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# Distance-regular Subgraphs in a Distance-regular Graph, I

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Let $\Gamma$ be a distance-regular graph with $r = l(1, 0, k-1) > 0$ and $c_{2r+1} = 1$. We show the existence of a Moore graph of diameter $r+1$ and valency $a_{r+1}+1$ as a subgraph in $\Gamma$. In particular, we show that either $a_{r+1} = 1$ or $r = 1$.



## 1. INTRODUCTION

All graphs considered in this paper are unidirected finite graphs without loops or multiple edges. Let $\Gamma$ be a connected graph. We identify $\Gamma$ with the set of vertices.

For two vertices $u,v$ in $\Gamma$, a *walk* of length $l$ connecting $u$ and $v$ is a sequence of the vertices $u = x_0, x_1, \ldots, x_l = v$ such that each $(x_i, x_{i+1})$ is an edge of $\Gamma$. If $x_{j-1} \neq x_{j+1}$ for $1 \leq j \leq l-1$, then we say that the walk is a *path*. We denote by $\partial_\Gamma(u, v)$ the distance between $u$ and $v$ in $\Gamma$, i.e. the length of a shortest path connecting $u$ and $v$ in $\Gamma$. Let

$$\Gamma_j(u) = \{x \in \Gamma \mid \partial_\Gamma(u, x) = j\},$$
$$k_\Gamma(u) = |\Gamma_1(u)|,$$
$$d_\Gamma(u) = \max\{\partial_\Gamma(u, x) \mid x \in \Gamma\}.$$

For two vertices $u$ and $x$ in $\Gamma$ with $\partial_\Gamma(u, x) = j$, let

$$C(u, x) = \Gamma_{j-1}(u) \cap \Gamma_1(x),$$
$$A(u, x) = \Gamma_j(u) \cap \Gamma_1(x)$$

and

$$B(u, x) = \Gamma_{j+1}(u) \cap \Gamma_1(x).$$

$\Gamma$ is said to be *distance-regular* if

$$c_j(\Gamma) = |C(u, x)|, \qquad a_j(\Gamma) = |A(u, x)| \qquad \text{and} \qquad b_j(\Gamma) = |B(u, x)|$$

depend only on $j = \partial_\Gamma(u, x)$ rather than on individual vertices. It is easy to see that if $\Gamma$ is a distance-regular graph, then $k_\Gamma(u)$ and $d_\Gamma(u)$ do not depend on the choice of $u$. Hence we write $k_\Gamma$ and $d_\Gamma$. They are called the *valency* and the *diameter* of $\Gamma$. Sometimes we omit the suffix when the concerning graph is clear. The numbers $c_i$, $a_i$ and $b_i$ are called the *intersection numbers* of $\Gamma$, and

$$\iota(\Gamma) = \begin{Bmatrix} * & c_1 & c_2 & \cdots & c_j & \cdots & c_{d-1} & c_d \\ a_0 & a_1 & a_2 & \cdots & a_j & \cdots & a_{d-1} & a_d \\ b_0 & b_1 & b_2 & \cdots & b_j & \cdots & b_{d-1} & * \end{Bmatrix}$$

is called the intersection array of $\Gamma$.

The following are basic properties of the intersection numbers which we use implicitly in this paper;
(1) $c_i + a_i + b_i = k$;
(2) $k = b_0 \geq b_1 \geq \cdots \geq b_{d-2} \geq b_{d-1} \geq 1$;
(3) $1 = c_2 \leq c_2 \leq \cdots \leq c_{d-1} \leq c_d \leq k$.

The reader is referred to [1, 3] for general theory of distance-regular graphs.



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We use the following notation in this paper:

$$l(\alpha, \beta, \gamma) = \#\{j \mid (c_j, a_j, b_j) = (\alpha, \beta, \gamma)\}.$$

For vertices $x, y \in \Gamma$, we write $x \sim y$ when they are adjacent. Let $X$ and $Y$ be sets of vertices. We denote $e(X, Y)$ the number of edges between $X$ and $Y$. We write $e(\{x\}, Y) = e(x, Y)$ when $X$ consists of a single element $x$. Let $x, y \in \Gamma$ with $\partial_\Gamma(x, y) = t$. We denote by $p[x, y]$ the unique shortest path connecting $x$ and $y$ when $c_t = 1$. We write $p[x, y] = \{z_j\}_j$ if

$$p[x, y] = \{x = z_0 \sim z_1 \sim \cdots \sim z_j \sim \cdots \sim z_{t-1} \sim z_t = y\}.$$

A *circuit* of length $l$ is a sequence of distinct vertices $x_0, x_1, \ldots, x_{l-1}$ such that $(x_i, x_{i+1})$ is an edge of $\Gamma$ for $0 \leq i \leq l-1$, where $x_l = x_0$ and $l \neq 2$. Let $g = g(\Gamma)$ denote the *girth* of $\Gamma$ that is the minimal length of a circuit in $\Gamma$.

In this paper, we prove the following result.

THEOREM 1.1. *Let $\Gamma$ be a distance-regular graph with $r = l(1, 0, k-1) > 0$. Assume that $c_{2r+1} = 1$. Then there exists a Moore graph of valency $a_{r+1} + 1$ and diameter $r + 1$ as a subgraph in $\Gamma$.*

A detailed description of Moore graphs will be found in [1, 3].

Using the classification of Moore graphs, we obtain the following:

COROLLARY 1.2. *Let $\Gamma$ be a distance-regular graph with $r = l(1, 0, k-1) > 0$. Assume that $c_{2r+1} = 1$. Then either $a_{r+1} = 1$ or $r = 1$.*

The theorem is clear for the case $a_{r+1} = 1$; in this case the subgraphs are mere circuits of length $2r + 3$.

Proposition 4.3.11 of [3] shows the existence of distance-regular subgraphs with $d = 2$ and $c_2 = 1$ in a distance-regular graph with $r = l(1, a_1, b_1) = 1$ and $c_3 = 1$. So our theorem is the result for general $r$, but with addtional condition $a_1 = 0$. In the subsequent paper [6], we treat the case $a_1 > 0$ and show that the existence of a collinearity graph of a Moore geometry as a subgraph in a distance-regular graph with $r = l(1, a_1, b_1)$ and $c_{2r+1} = 1$.

Here we conjecture the following:

CONJECTURE. Let $\Gamma$ be a distance-regular graph and let $r = l(1, a_1, b_1)$. If $c_s = 1$, then $s \leq 2r + N$ for some absolute constant $N$.

Our results show the conjecture is true on the assumption $r \geq 2$ and $a_{r+1} \neq 1$. For the remaining cases, we hope that our method is also applicable with some modification.

We believe that to find 'nice' distance-regular subgraphs in a distance-regular graph will be a key for the classification of distance-regular graphs.

## 2. PRELIMINARIES

In this section, we introduce the intersection diagram which will be our main tool.

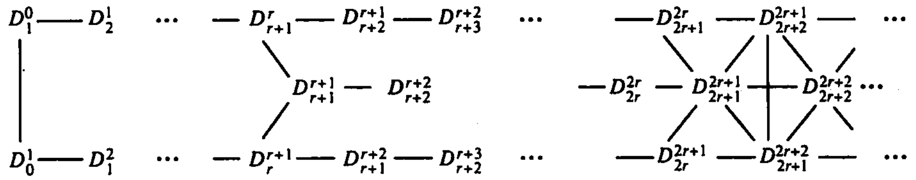
Let $\alpha$ and $\beta$ be adjacent vertices and $D_j^i = \Gamma_i(\alpha) \cap \Gamma_j(\beta)$. The *intersection diagram with respect to* $(\alpha, \beta)$ is the collection $\{D_j^i\}_{i,j}$ with lines between them. We draw a line

$$D_j^i \text{——} D_q^p$$

if there is a possibility of an existence of edges between them, and we erase the line when we know that there is no edge between them. Also, we erase $D^i_j$ when we know that it is empty. A detailed description of the intersection diagram will be found in [2], [4] and [5]. In this paper we say '$(\alpha, \beta)$-diagram' instead of 'the intersection diagram of the graph $\Gamma$ with respect to $(\alpha, \beta)$'.

In this section, we assume that $\Gamma$ is a distance-regular graph with $r = l(1, 0, k-1) \geq 2$, $c_{2r+1} = 1$ and $a_{r+1} = a \geq 2$.

Lemma 2.1. *Let $\alpha$ and $\beta$ be adjacent vertices in $\Gamma$. Let $q$ be an integer with $1 \leq q \leq r$. Then:*
(1) *The $(\alpha, \beta)$-diagram has the following shape:*



*Moreover, there exists no set of 3-vertices $\{x, y, z\}$ with $x \sim y \sim z$, $x \in D^{t-1}_{t-1}$, $y \in D^t_t$ and $z \in D^t_{t-1} \cup D^{t-1}_t$ for $t \leq 2r+1$.*
(2) *If $z_0 \sim z_1 \sim \cdots \sim z_r$ is a walk of length $r$ with $\partial_\Gamma(\alpha, z_j) = r+1+j$ for $0 \leq j \leq r$ and $z_0 \in \Gamma_{r+1}(\beta)$, then $\partial_\Gamma(\beta, z_j) = r+1+j$ for $1 \leq j \leq r$.*
(3) *If $z_0 \sim z_1 \sim \cdots \sim z_r$ is a walk of length $r$ with $\partial_\Gamma(\alpha, z_j) = r+1+j$ for $0 \leq j \leq r$ and $z_0 \in \Gamma_r(\beta)$, then $\partial_\Gamma(\beta, z_j) = r+j$ for $1 \leq j \leq r$.*
(4) *If $x_0 \sim x_1 \sim \cdots \sim x_l$ is a walk of length $l$ with $\partial_\Gamma(\alpha, x_j) = q-1+j$ for $0 \leq j \leq l$ and $x_0 \in \Gamma_q(\beta)$, then $\partial_\Gamma(\beta, x_j) = q+j$ for $0 \leq j \leq \min\{l, r+1-q\}$.*
(5) *If $x_0 \sim x_1 \sim \cdots \sim x_l$ is a path of length $l$ with $x_0 \in \Gamma_q(\beta)$ and $x_1 \in \Gamma_{q+1}(\beta)$, then $\partial_\Gamma(\beta, x_j) = q+j$ for $0 \leq j \leq \min\{l, r+1-q\}$.*

Proof. (1) See [2] and [5].
(2) Consider the $(\alpha, \beta)$-diagram. We have $z_0 \in D^{r+1}_{r+1}$ from our assumption. Since $z_1 \sim z_0$ and $\partial_\Gamma(\alpha, z_1) = r+2$, we obtain $z_1 \in D^{r+2}_{r+2}$. Inductively, we obtain $z_j \in D^{r+1+j}_{r+1+j}$ for $2 \leq j \leq r$. This implies our assertion.
(3, 4) We have our assertion, similar to (2).
(5) Since $c_{q+j} = 1$ and $a_{q+j} = 0$ for $1 \leq j \leq r-q$, we have the assertion. □

Let $X$ be a set of vertices. We identify $X$ with the induced subgraph on $X$.

Definition 2.2. Let $X, Y \subset \Gamma$ and $p, q$ be positive integers. A nonempty set $X$ is the *$(p, q)$-subgraph with respect to $Y$* if the following conditions hold:
(1) $k_X(z) \geq p$ for any $z \in X$;
(2) $\partial_\Gamma(x, y) \leq q$ for any $x \in X$ and $y \in Y$.

Lemma 2.3. *Let $w, z \in \Gamma$ and $\Lambda$ be an $(a, r+1)$-subgraph with respect to $\{w\}$. Then:*
(1) *if $z \in \Lambda \cap \Gamma_s(w)$ for some $1 \leq s \leq r$, there exists $x \in \Lambda_1(z) \cap \Gamma_{s+1}(w)$;*
(2) *$\Lambda \cap \Gamma_{r+1}(w) \neq \varnothing$.*

Proof. (1) Suppose that $\Lambda_1(z) \cap \Gamma_{s+1}(w) = \varnothing$. We have $\Lambda_1(z) \subset \Gamma_s(w) \cup \Gamma_{s-1}(w)$. Thus

$$2 \leq a \leq k_\Lambda(z) \leq a_s + c_s = 1.$$

This is a contradiction.

(2) Let $m = \max\{j \mid \Lambda \cap \Gamma_j(w) \neq \emptyset\}$. We have $1 \leq m \leq r+1$ from our assumption. From the maximality of $m$, we have $z \in \Lambda \cap \Gamma_m(w)$ and $\Lambda \cap \Gamma_{m+1}(w) = \emptyset$. Suppose that $m \leq r$. Then we have a contradiction from (1). Thus $m = r+1$. □

LEMMA 2.4. *Let $\alpha$, $\beta$ be adjacent vertices in $\Gamma$. Let $\Lambda$ be an $(a+1, r+1)$-subgraph with respect to $\{\alpha\}$ such that $\alpha$, $\beta \in \Lambda$. If $\partial_\Gamma(\alpha, x) = \partial_\Lambda(\alpha, x)$ for any $x \in \Lambda$, then $\partial_\Gamma(\beta, x) = \partial_\Lambda(\beta, x)$ for any $x \in \Lambda$.*

PROOF. Consider the $(\alpha, \beta)$-diagram. Let

$$L = \bigcup_{i=0}^{r+1} D_{i+1}^{i}, \qquad R = \bigcup_{i=0}^{r} D_{i}^{i+1}.$$

From our assumption, we have $D_j^i \cap \Lambda = \emptyset$ for $i \geq r+2$. Thus $\Lambda \subset L \cup R \cup D_{r+1}^{r+1}$. Take any $x \in \Lambda$. Note that $\partial_\Gamma(\alpha, x) = \partial_\Lambda(\alpha, x)$ iff $p[\alpha, x] \subset \Lambda$. Thus it is sufficient to show that $p[\beta, x] \subset \Lambda$. If $x \in L$, then

$$p[\beta, x] = \{\beta\} \cup p[\alpha, x] \subset \Lambda.$$

If $x \in R$, then

$$p[\beta, x] \subset p[\alpha, x] \subset \Lambda.$$

Assume that $x \in D_{r+1}^{r+1}$ and set $\{y\} = \Gamma_1(x) \cap D_r^{r+1}$. We show that $y \in \Lambda$. If $y \notin \Lambda$, then $\Lambda_1(x) \subset D_{r+1}^{r+1} \cup D_{r+1}^{r}$. Thus we have

$$a+1 \leq k_1(x) \leq e(x, D_{r+1}^{r+1} \cup D_{r+1}^{r}) = a.$$

This is a contradiction. Thus we obtain $y \in \Lambda$. Since $y \in R \cap \Lambda$, we have

$$p[\beta, x] = p[\beta, y] \cup \{x\} \subset \Lambda.$$

This is the desired result. □

## 3. PAIRS OF WALKS

Let $m$ and $n$ be positive integers and $X = (x_1, \ldots, x_m)$ and $Y = (y_1, \ldots, y_n)$ be sequences of vertices of $\Gamma$. The *distance matrix* $M[x_1, \ldots, x_m; y_1, \ldots, y_n]$ on $X$ and $Y$ is an $m \times n$ matrix the $(i, j)$th entry of which is $\partial_\Gamma(x_i, y_j)$. Let

$$J = \begin{pmatrix} r+1 & r+1 \\ r+1 & r+1 \end{pmatrix},$$

$$E_1 = \begin{pmatrix} r & r+1 \\ r+1 & r+1 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} r+1 & r \\ r+1 & r+1 \end{pmatrix},$$

$$E_3 = \begin{pmatrix} r+1 & r+1 \\ r & r+1 \end{pmatrix} \quad \text{and} \quad E_4 = \begin{pmatrix} r+1 & r+1 \\ r+1 & r \end{pmatrix}.$$

Let $s$, $t$ and $l$ be non-negative integers. Let

$$X: x_0 \sim x_1 \sim \cdots \sim x_{2l}, \qquad Y: y_0 \sim y_1 \sim \cdots \sim y_{l+s}$$

and

$$W: w_0 \sim w_1 \sim \cdots \sim w_t$$

be walks of length $2l$, $l+s$ and $t$, respectively.

In this paper, walks are ordered, i.e. the following are considered to be different:

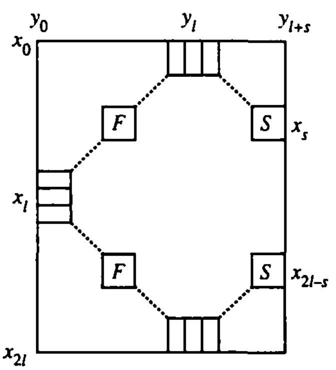
$$W: w_0 \sim w_1 \sim \cdots \sim w_t \quad \text{and} \quad W': w_t \sim w_{t-1} \sim \cdots \sim w_0.$$

DEFINITION 3.1. A pair of walks $(X, Y)$ is of *type C of size* $(l, s)$ if the following conditions hold:
(1) $0 \leq s < l$;
(2) $F_i = M[x_i, x_{i+1}; y_{l-i-1}, y_{l-i}] \in \{J, E_1, E_4\}$ for $0 \leq i \leq l-1$;
(3) $F_{l+i} = M[x_{l+i}, x_{l+i+1}; y_i, y_{i+1}] \in \{J, E_2, E_3\}$ for $0 \leq i \leq l-1$;
(4) $S_i = M[x_i, x_{i+1}; y_{l+i}, y_{l+i+1}] \in \{J, E_2, E_3\}$ for $0 \leq i \leq s-1$;
(5) $S_{2l-i} = M[x_{2l-i-1}, x_{2l-i}; y_{l+i}, y_{l+i+1}] \in \{J, E_1, E_4\}$ for $0 \leq i \leq s-1$.

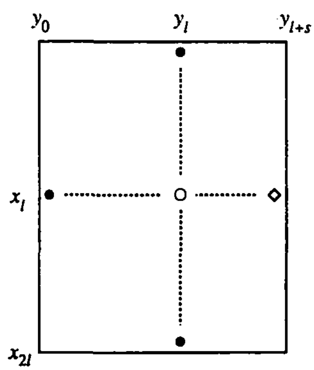
DEFINITION 3.2. A pair of walks $(X, Y)$ is of *type* $C^*$ *of size* $(l, s)$ if the following conditions hold:
(1) $0 \leq s < l \leq r$;
(2) $\partial_\Gamma(y_l, x_{l-j}) = \partial_\Gamma(y_l, x_{l+j}) = r+1-l+j$ for $0 \leq j \leq l$;
(3) $\partial_\Gamma(x_l, y_j) = r+1-j$ for $0 \leq j \leq l$;
(4) $\partial_\Gamma(x_l, y_{l+j}) = r+1-l+j$ for $0 \leq j \leq s$;
(5) $\partial_\Gamma(x_i, y_j) \leq r+1$ for any $i$, $j$.

REMARK. The above definitions give us the following entries of the distance matrix on $X$ and $Y$:

Type $C$




Type $C^*$




where $\bullet = r+1$, $\circ = r+1-l$ and $\diamond = r+1-l+s$.

For the rest of this paper we use the notation $(X, Y)$ for a pair of walks $(X, Y)$.

LEMMA 3.3. *Let* $(X, Y)$ *be of type* $C^*$ *of size* $(l, s)$. *Then the following hold*:
(1) $\partial_\Gamma(y_{l-i}, x_{l-j}) = r+1-l+i+j$ *for* $0 \leq i \leq l$ *and* $0 \leq j \leq l-i$;
(2) $\partial_\Gamma(y_{l-i}, x_{l+j}) = r+1-l+i+j$ *for* $0 \leq i \leq l$ *and* $0 \leq j \leq l-i$;
(3) $\partial_\Gamma(y_{l+i}, x_{l-j}) = r+1-l+i+j$ *for* $0 \leq i \leq s$ *and* $0 \leq j \leq l-i$;
(4) $\partial_\Gamma(y_{l+i}, x_{l+j}) = r+1-l+i+j$ *for* $0 \leq i \leq s$ *and* $0 \leq j \leq l-i$;
(5) $F_i = M[x_i, x_{i+1}; y_{l-i-1}, y_{l-i}] = E_4$ *for* $0 \leq i \leq l-1$;
(6) $F_{l+i} = M[x_{l+i}, x_{l+i+1}; y_i; y_{i+1}] = E_2$ *for* $0 \leq i \leq l-1$;
(7) $S_i = M[x_i, x_{i+1}; y_{l+i}, y_{l+i+1}] = E_3$ *for* $0 \leq i \leq s-1$;
(8) $S_{2l-i} = M[x_{2l-i-1}, x_{2l-i}; y_{l+i}, y_{l+i+1}] = E_1$ *for* $0 \leq i \leq s-1$.

PROOF. (1) We prove the assertion by induction on $i$. The case $i = 0$ follows from

Definition 3.2(2). Let $1 \leq h \leq l$ and set $q = r+1-l+h$. We have $x_l \in \Gamma_q(y_{l-h})$ from Definition 3.2(3). From the inductive assumption, we have $\partial_\Gamma(y_{l-h+1}, x_{l-j}) = q-1+j$ for $0 \leq j \leq l-h+1$. Thus we have $\partial_\Gamma(y_{l-h}, x_{l-j}) = q+j$ for $0 \leq j \leq l-h$ from Lemma 2.1(4). This is the desired result.

(2–4) We have our assertions, similar to (1).

(5) Let $0 \leq i \leq l-1$ and consider the $(x_i, x_{i+1})$-diagram. We have $y_{l-i} \in D_r^{r+1}$ and $y_{l-i-1} \in \Gamma_{r+1}(x_{i+1})$ from (1). Since $\partial_\Gamma(y_{l-i-1}, x_i) \leq r+1$ from Definition 3.2(5), we obtain $y_{l-i-1} \in D_{r+1}^{r+1}$. This is the desired result.

(6–8) We have our assertions, similar to (5). □

COROLLARY 3.4. *If $(X, Y)$ is of type $C^*$ of size $(l, s)$, then $(X, Y)$ is of type $C$ of size $(l, s)$.*

PROOF. This is a direct consequence of Lemma 3.3. □

DEFINITION 3.5. Let $u$, $\gamma_1$, $\gamma_2$, $\delta_1$, $\delta_2$ be mutually distinct vertices in $\Gamma$. The quintuple $(u, \gamma_1, \gamma_2, \delta_1, \delta_2)$ is a *basis* if the following conditions hold:

(1) $\gamma_1, \gamma_2, \delta_1, \delta_2 \in \Gamma_{r+1}(u)$.

(2) $\gamma_1 \sim \gamma_2$ and $\delta_1 \sim \delta_2$.

(3) Let

$$P_1 = p[u, \gamma_1] \cap p[u, \delta_1] = \{u = u_0 \sim u_1 \sim \cdots \sim u_l\},$$
$$P_2 = p[u, \gamma_2] \cap p[u, \delta_2] = \{u = v_0 \sim v_1 \sim \cdots \sim v_h\}.$$

Then $P_1 \cap P_2 = \{u\}$ and $l + h = r + 1$.

The following lemma shows that we obtain two pairs of paths of type $C^*$ from a basis.

LEMMA 3.6. *Let $(u, \gamma_1, \gamma_2, \delta_1, \delta_2)$ be a basis. Set*

$$p[u, \gamma_1] = \{u = u_0, u_1, \ldots, u_l = \beta_h, \beta_{h-1}, \beta_{h-2}, \ldots, \beta_0 = \gamma_1\},$$
$$p[u, \delta_1] = \{u = u_0, u_1, \ldots, u_l = \beta_h, \beta_{h+1}, \beta_{h+2}, \ldots, \beta_{2h} = \delta_1\},$$
$$p[u, \gamma_2] = \{u = v_0, v_1, \ldots, v_h = x_l, x_{l-1}, x_{l-2}, \ldots, x_0 = \gamma_2\},$$
$$p[u, \delta_2] = \{u = v_0, v_1, \ldots, v_h = x_l, x_{l+1}, x_{l+2}, \ldots, x_{2l} = \delta_2\},$$
$$U: u_l \sim u_{l-1} \sim \cdots \sim u_0, \qquad B: \beta_0 \sim \beta_1 \sim \cdots \sim \beta_{2h},$$
$$V: v_h \sim v_{h-1} \sim \cdots \sim v_0 \quad \textit{and} \quad X: x_0 \sim x_1 \sim \cdots \sim x_{2l}.$$

*Then*:

*(1) $(X, U)$ is of type $C^*$ of size $(l, 0)$;*

*(2) $(B, V)$ is of type $C^*$ of size $(h, 0)$.*

PROOF. Consider the distance matrices on $X$ and $U$ (resp. $B$ and $V$). It is easy to check the conditions of Definition 3.2. □

LEMMA 3.7. *Let $\alpha$ and $\beta$ be adjacent vertices in $\Gamma$. Let $\Lambda$ be an $(a, r+1)$-subgraph with respect to $\{\alpha, \beta\}$. If $x \in \Lambda \cap \Gamma_{r+1}(\alpha)$, then there exists a vertex $z \in \Lambda_1(x)$ with $M[z, x; \alpha, \beta] \in \{J, E_1, E_4\}$.*

PROOF. Since $2 \leq a \leq k_\Lambda(x)$, we have $z_1, z_2 \in \Lambda_1(x)$. Consider the $(\alpha, \beta)$-diagram. Note that $(D_{r+2}^{r+2} \cup D_{r+1}^{r+2} \cup D_{r+2}^{r+1}) \cap \Lambda = \varnothing$ and $x \in D_{r+1}^{r+1} \cup D_r^{r+1}$ from our assumption. First, we assume that $x \in D_r^{r+1}$. Since $e(x, D_{r-1}^r \cup D_{r+1}^r) = c_r + a_r = 1$ and $\Lambda \cap D_{r+1}^{r+2} = \varnothing$, at least one of $z_1$ and $z_2$ is in $D_{r+1}^{r+1}$. So we have $z \in D_{r+1}^{r+1} \cap \Lambda_1(x)$. Then it is easy to show that $M[z, x; \alpha, \beta] = E_4$. Second, we assume that $x \in D^{r+1}_{r+1}$. If $z_i \in D^{r+1}_{r+1}$ for some $i \in \{1, 2\}$, then we have $z = z_i \in \Lambda_1(x)$ with $M[z, x; \alpha, \beta] = J$. Suppose that $z_i \notin D^{r+1}_{r+1}$ for $i = 1,2$. Since $e(x, D^r_{r+1}) = e(x, D^{r+1}_r) = 1$ and $D^{r+2}_{r+2} \cap \Lambda = \emptyset$, we may assume that $z_1 \in D^r_{r+1}$ and $z_2 \in D^{r+1}_r$. Thus we have $z_1 \in \Lambda_1(x)$ with $M[z_1, x; \alpha, \beta] = E_1$. Hence we have our assertion. □

Next, we obtain a sufficient condition to yield a pair of walks of type $C$.

DEFINITION 3.8. A pair of walks $(X, Y)$ is *partially of type C of size* $(l, s)$ if the following conditions hold:
(1) Let $Y': y_0 \sim y_1 \sim \cdots \sim y_l$. Then $(X, Y')$ is of type $C^*$ of size $(l, 0)$.
(2) There exist $(a, r+1)$-subgraphs $\Lambda$, $\Lambda'$ with respect to $\{y_l, y_{l+1}, \ldots, y_{l+s}\}$ such that $x_0 \in \Lambda$ and $x_{2l} \in \Lambda'$, where $\Lambda$ and $\Lambda'$ need not to be different.

We construct a pair of walks of type $C$ from the one which is partially of type $C$. Let $(X, Y)$ be partially of type $C$ of size $(l, s)$. We may assume that $0 < s$. Let $\alpha_{j+s} = x_j$ for $0 \leq j \leq 2l$ and $\Lambda$ and $\Lambda'$ be the subgraphs as in Definition 3.8(2). Since $\partial_\Gamma(y_l, \alpha_s) = r + 1$, we obtain

$$\alpha_{s-1} \in \Lambda_1(\alpha_s) \qquad \text{with } M[\alpha_{s-1}, \alpha_s; y_l, y_{l+1}] \in \{J, E_1, E_4\}$$

from Lemma 3.7. Hence we have $\partial_\Gamma(\alpha_{s-1}, y_{l+1}) = r + 1$. Inductively, we can take

$$\alpha_{s-j} \in \Lambda_1(\alpha_{s-j+1}) \qquad \text{with } M[\alpha_{s-j}, \alpha_{s-j+1}; y_{l+j-1}, y_{l+j}] \in \{J, E_1, E_4\}$$

for $2 \leq j \leq s$. Similarly, we obtain

$$\alpha_{2l+s+j} \in \Lambda_1'(\alpha_{2l+s+j-1}) \qquad \text{with } M[\alpha_{2l+j}, \alpha_{2l+j-1}; y_{l+j-1}, y_{l+j}] \in \{J, E_1, E_4\}$$

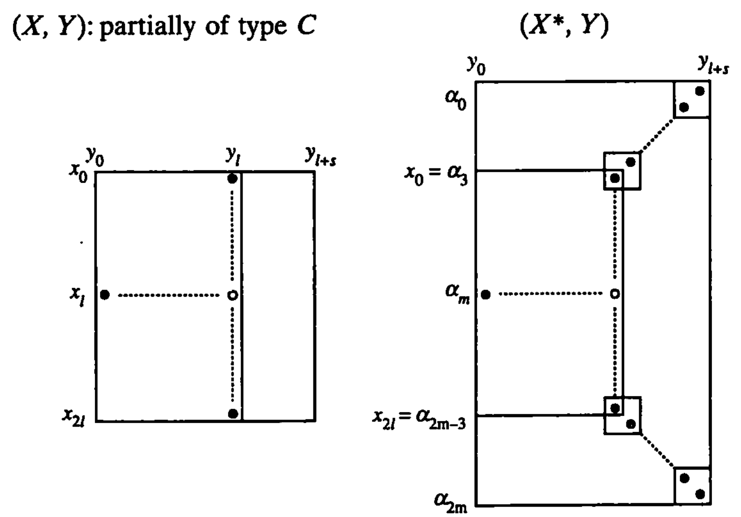
for $1 \leq j \leq s$. Note that

$$M[x, y, z, w] \in \{J, E_1, E_4\} \qquad \text{iff } M[y, x, z, w] \in \{J, E_2, E_3\}.$$

Let $m = l + s$ and $X^*$: $\alpha_0 \sim \alpha_1 \sim \cdots \sim \alpha_{2m}$ be a walk of length $2m$.

PROPOSITION 3.9. *Let $(X, Y)$ be partially of type C of size $(l, s)$, $m = l + s$ and $X^*$ be a walk of length $2m$ defined above. Then $(X^*, Y)$ is of type C of size $(m, 0)$.*

PROOF. This is a direct consequence of definition of $X^*$ and Lemma 3.3. □




where $\bullet = r + 1$ and $\circ = r + 1 - l$.

Definition 3.10. A pair of walks $(X, W)$ is *partially of type $C^*$ of size $(l, t, f)$* if the following conditions hold:
(1) $\partial_\Gamma(w_0, x_l) = r + 1$.
(2) $\partial_\Gamma(w_t, x_{l+j}) = \partial_\Gamma(w_t, x_{l-j}) = r + 1 - l + j$ for $0 \leq j \leq l$.
(3) $\partial_\Gamma(w_i, x_j) \leq r + 1$ for any $i, j$.
(4) There exists an integer $f$ with $0 \leq f \leq t$ and $l + f \leq r$ which satisfies $\partial_\Gamma(x_l, w_{t-j}) = r + 1 - l - j$ for $0 \leq j \leq f$, $\partial_\Gamma(x_l, w_{t-f-1}) = r + 2 - l - f$ and $\partial_\Gamma(w_0, w_{t-f}) = t - f$.
(5) There exist $(a, r+1)$-subgraphs $\Lambda$, $\Lambda'$ with respect to $\{w_0, w_1, \ldots, w_t\}$ such that $x_0 \in \Lambda$ and $x_{2l} \in \Lambda'$, where $\Lambda$ and $\Lambda'$ need not to be different.

We can make up a pair of walks of type $C^*$ from the one partially of type $C^*$. To show this, we investigate $(X, W)$ which is partially of type $C^*$.

Lemma 3.11. *Let $(X, W)$ be partially of type $C^*$ of size $(l, t, f)$. Then the following hold*:
(1) $l + 2f \leq t$;
(2) $\partial_\Gamma(x_l, w_{t-f+j}) = r + 1 - l - f + j$ *for* $0 \leq j \leq f$;
(3) $\partial_\Gamma(x_l, w_{t-f-j}) = r + 1 - l - f + j$ *for* $0 \leq j \leq l + f$;
(4) $\partial_\Gamma(x_{l+i}, w_{t-f}) = r + 1 - l - f + i$ *for* $0 \leq i \leq l$;
(5) $\partial_\Gamma(x_{l-i}, w_{t-f}) = r + 1 - l - f + i$ *for* $0 \leq i \leq l$.

Proof. (1) We have

$$r + 1 = \partial_\Gamma(w_0, x_l) \leq \partial_\Gamma(w_0, w_{t-f}) + \partial_\Gamma(w_{t-f}, x_l) = (t - f) + (r + 1 - l - f)$$

from Definition 3.10(1), (4). The assertion follows.

(2) The assertion follows from Definition 3.10(4).

(3) Since $\partial_\Gamma(w_0, w_{t-f}) = t - f$, we have $w_{t-f} \sim w_{t-f-1} \sim \cdots \sim w_0$ is a path of length $t - f$. Hence the assertion follows from Lemma 2.1(5).

(4) From Definition 3.10(2), we have $\partial_\Gamma(w_t, x_{l+i}) = r + 1 - l + i$. Thus

$$r + 1 - l + i = \partial_\Gamma(w_t, x_{l+i}) \leq \partial_\Gamma(w_t, w_{t-f}) + \partial_\Gamma(w_{t-f}, x_{l+i}) \leq f + \partial_\Gamma(w_{t-f}, x_{l+i}).$$

On the other hand, we obtain

$$\partial_\Gamma(w_{t-f}, x_{l+i}) \leq \partial_\Gamma(w_{t-f}, x_l) + \partial_\Gamma(x_l, x_{l+i}) \leq (r + 1 - l - f) + i$$

from (2). Hence the assertion follows.

(5) We have our assertion, similar to (4). □

Let $\delta_j = w_{t-l-2f+j}$ for $0 \leq j \leq l + 2f$ and $W^*$: $\delta_0 \sim \delta_1 \sim \cdots \sim \delta_{l+2f}$. For the case $f = 0$, we set $\eta_i = x_i$ for $0 \leq i \leq 2l$ and $X^* = X$. For the case $f \geq 1$, we set $\eta_{f+i} = x_i$ for $0 \leq i \leq 2l$ and $m = l + f$. Then we have

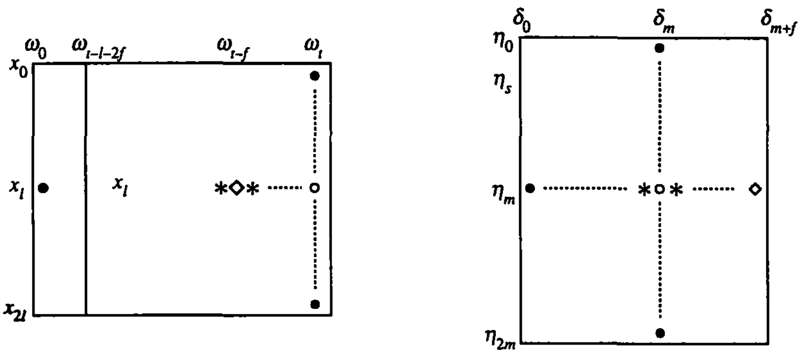
$$\partial_\Gamma(\eta_f, \delta_m) = \partial_\Gamma(x_0, w_{t-f}) = r + 1 - f \leq r$$

from Lemma 3.11(5). Let $\Lambda$ and $\Lambda'$ be subgraphs as in Definition 3.10(5). Since $\eta_f \in \Lambda \cap \Gamma_{r+1-f}(\delta_m)$, we obtain $\eta_{f-1} \in \Lambda_1(\eta_f) \cap \Gamma_{r+2-f}(\delta_m)$ from Lemma 2.3(1). Inductively, we have $\eta_{f-j} \in \Lambda_1(\eta_{f-j+1}) \cap \Gamma_{r+1-f+j}(\delta_m)$ for $2 \leq j \leq f$. Similarly, we have $\eta_{2l+f+j} \in \Lambda_1'(\eta_{2l+j-1}) \cap \Gamma_{r+1-f+j}(\delta_m)$ for $1 \leq j \leq f$. Let $X^*$: $\eta_0 \sim \eta_1 \sim \cdots \sim \eta_{2m}$ be a walk of length $2m$.

Proposition 3.12. *Let $(X, W)$ be partially of type $C^*$ of size $(l, t, f)$. Let $X^*$ and $W^*$ be walks which are defined above. Then $(X^*, W^*)$ is of type $C^*$ of size $(m, f)$ where $m = l + f$.*

PROOF. This is a direct consequence of Lemma 3.11 and the definition of $X^*$ and $W^*$. □

$(X, W)$: partially of type $C^*$ $\quad$ $(X^*, W^*)$




where ● $= r+1$, ○ $= r+1-l$, ◇ $= r+1-l-f$ and $*$ $= r+2-l-f$.

## 4. A Family of Minimal Circuits

In this section, we construct a nice family of minimal circuits from a pair of walks of type $C$ and we show that it gives us a lot of information about distance relations of their neighbors.

In this paper, we define a *minimal circuit* as a circuit of length $g = 2r+3$. Let $x$, $z \in \Gamma$ with $\partial_\Gamma(x, z) = r+1$. Set $H(x, z) = A(x, z)$. It is clear that $p[x, z] \cup p[x, w]$ forms a minimal circuit when $w \in H(x, z)$.

LEMMA 4.1. *Let $\alpha, \beta, \gamma, x_r \in \Gamma$ with $\alpha \sim \beta$, $\partial_\Gamma(\gamma, x_r) = r+1$ and $p[\gamma, x_r] = \{\gamma \sim x_0 \sim x_1 \sim \cdots \sim x_r\}$. Let $\xi \in H(\gamma, x_r)$ and $p[\gamma, \xi] = \{y_j\}_j$, then in the $(\alpha, \beta)$-diagram the following hold*:
(1) *if $\gamma \in D_{r+1}^{r+1}$ and $x_j \in D_{r+j}^{r+1+j}$ for $0 \leq j \leq r$, then $y_j \in D_{r+1+j}^{r+1+j}$ for $0 \leq j \leq r$;*
(2) *if $\gamma \in D_{r+1}^{r+1}$ and $x_j \in D_{r+1+j}^{r+1+j}$ for $0 \leq j \leq r$, then $y_j \in D_{r+1+j}^{r+1+j}$ for $0 \leq j \leq r$;*
(3) *if $\gamma \in D_{r+1}^{r}$ and $x_j \in D_{r+1+j}^{r+1+j}$ for $0 \leq j \leq r$, then $y_j \in D_{r+1+j}^{r+j}$ for $0 \leq j \leq r$.*

PROOF. (1) It is easy to see that $\xi \in D_{2r+1}^{2r+2} \cup D_{2r+1}^{2r+1}$ and $y_j \in D_{r+1+j}^{r+1+j}$ for $0 \leq j \leq r$ from Lemma 2.1(1).

(2) Let $\{\delta\} = \Gamma_1(\beta) \cap \Gamma_r(x_0)$ and consider the $(\beta, \delta)$-diagram. It is clear that $x_0 \in D_r^{r+1}$ and $\gamma \in D_{r+1}^{r+1}$. From our assumption and Lemma 2.1(3), we have $x_j \in D_{r+j}^{r+1+j}$ for $1 \leq j \leq r$. We obtain the locations of the $y_j$'s in the $(\beta, \delta)$-diagram from (1). In particular, we have $\partial_\Gamma(\beta, y_j) = r+1+j$ for $1 \leq j \leq r$. Since $y_0 = \gamma \in \Gamma_{r+1}(\alpha)$, we obtain $\partial_\Gamma(\alpha, y_j) = r+1+j$ for $1 \leq j \leq r$ from Lemma 2.1(2). This is the desired result.

(3) We have our assertion, similar to (2). □

PROPOSITION 4.2. *Let $(X, Y)$ be of type $C$ of size $(l, s)$ and $v \in \Gamma_1(y_{l+s})$. If $\partial_\Gamma(x_s, v) = r+2$, then $\partial_\Gamma(x_{2l-s}, v) = r+2$.*

PROOF. First, we assume that $s \geq 1$. Let $m = l+s$, $v_1^0 = v \in \Gamma_{r+2}(x_s)$ and take $v_i^0 \in \Gamma_{r+1+i}(x_s) \cap \Gamma_1(v_{i-1}^0)$ for $2 \leq i \leq r$. Then we have $\partial_\Gamma(y_{m-1}, v_r^0) = r+1$. Take $\xi_0 \in H(y_{m-1}, v_r^0)$ and let $p[y_{m-1}, \xi_0] = \{v_j^1\}_j$.

SUBLEMMA 4.3. $\partial_\Gamma(x_{s-1}, v_i^1) = r+1+i$ *for* $0 \leq i \leq r$.

PROOF. Note that $\partial_\Gamma(x_s, v_i^0) = r+1+i$ for $0 \leq i \leq r$ and

$$S_{s-1} = M[x_{s-1}, x_s; y_{m-1}, y_m] \in \{J, E_2, E_3\}.$$

Consider the $(x_s, x_{s-1})$-diagram. Suppose that $S_{s-1} = J$. We have $v_i^0 \in D_{r+1+i}^{r+1+i}$ for $0 \leq i \leq r$ from Lemma 2.1(2). Note that $v_0^1 = y_{m-1} \in D_{r+1}^{r+1}$, $\xi_0 \in H(y_{m-1}, v_r^0)$ and $p[y_{m-1}, \xi_0] = \{v_j^1\}_j$. We have $v_i^1 \in D_{r+1+i}^{r+1+i}$ for $0 \leq i \leq r$ from Lemma 4.1(2). Hence the assertion follows. Suppose that $S_{s-q} = E_2$. In the same manner, we have our assertion from Lemmas 2.1(3) and 4.1(1). Suppose that $S_{s-q} = E_3$. Similarly, we have the assertion from Lemmas 2.1(2) and 4.1(3). Thus we obtain the desired result. ■

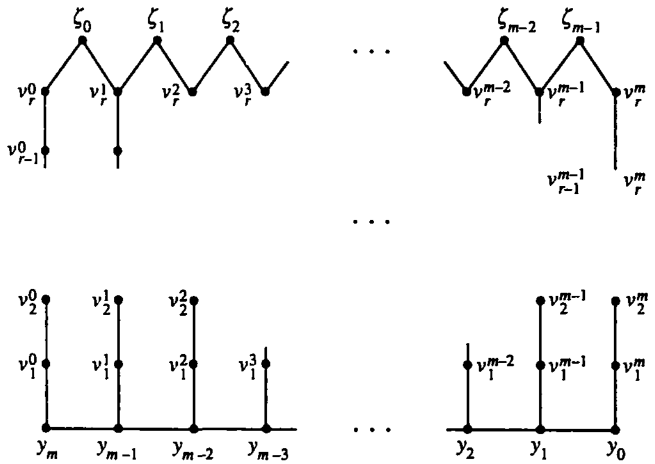
Now consider the distance of $y_{m-2}$ and $v_r^1$ from $x_{s-1}$: we have $\partial_\Gamma(y_{m-1}, v_r^1) = r+1$. Hence, inductively, the following results hold:

SUBLEMMA 4.4. *Take* $\xi_i \in H(y_{m-i-1}, v_r^i)$ *and let* $p[y_{m-i-1}, \xi_i] = \{v_j^{i+1}\}_j$ *for* $1 \leq i \leq m-1$. *Then:*

(1) $\partial_\Gamma(x_{s-f}, v_i^f) = r+1+i$ *for* $0 \leq f \leq s$, $0 \leq i \leq r$;
(2) $\partial_\Gamma(x_f, v_i^{s+f}) = r+1+i$ *for* $0 \leq f \leq l$, $0 \leq i \leq r$;
(3) $\partial_\Gamma(x_{l+f}, v_i^{l+s-f}) = r+1+i$ *for* $0 \leq f \leq l$, $0 \leq i \leq r$;
(4) $\partial_\Gamma(x_{2l-f}, v_i^{s-f}) = r+1+i$ *for* $0 \leq f \leq s$, $0 \leq i \leq r$.

PROOF. We have our assertions, similar to Sublemma 4.3. ■

Hence, we have $\partial_\Gamma(x_{2l-s}, v) = \partial_\Gamma(x_{2l-s}, v_0^0) = r+1$, by sublemma 4.4.
Second, we treat the case $s = 0$. In this case, we also obtain the same results in Sublemma 4.4(2)(3). Hence we obtain the desired result. □




COROLLARY 4.5. *Let* $(u, \gamma_1, \gamma_2, \delta_1, \delta_2)$ *be a basis and* $z \in \Gamma_1(u)$. *If* $\partial_\Gamma(\gamma_2, z) = r+2$, *then* $\partial_\Gamma(\delta_2, z) = r+2$.

PROOF. From Lemma 3.6, we have a pair of paths $(X, U)$ of type $C^*$ of size $(l, 0)$.

Then $(X, U)$ is of type $C$ of size $(l, 0)$ from Corollary 3.4. The assertion follows from Proposition 4.2. □

COROLLARY 4.6. *Let $(X, W)$ be partially of type $C^*$ of size $(l, t, f)$ and $\xi \in \Gamma_1(w_t)$. If $\partial_\Gamma(x_0, \xi) = r+2$, then $\partial_\Gamma(x_{2l}, \xi) = r+2$.*

PROOF. From Proposition 3.12, $(X^*, W^*)$ is of type $C^*$ of size $(m, f)$, where $m = l + f$, $X^*$: $\eta_0 \sim \eta_1 \sim \cdots \sim \eta_{2m}$ and $W^*$: $\delta_0 \sim \delta_1 \sim \cdots \sim \delta_{l+2f} = w_t$. Note that $x_0 = \eta_f$ and $x_{2l} = \eta_{2m-f}$. Hence our assertion follows from Proposition 4.2. □

## 5. PROOF OF THE THEOREM

Our purpose in this section is to prove Theorem 1.1.

Let $\Gamma$ be a distance-regular graph with $r = l(1, 0, k-1) \geq 2$, $c_{2r+1} = 1$ and $a_{r+1} = a \geq 2$. Fix a vertex $u \in \Gamma$. Let $G = G[u]$ be the subgraph induced by $\Gamma_{r+1}(u)$. Set $G = G_0 \cup G_1 \cup \cdots \cup G_e$, where the $G_j$'s are connected components of $G$. It is clear that each $G_j$ is a connected regular graph of valency $a$. Next we define a graph $\Omega$ as follows:

DEFINITION 5.1. (1) The vertex set of $\Omega$ is $\{G_j \mid 0 \leq j \leq e\}$.

(2) $G_q$ and $G_t$ are adjacent in $\Omega$ if $q \neq t$ and there exist $x_1, x_2 \in G_q$ and $y_1, y_2 \in G_t$ with $(u, x_1, x_2, y_1, y_2)$ as a basis.

When $G_q$ and $G_t$ are adjacent in $\Omega$, we write $G_q \approx G_t$. Set $\Omega = \Omega_0 \cup \Omega_1 \cup \cdots \cup \Omega_n$ where $\Omega_j$'s are connected components of $\Omega$. Let

$$\Psi = \bigcup_{G_q \in \Omega_0} G_q, \qquad \Delta = \bigcup_{x \in \Psi} p[u, x].$$

In order to prove Theorem 1.1, it is sufficient to show that the graph $\Delta$ is a regular graph of valency $a+1$, diameter $r+1$ and girth $2r+3$.

LEMMA 5.2. (1) $g(\Delta) = g(\Gamma) = 2r+3$.
(2) $\partial_\Gamma(u, x) = \partial_\Delta(u, x)$ *for any* $x \in \Delta$.
(3) $\Gamma_j(u) \cap \Delta = \emptyset$ *for any* $j \geq r+2$.
(4) $d_\Delta(u) = r+1$.
(5) $k_\Delta(w) = a+1$ *for any* $w \in \Psi$.
(6) $k_\Delta(u) \geq a+1$.
(7) $k_\Delta(x) \geq a+1$ *for any* $x \in \Delta$.

PROOF. (1)–(4). The assertions follow from the definition of $\Delta$.

(5) Let $w \in \Psi$. Note that $\Psi \subset \Gamma_{r+1}(u)$. We have $C(u, w) \subset p[u, w] \subset \Delta$, $A(u, w) \subset \Psi \subset \Delta$ and $B(u, w) \cap \Delta = \emptyset$ since (3). It follows that

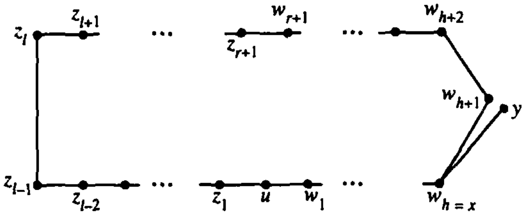
$$k_\Delta(w) = |\Delta_1(w)| = |C(u, w)| + |A(u, w)| = c_{r+1} + a_{r+1} = 1 + a.$$

(6) Take $w \in \Psi$. Let $\{x_1, x_2, \ldots, x_a\} = \Psi_1(w)$, $p[u, w] = \{w_j\}_j$ and $p[u, x_i] = \{z_j^i\}_j$. Since $g(\Gamma) = 2r+3$, $w_1 \neq z_1^i$ and $z_1^i \neq z_1^q$ for $1 \leq i, q \leq a$ and $i \neq q$. Note that $w_1 \in p[u, w] \subset \Delta$ and $z_1^i \in p[u, x_i] \subset \Delta$. Thus we have

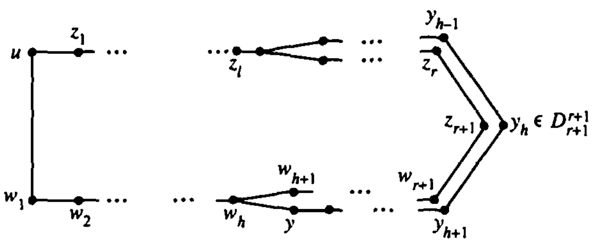
$$k_\Delta(u) \geq \#\{w_1\} + \#\{z_1^1, z_1^2, \ldots, z_1^a\} = 1 + a.$$

(7) Take $x \in \Delta$. Let $h = \partial_\Gamma(u, x)$. Then we have $h \leq r+1$ from (3). We may assume that $1 \leq h \leq r$ from (5) and (6). Since $x \in \Delta$, there exists $w_{r+1} \in \Psi$ with $x \in p[u, w_{r+1}] = \{w_j\}_j$. Note that $x = w_h$. Take $z_{r+1} \in A(u, w_{r+1})$ and let $p[u, z_{r+1}] = \{z_j\}_j$. Set

$l=r+1-h$. Then we have $\partial_\Gamma(x, z_l)=r+1$. Now we show that $A(z_l, x)\subset\Delta$. Take any $y\in A(z_l, x)$:




If $y=w_{h+1}$, then we have $y=w_{h+1}\in p[u, w_{r+1}]\subset\Delta$. Hence we may assume that $y\neq w_{h+1}$. Let $p[z_l, y]=\{y_j\}_j$. Note that $y_1\neq z_{l+1}$ because $g(\Gamma)=2r+3$. Now consider the $(u, w_1)$-diagram:




Since $z_{r+1}, w_{r+1}, y_h, y_{h+1}\in\Gamma_{r+1}(u)$, we have $z_{r+1}, w_{r+1}\in G_q$ and $y_h, y_{h+1}\in G_f$ for some $q$, $f$. Note that $G_q\in\Omega_0$ and $(u, z_{r+1}, w_{r+1}, y_h, y_{h+1})$ is a basis. If $G_q=G_f$, then $y_{h+1}\in G_q\subset\Psi$. If $G_q\neq G_f$, then we have $G_q\approx G_f$. Also we have $y_{h+1}\in G_f\subset\Psi$. Hence $y\in p[u, y_{h+1}]\subset\Delta$. Thus we have $A(z_l, x)\subset\Delta$, whence

$$k_\Delta(x)\geqslant\#\{w_{h-1}\}+\#A(z_l, x)=1+a. \qquad\square$$

PROPOSITION 5.3. *Let $z\in\Delta$. Then the following hold*:
(1) $\Gamma_j(z)\cap\Delta=\emptyset$ *for any* $j\geqslant r+2$;
(2) $\partial_\Gamma(z, x)=\partial_\Delta(z, x)$ *for any* $x\in\Delta$;
(3) $d_\Delta(z)=r+1$;
(4) $k_\Delta(x)=a+1$ *for any* $x\in\Delta_{r+1}(z)$.

We prove our assertion by induction on $h=\partial_\Gamma(u, z)$. For the case $h=0$, our assertion follows from Lemma 5.2. Let $0\leqslant t<r+1$. In the proof of the following lemmas we assume that our assertion is true for any $h\leqslant t$ and we show that our assertion is true for $h=t+1$. Take any $w\in\Delta_{t+1}(u)$ and set $p[u, w]=\{w_j\}_j$.

LEMMA 5.4. *Let $G_q\in\Omega_0$. In the $(w_t, w)$-diagram, exactly one of the following holds*:
(1) $G_q\cap(D^{r+1}_{r+1}\cup D^{r+1}_r)=\emptyset$;
(2) $G_q\cap D^{r+1}_{r+2}=\emptyset$.

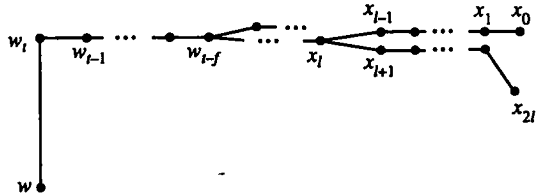
PROOF. From the inductive assumption, we have $\Gamma_i(w_t)\cap\Delta=\emptyset$ for $i\geqslant r+2$. Hence we obtain $D^i_j\cap\Delta=\emptyset$ for any $i\geqslant r+2$ and $\Delta$ is an $(a+1, r+1)$-subgraph with respect to $\{w_t\}$ from Lemma 5.2(7).

Suppose that both of (1) and (2) hold. Then we have a contradiction, from Lemma 2.3(2).

Suppose that neither (1) nor (2) holds. Take $\alpha \in G_q \cap D_{r+2}^{r+1}$ and $\beta \in G_q \cap (D_{r+1}^{r+1} \cup D_r^{r+1})$. Since $G_q$ is connected, there exists a path in $G_q$ connecting them:

$$\alpha = x_0 \sim x_1 \sim \cdots \sim x_p = \beta.$$

Without loss of generality, we may assume that $x_j \notin D_{r+2}^{r+1}$ for $j \neq 0$. Let $l = \max\{j \mid x_j \in D_{r+2-j}^{r+1-j}\}$. Then $x_0 \in D_{r+2}^{r+1}$, $x_{2l} \in D_{r+1}^{r+1}$ and $x_j$, $x_{2l-j} \in D_{r+2-j}^{r+1-j}$ for $1 \leq j \leq l$. Set $X: x_0 \sim x_1 \sim \cdots \sim x_{2l}$ and $W: w_0 \sim w_1 \sim \cdots \sim w_t$. Now we show that $(X, W)$ is partially of type $C^*$. Since $x_l \in G_q \subset \Gamma_{r+1}(u)$ and $u = w_0$, we have $\partial_\Gamma(w_0, x_l) = r+1$. We have $\partial_\Gamma(w_t, x_{l+j}) = \partial_\Gamma(w_t, x_{l-j}) = r+1-l+j$ for $1 \leq j \leq l$ from the locations of the $x_i$'s in the $(w_t, w)$-diagram. Since $w_i$, $x_j \in \Delta$, we obtain $\partial_\Gamma(w_i, x_j) = \partial_\Delta(w_i, x_j) \leq d_\Delta(w_i) = r+1$ for any $i, j$ from the inductive assumption. Let $f = \max\{i \mid w_{t-i} \in p[x_l, w_t]\}$ and $\Lambda = \Lambda' = G_q$. It is easy to see that they satisfy the conditions of Definition 3.10(4), (5). Hence we have that $(X, W)$ is partially of type $C^*$ of size $(l, t, f)$.




Note that $\partial_\Gamma(w, x_0) = r+2$ and $\partial_\Gamma(w, x_{2l}) \neq r+2$. This contradicts Corollary 4.6. Hence we obtain the desired result.

Now we say that $G_q$ is *of type* (1) (resp. *of type* (2)) with respect to $w$, if $G_q$ satisfies the condition of the case (1) (resp. (2)) in Lemma 5.4.

LEMMA 5.5. $\Gamma_{r+2}(w) \cap \Psi = \emptyset$.

PROOF. Let $W: w_0 \sim w_1 \sim \cdots \sim w_t$ be a path of length $t$. Since $w \in \Delta$, we have $y \in \Psi$ with $w \in p[u, y]$. Let $G_h \in \Omega_0$ such that $y \in G_h$. Consider the $(w_t, w)$-diagram. Then we have $y \in D_{r-t}^{r+1-t}$. It is easy to see that $G_h$ is of type (2) with respect to $w$. Suppose that there exists $x \in \Gamma_{r+2}(w) \cap \Psi$. Let $G_q \in \Omega_0$ such that $x \in G_q$. Since $x \in D_{r+2}^{r+1}$, $G_q$ is of type (1) with respect to $w$. Since $\Omega_0$ is connected, there exists a path in $\Omega_0$ connecting $G_q$ and $G_h$:

$$G_q = H_0 \approx H_1 \approx \cdots \approx H_p = G_h,$$

where each $H_i \in \Omega_0$. Now we have that $H_s$ is of type (1) and $H_{s+1}$ is of type (2) for some $0 \leq s < p$. Since $H_s \approx H_{s+1}$, there exist $\gamma_1, \gamma_2 \in H_s$ and $\delta_1, \delta_2 \in H_{s+1}$ with $(u, \gamma_1, \gamma_2, \delta_1, \delta_2)$ as a basis. Thus we obtain a pair of paths $(X, U)$ of type $C^*$ of size $(l, 0)$ from Lemma 3.6. Set

$$Q: u_l \sim \cdots \sim u_1 \sim u \sim w_1 \sim \cdots \sim w_t$$

to be a walk of length $q = l + t$. Then we have that $(X, Q)$ is partially of type $C$ of size $(l, t)$ with $\Lambda = H_s$, $\Lambda' = H_{s+1}$. From Proposition 3.9 we obtain that $X^*: \alpha_0 \sim \cdots \sim \alpha_{2q}$ is a walk of length $2q$ such that $(X^*, Q)$ is of type $C$ of size $(q, 0)$. Since $M[\alpha_0, \alpha_1; w_{t-1}, w_t] \in \{J, E_1, E_4\}$, we have that $\partial_\Gamma(\alpha_0, w_t) = r+1$. Since $\alpha_0 \in \Lambda = H_s$ and $H_s$ is of type (1) with respect to $w$, we have that $\alpha_0 \in D_{r+2}^{r+1}$. Hence we have that $\partial_\Gamma(\alpha_0, w) = r+2$. On the other hand, we have that $\partial_\Gamma(\alpha_{2q}, w_t) = r+1$ from $M[\alpha_{2q-1}, \alpha_{2q}; w_{t-1}, w_t] \in \{J, E_2, E_3\}$. Since $\alpha_{2q} \in \Lambda' = H_{s+1}$ and $H_{s+1}$ is of type (2) with respect to $w$, we have that $\alpha_{2q} \in D_r^{r+1} \cup D_{r+1}^{r+1}$. Thus we have that $\partial_\Gamma(\alpha_{2q}, w) \neq r+2$. This contradicts Proposition 4.2. Hence the lemma is proved. □

PROOF OF PROPOSITION 5.3. (1) Consider the $(w_t, w)$-diagram. From the inductive assumption, $D_j^i \cap \Delta = \emptyset$ for $i \geq r+2$, it is sufficient to show that $D_{r+2}^{r+1} \cap \Delta = \emptyset$. Suppose that $Z = D_{r+2}^{r+1} \cap \Delta \neq \emptyset$. Take any $x \in Z$. We have

$$a+1 \leq k_\Delta(x) = |\Delta_1(x)| = |D_{r+2}^{r+1} \cap \Delta_1(x)| + |D_{r+1}^r \cap \Delta_1(x)| \leq |Z_1(x)| + 1.$$

Since $\partial_\Gamma(u, x) \leq d_\Delta(u) = r+1$, we have that $Z$ is an $(a, r+1)$-subgraph with respect to $\{u\}$. From Lemma 2.3(2), we have that

$$\emptyset \neq Z \cap \Gamma_{r+1}(u) = D_{r+2}^{r+1} \cap \Delta \cap \Gamma_{r+1}(u) \subset D_{r+2}^{r+1} \cap \Psi \subset \Gamma_{r+2}(w) \cap \Psi.$$

This contradicts Lemma 5.5.

(2) Since $\Delta$ is an $(a+1, r+1)$-subgraph with respect to $\{w_t\}$, our assertion follows from Lemma 2.4.

(3) This follows from (1), (2) and Lemma 2.3(2).

(4) Let $x \in \Delta_{r+1}(w)$. Note that $\partial_\Gamma(w, x) = r+1$ from (2). We have $B(w, x) \cap \Delta = \phi$ from (1). Hence $\Delta_1(x) \subset C(w, x) \cup A(w, x)$. This implies that

$$k_\Delta(x) \leq C(w, x)| + |A(w, x)| = c_{r+1} + a_{r+1} = 1 + a.$$

Hence our assertion follows from Lemma 5.2(7). □

PROOF OF THEOREM 1.1. Note that $g(\Delta) = 2r+3$ from Lemma 5.2(1). Take any $x \in \Delta$. We have $d_\Delta(x) = r+1$ from Proposition 5.3(3). Thus we have $z \in \Delta_{r+1}(x)$. This implies that $x \in \Delta_{r+1}(z)$. Thus we have that $k_\Delta(x) = a+1$ from Proposition 5.3(4). This implies that $\Delta$ is a regular graph of valency $k_\Delta = a+1$, diameter $d_\Delta = r+1$ and girth $g(\Delta) = 2r+3$. Thus we obtain the desired result.

This completes the proof of Theorem 1.1. □

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