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COMBINATORICS OF JACOBI-CONFIGURATIONS III: THE SRIVASTAVA–SINGHAL GENERATING FUNCTION REVISITED

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A generalized version of Jacobi's generating function for the Jacobi polynomials has been presented by Srivastava and Singhal. The approach by Foata and Leroux to combinatorially prove the classical generating function is extended to cover (and even to generalize) this result.

1. Introduction

The study of combinatorial structures related to the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ was initiated by Foata and Leroux in [6], where they introduced the model of "Jacobi-endofunctions" in order to give a completely combinatorial proof of Jacobi's generating function:

$$\sum_{n\geq 0} P_n^{(\alpha,\beta)}(x)t^n = 2^{\alpha+\beta}\cdot R^{-1}\cdot(1-t+R)^{-\alpha}\cdot(1+t+R)^{-\beta},$$

where $R=(1-2xt+t)^{\frac{1}{2}}$.

Many proofs of this result are known, see e.g. Section 4.4 in [19], Chapter 16 in [13], or Askey's article [1]. But once you have learned to see the combinatorial pattern behind this identity, it appears that the Foata–Leroux proof certainly ranks among the most elementary (and elegant!) approaches. The present article is devoted to an extension of the Foata–Leroux technique which will lead to a combinatorial proof of the following beautiful result due to Srivastava and Singhal [14] (see also p. 82 and Chapter 7 in [16]):

$$\sum_{n\geq 0} P_n^{(\alpha-\lambda n,\beta-\mu n)}(x)t^n = (1+\xi)^{1+\alpha}\cdot(1+\eta)^{1+\beta}\cdot[1+\lambda\xi+\mu\eta-(1-\lambda-\mu)\xi\eta]^{-1}$$

where α, β, λ, μ are complex parameters and where ξ, η are functions (of x and t) implicitly defined by

$$\xi=\tfrac{1}{2}(x+1)\cdot t\cdot(1+\xi)^{1-\lambda}\cdot(1+\eta)^{1-\mu}$$
$$\eta=\tfrac{1}{2}(x-1)\cdot t\cdot(1+\xi)^{1-\lambda}\cdot(1+\eta)^{1-\mu}.$$

[All series and functions should be regarded as formal power series.]

The most prominent special cases of this result are:

(1) $\lambda = \mu = 1$, where it simplifies to

$$\sum_{n\geq 0} P_n^{(\alpha-n,\beta-n)}(x)t^n = [1+\tfrac{1}{2}(x+1)t]^\alpha \cdot [1+\tfrac{1}{2}(x-1)t]^\beta.$$

which is—up to a trivial modification of the variables—the generating function for the so-called Lagrange-polynomials, see p. 267 in [4] and p. 25 in [16]. Bergeron has already proposed a combinatorial model for these polynomials, see [2], but in his proof the generating function requires a cancellation-by-involution-type argument. It is interesting to note that his proof is not a special case of the one given below.

(2) $\lambda = \mu = 0$, where the implicit system for ξ and η leads to quadratic equation

$$\bar{\xi} = \left(1 + \left(\frac{x+1}{2}\right)t\bar{\xi}\right)\left(1 + \left(\frac{x-1}{2}\right)t\bar{\xi}\right),$$

where $\xi = (x+1/2)t\bar{\xi}$. The explicit solution for ξ and η thus obtainable gives back—after some simple transformations—the classical formula stated in the beginning.

Some other interesting particular cases, such as $\lambda = 1$, $\mu = b+1$ or $\lambda = -b$, $\mu = b+1$, are mentioned in [16], see e.g. p. 90.

Following a practice introduced in [6], see also [12, 17, 18], I will rewrite the Srivastava–Singhal result somewhat in order to get expressions which are easier to handle from the combinatorial point of view. In the statement above, the variable x will be replaced by $(X+Y)/(X-Y)$, where X and Y are new variables, and t will be replaced by $X-Y$. The generating function will be written as an exponential series (i.e. $P_n^{(\alpha,\beta)}$ will be multipled by $n!$), and finally λ (μ resp.) will be replaced by $-\lambda$ ($-\mu$ resp). What we will prove combinatorially then reads as follows:

Let $\mathcal{P}_n^{(\alpha,\beta)}(X, Y)$ denote the nth (homogeneous) Jacobi-polynomial, which is related to the classical Jacobi-polynomial $P_n^{(\alpha,\beta)}(x)$ as follows:

$$\mathcal{P}_n^{(\alpha,\beta)}(X, Y) = n! \cdot (X-Y)^n \cdot P_n^{(\alpha,\beta)}\left(\frac{X+Y}{X-Y}\right), \quad \text{or}$$

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{n!} \cdot \mathcal{P}^{(\alpha,\beta)}\left(\frac{x+1}{2}, \frac{x-1}{2}\right).$$

Theorem.

$$\sum_{n\geq 0} \frac{1}{n!} \mathcal{P}^{(\alpha+\lambda n,\beta+\mu n)}(X, Y) = (1+\xi)^{1+\alpha}(1+\eta)^{1+\beta}$$

$$\times [1 - \lambda\xi - \mu\eta - (1+\lambda+\mu)\xi\eta]^{-1},$$

where $\xi = \xi(X, Y)$ *and* $\eta = \eta(X, Y)$ *are implicitly defined by*

$$\xi = X \cdot (1+\xi)^{1+\lambda} \cdot (1+\eta)^{1+\mu},$$
$$\eta = Y \cdot (1+\xi)^{1+\lambda} \cdot (1+\eta)^{1+\mu}.$$

For the proof given below we will assume that λ and μ are (arbitrary) nonnegative integers.

As indicated above, the general idea of proof is very much the same as the one employed by Foata and Leroux. The combinatorial model will be more general, but fortunately it still falls into the class of "incomplete Jacobi-configurations", a concept introduced and extensively studied by Leroux and myself in [12]. (Only one of the results of this article will be used here without proof.)

I decided to present this work in the language of "species of structures", as proposed by Joyal in [8], and developed fruitfully by himself and his colleagues at the Université du Québec at Montréal. Among them I would like to mention J. and G. Labelle, P. Leroux, H. Décoste, and F. Bergeron, who initiated me into this theory, which provides a clean and transparent way to represent combinatorial facts and constructions which might get buried under heavy notation and/or epic descriptions otherwise. It would certainly be helpful if the reader of this article had a basic knowledge of Joyal's theory, Chapter 1, 2, 5, 6 from [8], or the nice introductory article [11] by J. Labelle, or the introductory part from G. Labelle's elegant treatment of Lagrange-inversion [9] should provide an appropriate background. The last mentioned article is of particular interest, since one of the basic facts used here is also at the heart of Labelle's combinatorial Lagrange-inversion. I take the opportunity to clarify one particular aspect of this, which, in my opinion, remained obscure in [9], [8], and [6]. This will be done in the next section, which deals with a much more general situation than the one encountered in the proof of the Srivastava–Singhal result. Section 3 treats the "unweighted" case (i.e. $\alpha = \beta = 0$) of the latter, whereas in Section 4 some auxiliary species are introduced, which are needed for the general case. In Section 5 all this will be pieced together in exactly the same way as Foata and Leroux did in [6] for the case $\lambda = \mu = 0$. Finally I will state a multivariable generalization of the Srivastava–Singhal generating function which—combinatorially—results from the multi-sorted analogue of the bi-sorted situation considered in this article (plus some simple transformations of the parameters).

The present article can be read independently from [17] and [18], although a certain familiarity with the concept of "Jacobi-endofunctions" is assumed—the reader may look at [6] or the introductory parts of [12]. Note, however, that in [18] a different generalization of Jacobi's generating function is presented, where the overall scheme of proof also follows the Foata-Leroux way.

Some notational remarks

The cardinality of a set E will always be written $\#E$. The notation $[a \cdots b]$ refers to the set of integers $\{a, a+1, a+2, \ldots, b\}$, provided that $a \leq b$. For

$a > b$ this set is empty, by convention. As far as species are concerned, an equality '$A = B$' always means 'A is isomorphic to B', and not just equipotence.

2. Endofunctions, trees and contractions

The following combinatorial fact is well known and variations of it have often been described and used in the literature (see e.g. [3] p. 69, [5] Ch. 6, [7] p. 175):

"Endofunctions are permutations of rooted trees".

To make this intuitive statement precise, one may employ one of the various models that have been proposed for the treatment of (labelled) structures, e.g. Joyal's theory of species of structures. Naturally, the fact stated above is among the first examples one encounters in introductory texts and lectures presenting this theory, e.g. in [8] this occurs as "example 12", which says:

$$F = S(A),$$

where the following (ordinary) species show up:
F: the species of endofunctions,
S: the species of permutations,
A: the species of rooted trees ("arborescences").
When it comes to less simple situations—e.g. considering endofunctions with additional properties, and to more sophicated applications, such as Lagrange-inversion—it turns out that this statement has to be revised somewhat. Indeed, for his treatment of Lagrange-inversion G. Labelle employs "R-enriched species" (where R is some suitable ordinary species) such as:
F_R: the species of R-enriched endofunctions,
A_R: the species of R-enriched rooted trees,
C_R: the species of R-enriched contractions,
where "contraction" means: endofunction with a single periodic point, which we also call "root". (In the sequel I will write "R-endofunction" instead of "R-enriched endofunction" etc. for short). Though A and C are isomorphic, the R-species A_R and C_R usually are not (and may not even be equipotent). The reason for this is simple enough to explain: for a rooted tree the root does not belong to its own fiber (=preimage, if edges are directed towards the root), whereas for a contraction it does. Thus, as soon as we put a restriction on the fibers of our structures, e.g. by requiring R-structures on fibers, we have to distinguish carefully between R-rooted trees and R-contractions. But there is a simple relation between A_R and C_R:

$$C_R = X \cdot R'(A_R),$$

where X denotes the singleton species, and R' signals the derivative of R. See [9] or [8] for a more detailed explication. Concerning the decomposition of

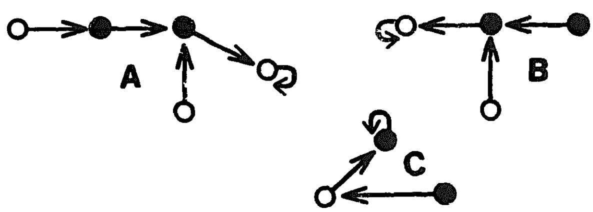
R-endofunctions we have (as stated correctly by Labelle and Joyal):

$$F_R = S(C_R).$$

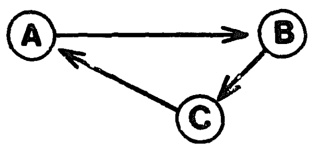
From the proofs of this identity, as given in [5] and [9], it is not quite clear what the isomorphism underlying "=" really looks like—both authors rely on "obvious" drawings. That there is an ambiguity lurking behind these all-too-obvious sketches came to my mind when reading through the article [6], where a similar situation arises, but now in the context of bi-sorted species. To make the point clear, let us look at the following example where, as in [6], we consider two-colored ("black" and "white", say) endofunctions with the property:

(∗) each point has at most one preimage colored black and at most one preimage colored white.

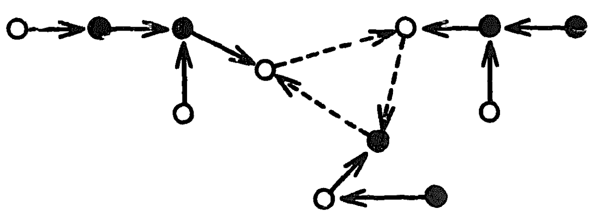
Here are three contractions of the required type:



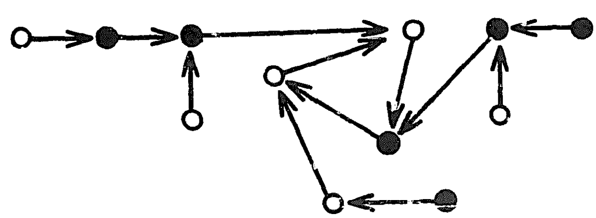
How should one piece them together in order to obtain a (∗)-endofunction if it is specified that the three components shall be permuted as follows:



The "obvious" idea of putting



is certainly not correct, since the result is not of the required type! What has to be done instead is·



In order to be able to preserve the (∗)-structures on fibers it is necessary to map the proper preimages of roots onto the root of the 'next' contraction—as one does with each root itself!

In the remainder of this section I will fix some notation concerning bi-sorted species and their generating functions. For notational convenience only bi-sorted species will be dealt with, though everything—including the specializations considered in the following sections—generalizes to multi-sorted species without any problems.

Let now R denote some bi-sorted species. I will write $R(X, Y)$ in order to indicate that there are now two sorts of points around: X-points and Y-points. R is then a functor which associates with each pair (U, V) of finite sets the set $R[U, V]$ of R-structures on (U, V). Throughout this article U (V resp.) will denote the set of X-points (Y-points resp.). The symbols x and y will act correspondingly as variables for our generating functions, e.g. in

$$\begin{aligned} R(x, y) &= \sum_{i,j\geq 0} \#R[[1\cdots i], [1\cdots j]] \frac{x^i y^j}{i!\,j!} \\ &= \sum_{n\geq 0} \frac{1}{n!} R_n(x, y), \end{aligned}$$

where

$$\begin{aligned} R_n(x, y) &= \sum_{0\leq i\leq n} \binom{n}{i} \#R[[1\cdots i], [i+1\cdots n]] x^i y^j \\ &= \sum \{\#R[u, v] x^{\#U} y^{\#V};\ U \mathbin{\dot\cup} V = [1\cdots n]\} \end{aligned}$$

is a homogeneous polynomial of degree n, representing those R-structures living on n points.

Using R one may now define the bi-sorted species $F_R(X, Y)$, $A_R(X, Y)$, and $C_R(X, Y)$ as usual. It will help to introduce the following auxiliary species;

$${}^XA_R = {}^XA_R(X, Y)\text{: } X\text{-rooted } R\text{-trees},$$
$${}^XC_R = {}^XC_R(X, Y)\text{: } X\text{-rooted } R\text{-contractions},$$

and similarly for ${}^YA_R$ and ${}^YC_R$. Besides the obvious identities

$$A_R = {}^XA_R + {}^YA_R \quad \text{and} \quad C_R = {}^XC_R + {}^YC_R$$

we have the following fundamental isomorphisms:

$$\begin{aligned} {}^XA_R &= X \cdot R({}^XA_R, {}^YA_R), & {}^YA_R &= Y \cdot R({}^XA_R, {}^YA_R), \\ {}^XC_R &= X \cdot (D_XR)({}^XA_R, {}^YA_R), & {}^YC_R &= Y \cdot (D_YR)({}^XA_R, {}^YA_R), \end{aligned}$$

where D_X (D_Y resp.) indicates the derivative w.r.t. X-points (Y-points resp.). Note that, as a solution of the implicit system given by the first of these two lines, $({}^XA_R, {}^YA_R)$ is uniquely determined—see [8] and [10]. [We should assume that $R[0, 0] \neq \emptyset$, since otherwise the concept of R-contractions and R-trees is not of much interest.]

The discussion from the first part of this section results in:

Proposition 1. *For any bi-sorted species $R(X, Y)$, there is an isomorphism between the species of R-endofunctions and the species of permutations of R-contractions, i.e. $F_R = S(C_R)$.*

As to generating functions, this proposition has the following consequence:

Corollary 1. *For any bi-sported species $R(X, Y)$, the generating function for the species of R-endofunctions is given by*

$$F_R(x, y) = \left[1 - x \cdot \left(\frac{\partial}{\partial x} R\right)(h(x, y), k(x, y)) - y \cdot \left(\frac{\partial}{\partial y} R\right)(h(x, y), k(x, y))\right]^{-1},$$

where $h(x, y)$ and $k(x, y)$ are functions implicitly defined by

$$h(x, y) = x \cdot R(h(x, y), k(x, y)),$$
$$k(x, y) = y \cdot R(h(x, y), k(x, y)).$$

The situation as given by the system for $h(x, y)$ and $k(x, y)$ calls for Lagrange-inversion, of course. But for the purpose of this article it is not necessary to enter further into this subject. This is in contrast to the original approach by Srivastava and Singhal, who used Lagrange-inversion (together with the Rodriguez-formula for the Jacobi polynomials) in order to prove their result.

3. A special case: Jacobi-endofunctions

In this section we will consider R-endofunctions for a particular class of bi-sorted species R. For integers $\lambda \geq 0$ let I_λ (I_λ^+ resp.) denote the (ordinary) species of injective mappings into $[0 \cdots \lambda]$ ($[1 \cdots \lambda]$ resp.). The corresponding generating functions are given by

$$I_\lambda(x) = (1 + x)^{1+\lambda} \quad \text{and} \quad I_\lambda^+(x) = (1 + x)^\lambda.$$

Let now λ, $\mu \geq 0$ be integers and define a bi-sorted species $I_{\lambda,\mu}(X, Y)$ by

$$I_{\lambda,\mu}(X, Y) = I_\lambda(X) \cdot I_\mu(Y).$$

In the sequel I will simply write $A_{\lambda,\mu}$ ($F_{\lambda,\mu}$ etc.) instead of $A_{I_{\lambda,\mu}}$ ($F_{I_{\lambda,\mu}}$ etc.). In this situation we get the following generating functions:

$$^XA_{\lambda,\mu}(x, y) = x \cdot (1 + {}^XA_{\lambda,\mu}(x, y))^{1+\lambda} \cdot (1 + {}^YA_{\lambda,\mu}(x, y))^{1+\mu},$$
$$^YA_{\lambda,\mu}(x, y) = (y/x) \cdot {}^XA_{\lambda,\mu}(x, y)$$
$$^XC_{\lambda,\mu}(x, y) = x \cdot (1 + \lambda) \cdot (1 + {}^XA_{\lambda,\mu}(x, y))^\lambda \cdot (1 + {}^YA_{\lambda,\mu}(x, y))^{1+\mu},$$
$$= (1 + \lambda) \cdot {}^XA_{\lambda,\mu}(x, y) \cdot [1 + {}^XA_{\lambda,\mu}(x, y)]^{-1},$$

and similarly for $^YC_{\lambda,\mu}(x, y)$.

From Corollary 1 we get in this particular case:

Corollary 2. *The species of $I_{\lambda,\mu}$-endofunctions has the generating function*

$$F_{\lambda,\mu}(x,y)=\frac{(1+\xi_{\lambda,\mu}(x,y))\cdot(1+\eta_{\lambda,\mu}(x,y))}{1-\lambda\cdot\xi_{\lambda,\mu}(x,y)-\mu\cdot\eta_{\lambda,\mu}(x,y)-(1+\lambda+\mu)\cdot\xi_{\lambda,\mu}(x,y)\cdot\eta_{\lambda,\mu}(x,y)}$$

where $\xi_{\lambda,\mu}$, $\eta_{\lambda,\mu}$ are implicitly defined by

$$\xi_{\lambda,\mu}(x,y)=x\cdot(1+\xi_{\lambda,\mu}(x,y))^{1+\lambda}\cdot(1+\eta_{\lambda,\mu}(x,y))^{1+\mu},$$
$$\eta_{\lambda,\mu}(x,y)=y\cdot(1+\xi_{\lambda,\mu}(x,y))^{1+\lambda}\cdot(1+\eta_{\lambda,\mu}(x,y))^{1+\mu}.$$

Looking now at the combinatorial picture one realizes that in the case $\lambda=\mu=0$ the restriction imposed on $I_{0,0}$-endofunctions is exactly condition (*) from the previous section: $I_{0,0}$-endofunctions are precisely the Jacobi-endofunctions of Foata–Leroux! From their work we know:

Proposition 2.

$$F_{0,0}(x,y)=\sum_{n\geq 0}\frac{1}{n!}\mathcal{P}_n^{(0,0)}(x,y).$$

As to the case $\lambda,\mu>0$, it helps to look at $I_{\lambda,\mu}$-endofunctions in a slightly different (but isomorphic!) way. For disjoint finite sets U, V the set $F_{\lambda,\mu}[U,V]$ is the set of all pairs (f,g) s.th.

- $f\in F[U,V]$,
- $g_U: U\to[0\cdots\lambda]$, $g_V: V\to[0\cdots\mu]$,
  where g_U (g_V resp.) is the restriction of g to U (to V resp.),
- g_U (g_V resp.) is injective on $f^{-1}(z)\cap U$ (on $f^{-1}(z)\cap V$ resp.) for all $z\in U\dot\cup V$.

Writing f_U (f_V resp.) for the part of f defined on U (V resp.), we may regard (f_U,g_U) ((f_V,g_V) resp.) as an injective mapping from U to $(U\dot\cup V)\times[0\cdots\lambda]$ (from V to $(U\dot\cup V)\times[0\cdots\mu]$ resp.), or—if we decide to identify $(U\dot\cup V)\times\{0\}$ with $U\cup V$ itself—as an injective mapping φ_U from U to $(U\dot\cup V)\dot\cup((U\dot\cup V)\times[1\cdots\lambda])$ (φ_V from V to $(U\dot\cup V)\dot\cup((U\dot\cup V)\times[1\cdots\mu])$). These objects are incomplete Jacobi-endofunctions in the sense of [12], and from their Proposition 2.3 [putting $A=U$, $B=V$, $C=(U\dot\cup V)\times[1\cdots\lambda]$, $D=(U\dot\cup V)\times[1\cdots\mu]$, $E=\emptyset$, $\alpha=0$, $\beta=0$] we get

Proposition 3. $(F_{\lambda,\mu})_n(x,y)=\mathcal{P}_n^{(n\lambda,n\mu)}(x,y)$ $(n\geq 0)$.

Corollary 3. *For integers $\lambda,\mu\geq 0$, the generating function for $I_{\lambda,\mu}$-endofunctions is given by*

$$F_{\lambda,\mu}(x,y)=\sum_{n\geq 0}\frac{1}{n!}\mathcal{P}_n^{(n\lambda,n\mu)}(x,y).$$

This corollary, together with Corollary 2, yields the Srivastava–Singhal result in the case $\alpha=\beta=0$. The case where α, β are treated as parameters will be dealt with in the following section.

4. An extension of the foregoing: counting "pure" cycles

The work done in the previous section did not use Proposition 2.3 of [12] in its full generality. What is necessary in order to get the complete Srivastava–Singhal result is: leaving A, B, C, D, E as they are, but treating α and β as arbitrary parameters. What this amounts to combinatorially is: associating with each pair (φ_U, φ_V) of injective functions, as described above, a weight or valuation

$$w_{\alpha,\beta}(\varphi_U, \varphi_V)=(1+\alpha)^{\mathrm{cyc}(\varphi_U)}\cdot(1+\beta)^{\mathrm{cyc}(\varphi_V)}.$$

Here $\mathrm{cyc}(\varphi_U)$ ($\mathrm{cyc}(\varphi_V)$ resp.) means the number of cycles which the injective function φ_U (φ_V resp.) has within U (V resp.)—I will call them pure-X-cycles (pure-Y-cycles resp.). We then have the following result;

Proposition 4.

$$\mathcal{P}_n^{(\alpha+\lambda n,\beta+\mu n)}(x,y)=\sum\{w_{\alpha,\beta}(\varphi_U,\varphi_V)x^{\#U}y^{\#V};(\varphi_U,\varphi_V)\in F_{\lambda,\mu}[U,V],\ U\cup V=[1\cdots n]\},\quad(n\geqslant 0).$$

As a consequence, if we denote by $F_{\lambda,\mu}^{(\alpha,\beta)}$ the $w_{\alpha,\beta}$-weighted species of $I_{\lambda,\mu}$-endofunctions, then

Corollary 4.

$$F_{\lambda,\mu}^{(\alpha,\beta)}(x,y)=\sum_{n\geqslant 0}\frac{1}{n!}\mathcal{P}_n^{(\alpha+\lambda n,\beta+\mu n)}(x,y).$$

If we go now back to the original description of $I_{\lambda,\mu}$-endofunctions, then we find that for any $(f,g)\in F_{\lambda,\mu}[U,V]$ we count via the valuation $w_{\alpha,\beta}$ the number of f-components such that all the f-periodic elements of this component are contained in U (or in V) and are all mapped onto 0 by g. In this sense we may speak of pure-X-cycles (or pure-Y-cycles) of (f,g). We are led to introduce some auxiliary species:

$I_{\lambda,\mu}^{+}(X,Y)=I_{\lambda}^{+}(X)\cdot I_{\mu}(Y)$, $\quad I_{\lambda,\mu}^{-}(X,Y)=I_{\lambda}(X)\cdot I_{\mu}^{+}(Y)$

$A_{\lambda,\mu}^{+}(X,Y)=X\cdot I_{\lambda,\mu}^{+}({}^{X}A_{\lambda,\mu},{}^{Y}A_{\lambda,\mu})$, $\quad A_{\lambda,\mu}^{-}(X,Y)=Y\cdot I_{\lambda,\mu}^{-}({}^{X}A_{\lambda,\mu},{}^{Y}A_{\lambda,\mu})$

${}^{X}F_{\lambda,\mu}(X,Y)$: $I_{\lambda,\mu}$-endofunctions, where all cycles are pure-X-cycles,

${}^{X}C_{\lambda,\mu}(X,Y)$: $I_{\lambda,\mu}$-contractions, where the root is a pure-X-cycle, and similarly for ${}^{Y}F_{\lambda,\mu}$ and ${}^{Y}C_{\lambda,\mu}$.

As before, we have

$$^X F_{\lambda,\mu} = S(^X C_{\lambda,\mu}) \quad \text{and} \quad ^Y F_{\lambda,\mu} = S(^Y C_{\lambda,\mu}),$$

but, as $^X C_{\lambda,\mu}$ ($^Y C_{\lambda,\mu}$ resp.) and $A^+_{\lambda,\mu}$ ($A^-_{\lambda,\mu}$ resp.) are isomorphic, we get

Proposition 5. *For integers $\lambda, \mu \geq 0$*

$$^X F_{\lambda,\mu} = S(A^+_{\lambda,\mu}) \quad \textit{and} \quad ^Y F_{\lambda,\mu} = S(A^-_{\lambda,\mu}).$$

But now, since

$$\begin{aligned} A^+_{\lambda,\mu}(x, y) &= x \cdot (1 + {}^X A_{\lambda,\mu}(x, y))^{\lambda} \cdot (1 + {}^Y A_{\lambda,\mu}(x, y))^{1+\mu} \\ &= {}^X A_{\lambda,\mu}(x, y) \cdot [1 + {}^X A_{\lambda,\mu}(x, y)]^{-1} \\ A^-_{\lambda,\mu}(x, y) &= y \cdot (1 + {}^X A_{\lambda,\mu}(x, y))^{1+\lambda} \cdot (1 + {}^Y A_{\lambda,\mu}(x, y))^{\mu} \\ &= {}^Y A_{\lambda,\mu}(x, y) \cdot [1 + {}^Y A_{\lambda,\mu}(x, y)]^{-1} \end{aligned}$$

we arrive at

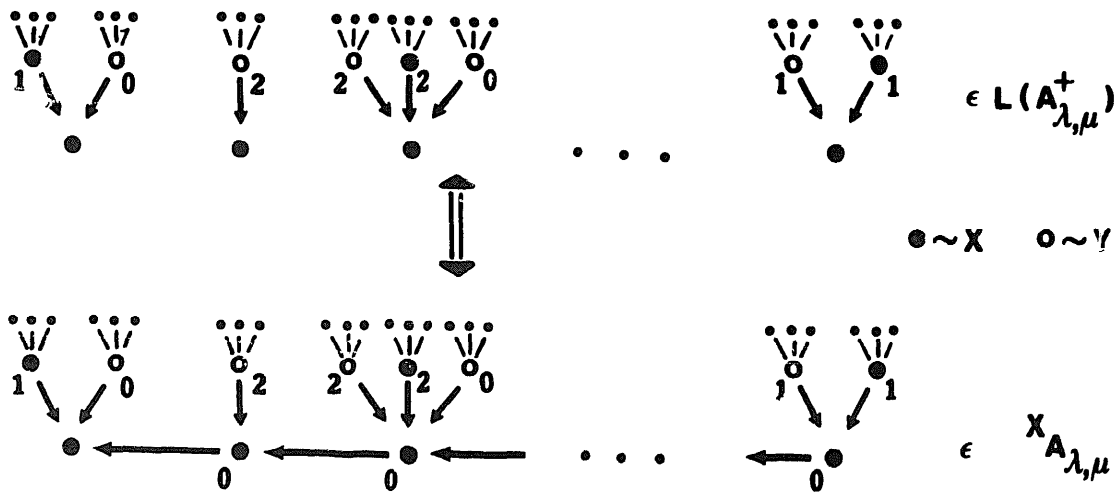
Corollary 5. *The generating functions for pure-X-(-Y resp.)-$I_{\lambda,\mu}$-endofunctions are given by*

$$^X F_{\lambda,\mu} = 1 + {}^X A_{\lambda,\mu}(x, y) \quad \textit{and} \quad ^Y F_{\lambda,\mu}(x, y) = 1 + {}^Y A_{\lambda,\mu}(x, y).$$

Remark. This last result can be obtained directly in a rather simple way by making two observations:

(1) $^X F_{\lambda,\mu} \approx L(A^+_{\lambda,\mu})$, where L denotes the ordinary species of linear lists, and where '$\approx$' means equipotence, but not necessarily isomorphism.

(2) $L(A^+_{\lambda,\mu}) = 1 + {}^X A_{\lambda,\mu}$, a fact which is easily visualized (rather than requiring an elaborate proof):



Note that by the definition of $A^+_{\lambda,\mu}$ none of the indicated arrows comes from an X-point which carries a g-value equal to 0, whereas the label 0 attached to the points of the last line indicates that the g-value of these points is set to 0.

The drawback of this proof is the inherent non-naturalness of (1), where we have an underlying bijection which transforms the linear list of roots into a permutation of recurrent elements by making use (e.g. via left-to-right-maxima-decomposition) of an assumed total order of the elements of the base set.

5. Proof and extension of the Srivastava–Singhal result

Using results from the previous sections it is now an easy task to finish the proof of the theorem, as stated in Section 1. The basic idea goes back to Foata–Leroux, and it works here in exactly the same way as it did in [6].

We first note that $F_{\lambda,\mu}$ is the product of three species:

$$F_{\lambda,\mu} = {}^{X}F_{\lambda,\mu} \cdot {}^{Y}F_{\lambda,\mu} \cdot {}^{m}F_{\lambda,\mu},$$

where ${}^{m}F_{\lambda,\mu}$ takes care of all those $I_{\lambda,\mu}$-endofunctions having no 'pure' cycles at all. If we put our standard valuation $w_{\alpha,\beta}$ on each of these species, then we get on the generating function level:

$$\begin{aligned} F_{\lambda,\mu}^{(\alpha,\beta)}(x,y) &= {}^{X}F_{\lambda,\mu}^{(\alpha,\beta)}(x,y) \cdot {}^{Y}F_{\lambda,\mu}^{(\alpha,\beta)}(x,y) \cdot {}^{m}F_{\lambda,\mu}^{(\alpha,\beta)}(x,y) \\ &= [{}^{X}F_{\lambda,\mu}(x,y)]^{1+\alpha} \cdot [{}^{Y}F_{\lambda,\mu}(x,y)]^{1+\beta} \cdot [{}^{m}F_{\lambda,\mu}(x,y)] \\ &= [{}^{X}F_{\lambda,\mu}(x,y)]^{\alpha} \cdot [{}^{Y}F_{\lambda,\mu}(x,y)]^{\beta} \cdot [{}^{X}F_{\lambda,\mu}(x,y) \cdot {}^{Y}F_{\lambda,\mu}(x,y) \cdot {}^{m}F_{\lambda,\mu}(x,y)] \\ &= [{}^{X}F_{\lambda,\mu}(x,y)]^{\alpha} \cdot [{}^{Y}F_{\lambda,\mu}(x,y)]^{\beta} \cdot F_{\lambda,\mu}(x,y). \end{aligned}$$

Here we have made use of a general principle for the enumeration of "exponential structures", which in our situation says:

$${}^{X}F_{\lambda,\mu}^{(\alpha,\beta)}(x,y) = \exp({}^{X}G_{\lambda,\mu}^{(\alpha,\beta)}(x,y)) = \exp[(1+\alpha){}^{X}G_{\lambda,\mu}(x,y)] = [{}^{X}F_{\lambda,\mu}(x,y)]^{1+\alpha}$$

where ${}^{X}G_{\alpha,\beta}$ denotes the species of connected ${}^{X}F_{\alpha,\beta}$-structures. The Srivastava–Singhal result now follows from Corollaries 2, 4, and 5 together with the above decomposition identity.

At this point one might be tempted to employ the combinatorial machinery developed so far in similar situations, e.g. for the extensions of the theorem as given by Srivastava and Singhal in [15]. There are also numerous generating functions of the same kind to be found in [16] which, at least to a large extent, can be attacked by the same or similar combinatorial methods—but this will be done elsewhere. Here I content myself to present a multivariable generalization of the Srivastava–Singhal result—which simply comes out of our combinatorial set-up if considering the multi-sorted situation. Since really no new proof is necessary, I will state it as

Corollary 6. *Let $k \geq 1$ and let $(\alpha_1, \ldots, \alpha_k)$, $(\lambda_1, \ldots, \lambda_k)$ denote two k-tuples of (complex) parameters. Then*

$$\sum_{n_1,\ldots,n_k \geq 0} \prod_i \left(\alpha_i + \lambda_i \sum_j n_j\right)_{n_i} \cdot x_i^{n_i}/n_i! = \prod_i (1+\xi_i)^{\alpha_i} \cdot \left[1 - \sum_j \lambda_j \xi_j\right]^{-1}$$

where the ξ_j are determined by the implicit system

$$\xi_j = x_j \cdot (1+\xi_j) \cdot \prod_i (1+\xi_i)^{\lambda_i} \quad (1 \leqslant j \leqslant k).$$

[In $\sum_j$ and $\prod_i$ the indices i, j are running from 1 to k.]

References

[1] R. Askey, Jacobi's generating function for Jacobi polynomials, Proc. Amer. Math. Soc., 71 (1978) 243–246.
[2] F. Bergeron, Modèles combinatoires de familles de polynômes orthogonaux, Rapport de Recherche #3 (Dept. Math. et Info., UQAM, Montréal, 1984).
[3] L. Comtet, Advanced Combinatorics (Reidel, Dordrecht and Boston, 1974).
[4] A. Erdelyi (ed.), Higher Transcendental Functions, vol III. (McGraw-Hill, New York, 1953).
[5] D. Foata, La série exponentielle dans les problèmes d'énumeration (Presses de l'Université de Montréal, 1974).
[6] D. Foata and P. Leroux, Polynômes de Jacobi, interpretation combinatoire et fonction génératrice, Proc. Amer. Math. Soc. 87 (1983) 47-53.
[7] I. Goulden and D. Jackson, Combinatorial Enumeration (J. Wiley, New York, 1983).
[8] A. Joyal, Une théorie combinatoire des séries formelles, Advances in Mathem. 42 (1981) 1–82.
[9] G. Labelle, Une nouvelle demonstration combinatoire des formules d'inversion de Lagrange, Advances in Mathemat. 42 (1981) 217–247.
[10] G. Labelle, Une combinatoire sous-jacente au theorème des fonctions implicites, J. Combinat. Theory 40 (1985) 377–393.
[11] J. Labelle, Applications diverses de la théorie combinatoire des espèces de structures, Ann. Sc. Math. Québec 7 (1983) 59–94.
[12] P. Leroux and V. Strehl, Jacobi polynomials: Combinatorics of the basic identities, Discrete Mathemat. 57, (1985) 167–187.
[13] E.D. Rainville, Special Functions (Macmillan, New York, 1960).
[14] H.M. Srivastava and J.P. Singhal, New generating functions for the Jacobi and related polynomials, J. Math. Anal. Appl. 41 (1972) 725–748.
[15] H.M. Srivastava and J.P. Singhal, A unified presentation of certain classical polynomials, Mathemat. Comput. 26 (1972) 969–975.
[16] H.M. Srivastava and H.L. Manocha, A Treatise on Generating Functions (J. Wiley, New York, 1984).
[17] V. Strehl, Combinatorics of Jacobi-configurations I: Complete oriented matchings, Proc. Coll. Combinatoire Enumerative (UQAM 1985) Springer Lecture Notes in Math. vol. 1234, 294–307 (Springer Verlag, Berlin 1986).
[18] V. Strehl, Combinatorics of Jacobi-configurations II: A rational approximation via matching polynomials, Actes du Seminaire Lotharingien de Combinatoire XIII (Bologne 1985) 113–133 (Publications de l'IRMA, Strasbourg 1986).
[19] G. Szego, Orthogonal Polynomials, AMS Colloquium Publications, 23 (Providence, 1975) 4th ed.