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Weighing Stones in Ancient Mesopotamia

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The Old Babylonian mathematical tablet YBC 4652 contains a series of related problems stated together with their answers. In this paper, we propose a procedure for determining the solutions to these problems and consider the pedagogy underlying the mathematics. © 2002 Elsevier Science (USA)

L'ancienne tablette mathématique babylonienne YBC 4652 contient une série de problèmes avec leurs réponses. Dans cet article, nous proposons une procédure pour déterminer les solutions de ces problèmes et considérons la démarche pédagogique sous-jacente aux mathématiques. © 2002 Elsevier Science (USA)

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The mathematical cuneiform tablet YBC 4652[2] contains a series of problems ostensibly to do with findin the original weight of a stone[3] given its weight after a series of modifications The modification have nothing to with the physical reality of stones and it is clear that these problems are purely exercises in mathematical techniques, thinly disguised by a practical overlay. However, in the text, although the solutions are given for each of the problems, the method for obtaining the solutions is not explained. In this paper we elucidate the mathematical techniques underlying these problems and use this information to sharpen our understanding of the depth and boundaries of Mesopotamian mathematics. In particular, we consider the pedagogy underlying the problems set.

Almost 200 cuneiform mathematical problem texts are known, and almost all of them, including this one, are from the Old Babylonian period (ca. 2000–1600 B.C.). They range from tablets containing but one problem to tablets containing over 200 abbreviated exercises

[1] Research supported in part by a Faculty Research Grant from St. Lawrence University.

[2] Tablet number 4652 in the Yale Babylonian Collection, Yale University, New Haven. YBC 4652 is an Old Babylonian tablet of unknown provenance. First published in *MCT* as Problem Text R [Neugebauer and Sachs 1945, 100–103, photograph Plate 13, copy Plate 39]. The tablet is regularly mentioned in surveys of Mesopotamian mathematics (see, for example, [Goetsch 1968–1969, 109; Friberg 1987–1990, 572]) as an exemplar of linear equations in one variable. Despite these casual mentions, it has not before been subject to detailed study.

[3] By a "stone" is meant a stone weight (usually crafted from hematite) of the sort widely used as a standard of measurement throughout the Near East. Stone weights were often carefully shaped, sometimes into nonrepresentational figure such as ellipsoids; others, especially larger weights, were carved into the shape of ducks. For detailed analysis of these types of weights, see [Powell 1979].

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packed into a modest space.[4] Some texts have diverse groups of problems with no obvious connection, but others provide collections of thematically and technically linked exercises.

The "weighing stones" tablet has a partially preserved colophon indicating that it originally contained 22 problems. The tablet is now broken and we have remains of 11 problems, although only six have been fully restored. It is clear from what remains that all of the problems began with the formulaic opening[5] "na_4 ì-pà ki-lá nu-na-tag." (I found a stone. I did not weigh it.[6]) The poor stone undergoes various mathematical manipulations after which the weight of the modif ed stone is (always) found to be 1 mana. The question for the student is to determine the original weight of the stone. As noted above, the stone is a weight stone. This is a typical example of Mesopotamian donnish humour.

The tablet was originally published by Neugebauer and Sachs in *Mathematical Cuneiform Texts*[7] (MCT) [Neugebauer and Sachs 1945], where problem numbers were assigned as if the whole tablet were extant. We give here the f rst surviving complete problem, retaining Neugebauer and Sachs' numbering and transliteration, but supplying our own translation.[8]

7. na_4 ì-pà ki-lá [nu-na-ta]g igi-7-gál bí-dah igi-11-gá[1 b]í-[da]h[9]
   I found a stone. I did not weigh it. A seventh I added. An eleventh I added.

   ì-lá 1 ma-na sag na_4 en-nam sag na_4 $\frac{2}{3}$ ma-na 8 gín $22\frac{1}{2}$ še
   I weighed it. 1 mana. What was the original (weight of the) stone? The original stone was $\frac{2}{3}$ mana 8 gín $22\frac{1}{2}$ še.

In his monumental paper "Algebra and Naive Geometry," Høyrup has stated of Old Babylonian mathematics that "normal mathematical problems were constructed backwards from known solutions" [Høyrup 1990a, 53], a view reiterated by Robson, "Problems were constructed from answers known beforehand" [Robson 1995, 11]. Clearly, that is not the case here. It is hard to imagine a scribe picking a number such as $\frac{2}{3}$ mana 8 gín $22\frac{1}{2}$ še and then f guring out a series of algebraic procedures that would lead to the result 1 mana—and repeating this process 22 times. Rather, the scribe who prepared the tablet was in a position familiar to any teacher of mathematics: he[10] had a structure of problem in mind and some techniques he wanted the students to practice and he needed to make up a series of problems admitting a solution. A careful study of the tablet reveals the purposes of the exercises and the techniques used to solve them.

[4] The total count partly depends on one's def nition of a problem text. Nemet-Nejat claims 180 [Nemet-Nejat 1993, 21], while Robson counts "over 160" [Robson 1995, 9], of which only 12 are from later than the Old Babylonian period. A 24194 (*MCT* Problem Text T) has "200 or more examples on a single tablet measuring about 3 by 4 inches" [Neugebauer and Sach 1945, 116]. The remaining fragment of YBC 4652 measures approximately 83 by 68 mm. The original complete tablet would have f tted comfortably in the palm of the hand.

[5] Lists of statements with formulaic openings are a common feature of Mesopotamian scribal practice. They occur not just in mathematics, but in other f eld such as medical and legal texts. The Law Code of Hammurabi is probably the most famous example.

[6] See [Neugebauer and Sachs 1945, 102] for a discussion of the translation of ki-lá nu-na-tag as to do with weight.

[7] See footnote 2.

[8] The differences between our translation and that of Neugebauer and Sachs are mostly stylistic.

[9] That part within square brackets [....] has been restored by Neugebauer and Sachs where the tablet has been too damaged to read the original signs. Given the limited vocabulary and restricted structure of the problems, we can be quite conf dent that all these restorations ref ect the original closely.

[10] We will tacitly assume the author to be male; most scribes were.

First, let us mention the metrological aspect. The answers to the problems, the original weight of the stone, are stated in the appropriate units. The abstract structure of the problem has nothing to do with stones and could equally well have been applied to length of a stick, area of a field, or volume of an excavation (and indeed similar problems were set for finding these quantities[11]). The Mesopotamians had no general notation for an unknown as we would use the symbol x and so tended to cast these problems as word problems involving standard everyday items. More importantly, though, while intermediate computations would doubtless have been carried out in the abstract "scientific" sexagesimal system,[12] at the end of the problem the student had to write out the answer in the correct units.

By the Old Babylonian period, Mesopotamian metrology was greatly simplified from the earlier Sumerian systems,[13] but it still employed a rich set of units. A standard part of the mathematical corpus is metrological tables giving unit conversions and multiplications within the metrological system.[14] One aim of word problems of the "weighing stones" type was to give students practice in working with units of weights and measures. In this particular case, we need only to know that the standard unit of weight measurement was the gín and that there are 60 gín in 1 mana and 180 še in 1 gín. The še is usually considered as a barleycorn, and 1 mana is about 1/2 a kilogramme in modern units. In regular sexagesimal notation,[15] the answer to Problem 7 above would have been written as 48; 7,30 gín.[16]

Let us now consider the structure of the problem. We are to find the original weight of the stone and, rather anachronistically, we will denote this unknown weight by w. We start with the stone of weight w. We add a 7th to it. So now we have the original stone plus a 7th, or a stone of weight $w + \frac{1}{7}w$. Next we add an 11th (of what we now have, to what we now have), after which we are holding a stone weighing $(w + \frac{1}{7}w) + \frac{1}{11}(w + \frac{1}{7}w)$ of the weight of the original. This modified stone weighs 1 mana, or 1,0 gín. In their original publication of the tablet, Neugebauer and Sachs translate dah and zi as "add" and "subtract," respectively and comment that "the main difficulty encountered in interpreting the text of the problem consists in placing the parentheses correctly" [Neugebauer and Sachs 1945, 103]. However, a more sensitive use of mathematical terminology obviates some of the confusion.

[11] For example, the text Str. 368 begins, "I took a reed. I do not know its length" [Neugebauer 1935 I, 311].

[12] The place value base-60 system associated with the Babylonians. Its rigorous use was mostly confined to scientific and educational purposes. In everyday mathematics, people would stick more closely to the underlying metrology of the situation.

[13] For an excellent discussion of Sumerian metrological numeration systems and the transition to cuneiform, see [Nissen *et al*. 1993].

[14] See [Friberg 1987–1990, Sect. 5.1] for a brief summary of published metrological texts and tables.

[15] There are a number of ways of writing sexagesimal numbers. In computations, we will use the convention of separating sexagesimal places by a comma. When we need to emphasize a sexagesimal point, we will use a semicolon. Most of the time, we will represent numbers in floating-point form with no indication of the absolute value in underlying units.

[16] Powell notes that the weight 40 gín was usually denoted as $\frac{2}{3}$ mana, commenting that "exceptions to this rule from Ur III to NB are extremely rare" [Powell 1979, 103]. In the Old Babylonian period, there were special signs for the fractions 1/2, 1/3, 2/3, and 5/6. All of them occur in this tablet. Interestingly, the numbers "7" and "8" are always written in the "everyday" notation of two rows of wedges, rather than in the scientific form of three rows, and "4" is written as two rows of two wedges, rather than one row of three wedges with a single wedge underneath. We also note that the entire tablet is in Sumerian. There is not a single syllabic Akkadian word. These features are unusual although by no means unique among Old Babylonian mathematical texts.

A full use of the "conformal translation" as proposed by Friberg and Høyrup[17] renders the translation almost unintelligible, but in small doses it can be helpful. For example, Høyrup renders igi-n-gál as "the nth part" and suggests that dah be interpreted as "identity-conserving addition" and preferably translated as "to append." Thus, in Problem 7, Høyrup would translate "igi-7-gál bí-dah" as "I appended the seventh part." Clearly, the "part" refers to the original stone and so the weight of the modif ed stone is now $w + \frac{1}{7}w$ of the original weight w. The Mesopotamian student is now standing with a stone and a seventh in hand. To the procedurally based Mesopotamian mathematician, the next addition or subtraction takes place with respect to the stone currently in hand. It is this sense of time, motion, and change inherent in Old Babylonian mathematics that is lost in the static picture induced by describing the problem in the language of algebraic equations.

To modern eyes, the problem above constructs a reasonably complicated linear equation in one variable. So we have as a f rst approximation that the original scribe wanted students to practice f nding solutions to linear equations in one variable. But this modern viewpoint obscures some of the subtler aspects of Babylonian mathematics.

The modern approach to solving such a problem is to begin with an algebraic equation and reduce it to simpler and simpler equations through a series of algebraic manipulations until the simplest possible identity is reached; i.e., the equation is "solved." This procedure is alien to Babylonian mathematics. Mesopotamian mathematicians (and their students) started with known (or guessed) numbers and followed prescribed procedures or algorithms until arriving at the sought after number(s). This is the thrust behind Høyrup's comment quoted above. Normal Babylonian mathematical problems were constructed backwards from known solutions because in that way the algorithm was guaranteed to be successful. In making up problems for our students, sometimes the initial values can be chosen at random because the computational techniques are powerful enough to tackle all cases, and sometimes greater care must be taken in setting up the problem in order to simplify calculation and avoid (or choose) special cases (try asking students to f nd the inverse of an arbitrary 4×4 matrix on an exam). On the face of it, any linear equation in one variable should be solvable by the student and we might suppose (especially given the complexity of the answers) the problems set to be quite arbitrary. Yet this is not quite the case, as we will demonstrate below, even though Babylonian mathematics contains much harder problems than these.

In Babylonian mathematics, division is always achieved by multiplication by the reciprocal. Reciprocals of numbers, the igi-n-gál, play a central role in Babylonian mathematics. We have numerous examples of reciprocal tables listing the reciprocal pairs of a canonical set of numbers.[18] The reciprocal of a number n has a f nite sexagesimal representation so long as all its prime divisors are divisors of the base, 60. These numbers, where n is of the form $n = 2^a 3^b 5^c$, are called regular numbers. In mathematical exercises, Babylonian scribes went to a great deal of trouble to ensure the student only needed to f nd reciprocals of regular numbers, and preferably ones in the table.[19] Yet in Problem 7 both the reciprocals,

[17] See [Friberg 1987–1990] on transliterating numbers and [Høyrup 1990a] on vocabulary.

[18] The standard table gave the reciprocals of all regular numbers from 2 to 81. See, for example [Neugebauer 1935 I, 32–67, II, 36–37, III, 49–50, and Neugebauer and Sachs 1945, 19–33], as well as many other publications.

[19] For regular numbers not in the standard table of reciprocals, there was a procedure for computing the reciprocal, known (to us) as The Technique, and explained by A. Sachs [Sachs 1947]. The skills required to operate The Technique successfully are perhaps more sophisticated than those needed to solve the problems in this tablet. The Technique contains more steps than the algorithm we shall explain and there is a possibility of needing to repeat certain parts of it if the number is complicated.

1/7 and 1/11, are irregular. As can be seen from the rest of the problems on the tablet (given below), this is not just a coincidence, but the whole point of the exercises.

If a student had to take, say, the 9th part of a number in a computation, he would look up the reciprocal of 9 (which is 6,40), then multiply the given number by 6,40. This technique is not available to the student facing the weighing stones problems.

In order to be able to take an irregular fraction of a quantity and obtain a finite result, the quantity must be a multiple of the irregular number (here we are exploiting the "floating-point" representation of sexagesimal numbers). For convenience, let us simplify Problem 7 so that instead of two steps, we have only one.[20] Consider the following problem:

> I found a stone. I did not weigh it. A seventh I added.
> I weighed it. 1 mana. What was the original weight of the stone?

The corresponding modern equation would be $w + \frac{1}{7}w = 1$.

An Old Babylonian student does not write an unknown w, but instead uses a number. Furthermore, it must be a number divisible by 7 (or, rather, a multiple of 7). The natural thing to do is to choose w to have the value 7—that is, to use the method of "false position." The technique of false position is well attested in other situations in Babylonian mathematics.[21] With the original weight of the stone set to 7, the weight of the seventh part is 1 and the weight of the modified stone is 8, instead of the 1 mana it should be.

The original guess was 8 times too large and must be divided by 8. Following standard procedure, we must take the reciprocal of 8 and multiply that by our original estimate. But 8 is a regular number where 7 was not, and so we can proceed. The reciprocal of 8 is 7,30 and 7 times 7,30 is 52,30. Thus, the original weight of the stone was 52,30 gín.

The original Problem 7 was constructed in two stages: "A seventh I added. An eleventh I added." We propose that the solution, too, proceeded in two stages, reversing the order of the manipulations in the statement. As is shown below in the more complicated later problems, assuming a multi-stage procedure allows a uniform technique for solving all of the extant problems on the tablet and simplifies the calculations. In modern terminology and approach, this is equivalent to having a student first make the substitution $w + \frac{1}{7}w = v$, solve the simpler equation $v + \frac{1}{11}v = 1$ for v, and then solve $w + \frac{1}{7}w = v$ for w, instead of directly solving the full equation $(w + \frac{1}{7}w) + \frac{1}{11}(w + \frac{1}{7}w) = 1$ for w. The technique of substitution is well attested in Babylonian mathematics and, indeed, Goetsch uses this very problem as an example: "Die Vorliebe der Babylonier für Substitutionen bei der Lösung quadratischer Gleichungen läßt vermuten, daß sie auch die hier erwähnten Beispiele mit dieser Technik lösten" [Goetsch 1968–1969, 109].

The two-stage procedure introduces one slight complication we avoided in the first simple example; the intermediate number v will no longer necessarily be 1, and the reciprocal must first be multiplied by v before the product multiplies the guess. The procedure for Problem 7 goes thus:

[20] The problems in our text proceed from the simpler to the more difficult. Since the first extant problem is number 7, it is reasonable to suppose that, say, Problem 1 would have been of the same type, but easier.

[21] Indeed, it was noted by Thureau-Dangin as long ago as 1938 [Thureau-Dangin 1938] for problems involving reciprocals of irregular numbers. For a partial summary of attested uses in Mesopotamian mathematics, see [Friberg 1987–1990, Sect. 5.7d]; more detailed discussions of its usage can be found in [Vogel 1960] and [Li Ma 1993].

Round 1: Take 11. Add 1 to get 12.
The reciprocal of 12 is 5.
5 times 1 is 5.
5 times 11 is 55.

That is, we have $v = 55$ gín.

Round 2: Take 7. Add 1 to get 8.
The reciprocal of 8 is 7,30.
7,30 times 55 is 6,52,30.
6,52,30 times 7 is 48,7,30.

Hence, $w = 48;7,30$ gín.

Two rounds of a four-line procedure solve the problem. What is crucial of course is that the number obtained at the end of Line 1 in each round be always a regular number. This is where the care of construction of these problems is displayed, as we shall see below.

In a graded sequence of exercises, we expect Problem 8 to be a little harder than Problem 7, and indeed it is: it uses subtraction.

8. na$_4$ ì-pà ki-lá nu-na-tag igi-7-gál ba-zi igi-13-gál ba-zi-ma
I found a stone. I did not weigh it. A seventh I took away. A thirteenth I took away.

ì-lá 1 ma-na sag na$_4$ en-nam sag na$_4$ 1 ma-na $15\frac{5}{6}$ gín
I weighed it. 1 mana. What was the original weight of the stone? The original stone was 1 mana $15\frac{5}{6}$ gín.

Applying the same procedure in two rounds, we need change only the addition in the f rst line of each round to a subtraction.

Round 1: Take 13. Subtract 1 to get 12.
The reciprocal of 12 is 5.
5 times 1 is 5.
5 times 13 is 1,5.

Round 2: Take 7. Subtract 1 to get 6.
The reciprocal of 6 is 10.
10 times 1,5 is 10,50.
10,50 times 7 is 1,15,50.

Solution: The original weight of the stone was 1,15;50 gín, or 1 mana $15\frac{5}{6}$ gín.

Problem 9 presents us with three steps instead of two and mixes addition and subtraction. To f nd the solution we need merely repeat the procedure three times. This strengthens the assumption that the solution algorithm mirrors the steps of the construction of the problem.

9. na$_4$ ì-pà ki-lá nu-na-tag igi-7-gál ba-zi igi-11-gál bí-dah
[igi-1]3-gál ba-zi ì-lá 1 ma-na sag na$_4$ en-nam
[sag] na$_4$ 1 ma-na $9\frac{1}{2}$ gín $2\frac{1}{2}$ še

I found a stone. I did not weigh it. A seventh I took away. An eleventh I added.
A thirteenth I took away. I weighed it. 1 mana. What was the original weight of the stone? The original stone was 1 mana $9\frac{1}{2}$ gín $2\frac{1}{2}$ še.

Round 1: Take 13. Subtract 1 to get 12.
The reciprocal of 12 is 5.
5 times 1 is 5.
5 times 13 is 1,5.
Round 2: Take 11. Add 1 to get 12.
The reciprocal of 12 is 5.
5 times 1,5 is 5,25.
5,25 times 11 is 59,35.
Round 3: Take 7. Subtract 1 to get 6.
The reciprocal of 6 is 10.
10 times 59,35 is 9,55,50.
9,55,50 times 7 is 1,9,30,50.

Solution: The original weight of the stone was 1,9;30,50 gín, or 1 mana $9\frac{1}{2}$ gín $2\frac{1}{2}$ še.

Problem 9 is the last of the fully restorable problems on the obverse of the tablet. Problem 10 again mixes addition and subtraction and originally had at least three steps, but only the f rst two are left. Problem 11 is completely missing. On the reverse, we start up again with Problem 19, and the last four problems present some interesting challenges.

By Problem 19, the student is expected to have a good understanding of the basic procedure, and the problems have gained two new wrinkles. The f rst is addition of absolute rather than relative quantities (adding 2 gín), the second is a matter of compounding the fractions.[22] The f rst complication means a student must keep track of the absolute sizes of the sexagesimal computations. The second allows for a couple of different approaches.

19. na_4 ì-pà ki-lá nu-na-tag 6-bi ì-lá 2 gín [bí-dah-ma]
igi-3-gál igi-7-gál a-rá-24-kam tab bí-dah-ma
ì-lá 1 ma-na sag na_4 en-nam sag na_4 $4\frac{1}{3}$ gín
I found a stone. I did not weigh it. 6 times the weight and 2 gín I added.
A third of a seventh of 24 times I added.
I weighed it. 1 mana. What was the original weight of the stone? The original stone was $4\frac{1}{3}$ gín.

Round 1: Proceeding normally, the student multiplies 3 and 7 to begin the procedure with 21.

Take 21. Add 24 to get 45.
The reciprocal of 45 is 1,20.
1,20 times 1 is 1,20.
1,20 times 21 is 28.

Round 1a: Alternatively, one could notice that 24 is 3 times 8 and simplify the problem.

Take 7. Add 8 to get 15.
The reciprocal of 15 is 4.
4 times 1 is 4.
4 times 7 is 28.

Both approaches yield the same correct solution. The second has slightly simpler computations. To solve the second part of the problem, the student proceeds as follows:

[22] Høyrup discusses compounding of fractions under the terminology "parts of parts" in [Høyrup 1990b].

Take 28. Subtract 2 to get 26.
The reciprocal of 6 is 10.
10 times 26 is 4,20.

Solution: The original weight of the stone was $4\frac{1}{3}$ gín.

Note how the introduction of the constant term (2 gín) in the linear equation forces the multiplicative coeff cient to be a regular number (in this case, 6). The student must take the reciprocal of the coeff cient: it would not be an irregular number. Within Babylonian mathematics, the only way of solving this type of complex linear problem with constant term is a combination of false position and substitution. The complexity of the problem lends conf dence to our interpretation.

Problem 20 is very similar to Problem 19. Again there are two reciprocals and a product; again there is a convenient simplif cation (but this time there is a trick). There are a linear term and a regular multiplicative factor.

20. na_4 ì-pá ki-lá nu-na-tag 8-bi ì-lá 3 gín bí-dah-ma
igi-3-gál igi-13-gál a-rá-21 e-tab bí-dah-ma
ì-lá 1 ma-na sag na_4 en-nam sag na_4 $4\frac{1}{2}$ gín
I found a stone. I did not weigh it. 8 times the weight and 3 gín I added.
A third of a thirteenth of 21 times I added.
I weighed it. 1 mana. What was the original weight of the stone? The original stone was $4\frac{1}{2}$ gín.

Round 1: Take 39. Add 21 to get 1.
The reciprocal of 1 is 1.
1 times 1 is 1.
1 times 39 is 39.

Round 1a: In parallel to the alternative approach in Problem 19, one could notice that 21 is 7 times 3, simplify and get (slightly) tricked.

Take 13. Add 7 to get 20.
The reciprocal of 20 is 3.
3 times 13 is 39.

To f nish the problem, the student follows a procedure similar to that of the previous problem.

Take 39. Subtract 3 to get 36.
The reciprocal of 8 is 7,30.
7,30 times 36 is 4,30.

Solution: The original weight of the stone was $4\frac{1}{2}$ gín.

Problem 20 is a truly splendid problem. Although it looks complicated, the coeff cients have been chosen carefully so that a correct application of the procedure yields the solution with quite simple computations. Together, these two problems display an amazing sensitivity to, and understanding of, the properties of small numbers in the sexagesimal system.

Problem 21 is the f rst to contain three reciprocals in a two-step problem (as opposed to the three iterations of a single reciprocal procedure we saw in Problem 9). It is the f rst to use reciprocals of regular numbers in expressions that cannot be simplif ed, and has a solution of greater metrological complexity that the preceding problems, which have very "nice" numbers to work with. It also mixes addition and subtraction. It is certainly a much more advanced problem than those found earlier in the exercise set.

21. na$_4$ ì-pà ki-lá nu-na-tag igi-6-gál ba-zi
igi-3-gál igi-8-gál bí-dah-ma ì-lá 1 ma-na
sag na$_4$ en-nam sag na$_4$ 1 ma-na 9 gín $21\frac{1}{2}$ še
ù⟨igi-⟩ 10-gál še kam
I found a stone. I did not weigh it. A sixth I subtracted.
A third of an eighth I added. I weighed it. 1 mana.
What was the original weight of the stone? The original stone was 1 mana 9 gín $21\frac{1}{2}$ še and a tenth of a še.

Applying the proposed procedure, the student performs the following computations:

Round 1: Take 24. Add 1 to get 25.
The reciprocal of 25 is 2,24.
2,24 times 1 is 2,24.
2,24 times 24 is 57,36.
Round 2: Take 6. Subtract 1 to get 5.
The reciprocal of 5 is 12.
12 times 57,36 is 11,31,12.
11, 31,12 times 6 is 1,9,7,12.

Solution: The original weight of the stone was 1 mana 9 gín $21\frac{1}{2}$ še and a tenth of a še.

Again, the proposed procedure solves the problem. Note that in this problem the coeff cients are all regular, as are the induced quantities. Thus, this particular problem could be solved without reference to the proposed procedure. For example,[23] since the end result is 1 mana, one could begin by posing a false value of 1 and proceed in a straightforward fashion through the problem. However, one would then end up with the reciprocal of the correct result and hence the f nal step would involve determining 1,9,7,12 as a reciprocal. This involves much more complicated computations than the relatively simple ones required by the proposed procedure. Further, given the uniform character of the problems up to this point, it seems unnecessary to introduce a special technique to solve only one problem (or two; Problem 22 is similar) when it can be solved in the same manner as all the others.

Any such reconstruction of a mathematical procedure is of course speculative. However, the construction described above requires only the use of two standard techniques of Mesopotamian mathematics, the method of false position and the use of substitution. Both of these techniques are well within the grasp of a student dealing with problems of the given level of complexity. Furthermore, the suggested procedure has the advantage of testability. The key point in the procedure, as indeed it is the crux of much of Mesopotamian mathematics, is the taking of reciprocals at each stage of the problem. In order for the problem to be solvable within the Old Babylonian context, the numbers for which reciprocals are to be found must be regular sexagesimals. The proposed method cannot work if we attempt to f nd the reciprocal of an irregular number. Any future f nd, or restoration of the current tablet, producing a problem which involves forming an irregular number as one of the steps would prove that this method was not (universally) followed.

[23] This observation was pointed out to me by an anonymous referee of an earlier version of this paper.

The smallest, and most common, irregular numbers in use in Old Babylonian mathematics are 7, 11, 13, 14, 17, and 19.[24] If a scribe forms a problem involving addition or subtraction of a seventh of a quantity, the method given above is guaranteed to work, as the procedure would begin. "Take 7. Subtract 1 to get 6" or "Take 7. Add 1 to get 8." Both 6 and 8 are regular numbers, and so their reciprocals can be obtained in the second line of the procedure. Similarly, addition or subtraction of an 11th, 17th, or 19th part presents no difficulty. The only trouble arises in using one of the adjacent pair of irregular numbers, 13 and 14. Subtraction of a 13th part would give "Take 13. Subtract 1 to get 12," and 12 is regular, but addition of a 13th would lead to "Take 13. Add 1 to get 14," and 14 is not regular and so has no finite sexagesimal reciprocal. Similarly, addition of a 14th part leads to a solvable problem, but subtraction of a 14th does not. In the six extant problems, 14 does not occur and 13 occurs only once (in Problem 8), where indeed the 13th part is subtracted.

An interesting complication arises when the "parts" are regular, as for example the "igi-6-gál" in Problem 21. Addition of a sixth part would give 7 in the first line and 7 has no finite reciprocal.[25] Subtraction of a sixth, however, leads to 5, and the method will provide the solution. In Problem 21, the sixth is subtracted.

Finally, we turn to the last, partially broken problem on the tablet, Problem 22, where we can now complete the restoration. Neugebauer and Sachs give the following transliteration:

22. na_4 ì-pà ki-lá nu-na-tag $\frac{2}{3}$ igi-6[-gál ....]
igi-3-gál igi-8-gál bí-dah-ma ì-l[á 1 ma-na]
sag na_4 en-nam sag na_4 1 m[a-na ......]

Given the high degree of similarity of the problems on the tablet, their restorations appear quite certain. In particular, we will assume the restoration of the final weight of the stone as 1 mana is correct. The only question is whether the modification in the first line is an addition or subtraction. Problems 21 and 22 are very similar, which might suggest the first line of the problem end "ba-zi" for a subtraction. In either case, the first round of the procedure is the same, and in fact is identical to that of the preceding problem.

Round 1: Take 24. Add 1 to get 25.
The reciprocal of 25 is 2,24.
2,24 times 1 is 2,24.
2,24 times 24 is 57,36.

For Round 2, the student must know that the reciprocal of $\frac{2}{3}$ is 1,30[26] and that 1,30 times 6 is 9. Next, consider the two alternatives of subtraction and addition.

Round 2a: Take 9. Subtract 1 to get 8.
Reciprocal of 8 is 7,30.
7,30 times 57,36 is 7,12.
7,12 times 9 is 1,4,48.

Solution: The original weight of the stone was 1 mana 4$\frac{2}{3}$ gín 24 še.

[24] For a discussion of the uses of these numbers, see [Høyrup 1993].
[25] Of course, the student would know that the reciprocal of 6 is 10 and be able to proceed in an alternate fashion. The point is that the procedure described here could not be followed.
[26] A standard piece of Mesopotamian mathematical knowledge.

Round 2b: Take 9. Add 1 to get 10.
Reciprocal of 10 is 6.
6 times 57,36 is 5,45,36.
5,45,36 times 9 is 51,50,24.

Solution: The original weight of the stone was $51\frac{5}{6}$ gín 1 še and one-fifth of a še.

Hence, if the modification in the second round is a subtraction, the original weight of the stone is more than 1 mana, while if the modification is an addition, the original weight is less than 1 mana. However, the last line of the problem clearly states that the original weight is more than 1 mana before the rest is broken. Hence, the modification must indeed be a subtraction. Problem 22 requires a little more mathematical sophistication on the part of the student than the earlier problems, but yields to the same procedure.

Some additional pedagogical points should be noted with reference to the proposed procedure. First, if a student follows this procedure, then every needed reciprocal not only is a regular number, but occurs in the standard table of reciprocals. Second, almost always, at least one of the factors in a product occurs in the standard set of multiplication tables.[27] Out of the whole tablet, there are just two exceptions to this observation. The first is the product 1,20 times 21 in Problem 19, and we already noted that there was an alternate route that led to the product 4 times 7 here. Further, there is a 20 times table. The second difficult multiplication is 5,25 times 11 in Problem 9, and there are multiplication tables for both 5 and 25. In short, the problems have been set so that the computations are as easy as possible for the student, leaving the student able to concentrate on learning the correct method. The simplicity of the calculations and the fact that a single method will solve all the extant problems argues in favor of this interpretation.[28]

A close reading of the tablet not only reveals a plausible procedure for finding the solutions to the given problems, but, perhaps more importantly, illustrates the enormous care and pedagogical precision that went into their construction. Whoever originally set these problems knew exactly what he wanted his students to learn and created a set of exercises that fulfils this task admirably.

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[27] The canonical list of multiplication tables gave tables for 40 principal numbers, many of which were also in the reciprocal table. The student could be expected to have both reciprocal and multiplication tables to hand as aids in computation.

[28] A referee for an earlier version of this article suggested that this tablet may in fact be “a catalogue of the statements of a previously written procedure text,” citing the similar tablet YBC 4657 and its accompanying procedure texts YBC 4662 and YBC 4663 (published as MCT Texts G, H and J [Neugebauer and Sachs 1945, 66–75]).

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