{vol}\,(A) = \deg(I_A)$ when the underlying toric ideal I_A is Cohen–Macaulay.

The first evidence that these methods will not work is that the generic rank of the Horn system $H_{\mathcal{B}}(c)$ is *not* $\deg(I) = d_1 \cdot d_2$, unless we make the assumption that $\mathcal{B}$ has no linearly dependent rows in opposite open quadrants of $\mathbb{Z}^2$.

If we follow the arguments that proved [25, Lemma 4.3.7], which is the main ingredient needed to prove that, when I_A is Cohen–Macaulay, $\mathrm{rank}\,(H_A(A \cdot c)) = \mathrm{vol}\,(A)$ for all c, we see that the crucial point is whether the $n - m$ polynomials

$$\sum_{j=1}^{n} a_{ij}x_jz_j \in \mathbb{C}[x_1, \ldots, x_n, z_1, \ldots, z_n], \quad i = 1, \ldots, n-m, \tag{37}$$

form a regular sequence in $\mathbb{C}(x_1, \ldots, x_n)[z_1, \ldots, z_n]/I$, where here we think of I as an ideal in the variables $z_1, \ldots, z_n$. But if $\mathcal{B}$ has linearly dependent rows in opposite open quadrants, the ring

$$\frac{\mathbb{C}(x_1, \ldots, x_n)[z_1, \ldots, z_n]}{I + \langle \sum_{j=1}^{n} a_{ij} x_j z_j \in \mathbb{C}[x_1, \ldots, x_n, z_1, \ldots, z_n],\ i = 1, \ldots, n-m \rangle}$$

is not artinian!

Lemma 12.1. *Let $m = 2$. If $\langle A \cdot xz \rangle$ is ideal generated by the polynomials* (37), *then the ideal $I + \langle A \cdot xz \rangle$ is artinian in $\mathbb{C}(x_1, \ldots, x_n)[z_1, \ldots, z_n]/I$, if and only if $\mathcal{B}$ has no linearly dependent rows in opposite open quadrants of $\mathbb{Z}^2$.*

Proof. We need to investigate the intersection of the zero locus of $\langle A \cdot xz \rangle$ over $\mathbb{C}(x)$ with the zero locus of I over $\mathbb{C}(x)$. Specifically, we want to show that this intersection is a finite set if and only if $\mathcal{B}$ contains no linearly dependent rows in opposite open quadrants of $\mathbb{Z}^2$. We can perform this intersection irreducible component by irreducible component of I, recalling the primary decomposition of I from Proposition 4.2.

The toric irreducible components of I we can deal with all at the same time: we know that $\mathbb{C}(x)[z]/(I_{\mathcal{B}} + \langle A \cdot xz \rangle)$ is zero dimensional. That just leaves the primary components of I corresponding to associated primes $\langle z_i, z_j \rangle$, where b_i and b_j lie in the interior of open quadrants of $\mathbb{Z}^2$. But now it is clear that such a component will meet the zero locus of $\langle A \cdot xz \rangle$ in an infinite set if and only if b_i and b_j are linearly dependent. $\square$

As a consequence of Lemma 12.1 and the arguments in [25, Section 4.3], we have one case when the fact that I is a complete intersection will imply that the rank of $H_{\mathcal{B}}(c)$ does not depend on c.

Theorem 12.2. *If $\mathcal{B}$ has no linearly dependent rows in opposite quadrants of $\mathbb{Z}^2$ then*

$$\operatorname{rank}(H_{\mathcal{B}}(c)) = d_1 \cdot d_2 \quad for\ all\ c \in \mathbb{C}^n.$$

Note that this result holds even when the rows of $\mathcal{B}$ do not add up to zero.

Remark that the case in which no pair of (linearly dependent or not) rows lie in the interior of opposite quadrants corresponds precisely to the case in which the lattice ideal $I_{\mathcal{B}}$ is a complete intersection. This agrees with the characterization in [10].

There is another situation when we can apply the arguments from [25, Section 4.3] to prove that a certain holonomic rank does not depend on c. Let J be the ideal in $\mathbb{C}[\partial_1, \ldots, \partial_n]$ obtained by saturating from I the components I_{ij} corresponding to linearly dependent rows of $\mathcal{B}$. Then

$$\deg(J) = d_1 \cdot d_2 - \sum v_{ij},$$

where the sum runs over the linearly dependent rows of $\mathcal{B}$ that lie in opposite open quadrants of $\mathbb{Z}^2$. As before, the methods in [25, Section 4.3] prove the following result.

Lemma 12.3. *If J is Cohen–Macaulay,*

$$\operatorname{rank}\left(J + \langle A \cdot \theta - A \cdot c \rangle\right) = \deg(J).$$

The previous lemma and our rank formula for Horn systems have the following consequence.

Corollary 12.4. *If J is Cohen–Macaulay and c is generic, the solution spaces of $H_{\mathcal{B}}(c)$ and $J + \langle A \cdot \theta - A \cdot c \rangle$ coincide.*

We believe that Corollary 12.4 holds even when J is not Cohen–Macaulay. It would be desirable to obtain an independent proof of this, since in that case we would have a proof of our rank formula in the case that J is Cohen–Macaulay that does not rely on a precise description of the solution space.

The natural question at this point is whether we can extend arguments in Section 9 to give an algebraic formula for the rank of a Horn system for any m. However, in order to use those methods, several ingredients are missing. First, we need to assume that the lattice basis ideal I is a complete intersection, since this is not necessarily true if $m > 2$. Moreover, it is not true in general that given a toric ideal I_A, one can find a lattice basis ideal contained in I_A that is a complete intersection [3]. Moreover, our techniques for finding the form of the solutions of $H_{\mathcal{B}}(c)$ for $m = 2$ do not directly generalize to higher m. In any case, in order to obtain an explicit rank formula in the case that $m > 2$, combinatorial expressions for the multiplicities of the minimal primes of any lattice basis ideal are needed.

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