J. Math. Anal. Appl. 367 (2010) 451–460

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On solutions of linear differential equations with entire coefficients

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ARTICLE INFO

Article history:
Received 6 January 2010
Available online 1 February 2010
Submitted by D. Khavinson

Keywords:
Nevanlinna theory
Linear differential equations
Zeros

ABSTRACT

The first result of the paper concerns the effect of perturbation of the entire coefficients of certain linear differential equations on the oscillation of the solutions. Subsequent results involve the separation of the zeros of a Bank–Laine function.

1. Introduction

This paper will assume familiarity with the standard notation and concepts of Nevanlinna theory as set out in [14] and [18]. Our primary concern is the homogeneous linear differential equation

$$y^{(k)} + \sum_{j=0}^{k-2} A_j y^{(j)} = 0, \tag{1}$$

where $k \geqslant 2$ and $A_0, \dots, A_{k-2}$ are entire functions. It is well known [17] that every solution of (1) is entire. Recent years have seen extensive research (see, for example, [1,3–8,18–20,24,25]) involving the connection between the order of growth $\rho(A_j)$ of the coefficients A_j and the exponent of convergence $\lambda(f)$ of the zeros of a solution f, these being defined [14] by

$$\rho(A_j) = \limsup_{r\to\infty} \frac{\log^+ T(r, A_j)}{\log r}, \qquad \lambda(f) = \limsup_{r\to\infty} \frac{\log^+ N(r, 1/f)}{\log r}.$$

In particular it was shown in [3,24,25] that, if $k = 2$ and A_0 is transcendental of order at most $1/2$, then (1) cannot have two linearly independent solutions f_1 and f_2, each with $\lambda(f_j)$ finite, and a comparable result was proved for higher order equations in [20]. On the other hand, it is possible to have one solution f of (1) with no zeros at all, even for coefficients of very small growth. To see this set $f = e^B$ where B is an entire function. Then f solves (1) with $k = 2$ and $-A_0 = f''/f = B'' + (B')^2$, as well as similar equations of higher order obtained by computing $f^{(k)}/f$ in terms of B. Our main result shows, however, that small perturbations of such equations lead to solutions whose zeros must have infinite exponent of convergence.

Theorem 1.1. *Let $k \geqslant 2$ and let $A_0, \dots, A_{k-2}$ be entire functions of finite order with the following property. There exists a set $E_1 \subseteq [1, \infty)$, of infinite logarithmic measure, such that, with $A = A_0$,*

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doi:10.1016/j.jmaa.2010.01.054

$$\log r + \sum_{1 \leqslant j \leqslant k-2} \log^+ M(r, A_j) = o\Big(\inf_{|z|=r} \log^+ \big|A(z)\big|\Big) \tag{2}$$

as $r \to \infty$ with $r \in E_1$. Assume that (1) *has a solution f with $\lambda(f) < \infty$ and*

$$\lim_{r \to \infty,\, r \in E_1} \frac{n(r, 1/f) \log r}{T(r, A)} = 0. \tag{3}$$

Let h and B_j (for $1 \leqslant j \leqslant k-2$) be entire functions such that $h \not\equiv 0$ and

$$\log^+ M\big(r, h'\big) + \sum_{1 \leqslant j \leqslant k-2} \log^+ M(r, B_j) = o\Big(\inf_{|z|=r} \log^+ \big|A(z)\big|\Big) \tag{4}$$

as $r \to \infty$ with $r \in E_1$. Then, with

$$B = A + h, \tag{5}$$

the differential equation

$$y^{(k)} + \sum_{1 \leqslant j \leqslant k-2} B_j y^{(j)} + By = 0 \tag{6}$$

cannot have a solution g with $\lambda(g) < \infty$.

For a related result involving the effect of perturbation on the oscillation of solutions see [7, Theorem 3.1]. In Section 2 it will be deduced from well-known minimum modulus results that (2), (3) and (4) are satisfied if either of the following two hypotheses hold:

(a) The function A is transcendental of order 0, while h and the A_j and B_j (for $1 \leqslant j \leqslant k-2$) are polynomials, and f has finitely many zeros.
(b) The function A has order $\rho(A) < 1/2$, while h and the A_j and B_j (again for $1 \leqslant j \leqslant k-2$) have order less than $\rho(A)$, and $\lambda(f) < \rho(A)$.

In particular Theorem 1.1 improves a result of the first author: it was proved in [1] that, if $k = 2$ and (1) has a solution f with $\lambda(f) < \rho(A_0) < 1/2$, while $h \not\equiv 0$ but $\rho(h) < \rho(A_0)$, then Eq. (6) cannot have a solution g with $\lambda(g) < \rho(A_0)$.

Our remaining results concern the zero sequence of a Bank–Laine function of finite order. An entire function E is called a Bank–Laine function if $E(z) = 0$ implies $E'(z) = \pm 1$: such functions arise precisely as products of linearly independent solutions of the second order equation

$$w''(z) + A(z) w(z) = 0, \tag{7}$$

with A entire and the normalisation that the solutions have Wronskian equal to 1 [5]. Here E and A are connected by the Bank–Laine formula [3]

$$4A = \left(\frac{E'}{E}\right)^2 - 2\frac{E''}{E} - \frac{1}{E^2}, \tag{8}$$

which was introduced in [3] and has inspired much subsequent research: see [18,20] for references. It is conjectured that if the zeros of E have finite exponent of convergence then $\rho(A)$ is infinite or a positive integer.

Now Shen [26] has shown that in general a Bank–Laine function E may have an arbitrary zero sequence (z_n), provided of course that z_n tends to infinity without repetition. On the other hand, if it is assumed that E has finite order then it is possible to prove that certain zero sequences cannot arise for E [11,21]. The following theorem was proved by ElZaidi [12,13].

Theorem 1.2. *(See [12,13].) Let E be a Bank–Laine function of finite order, with zero sequence (z_n). Then there exists $M > 0$ such that $|z_m - z_n| \geqslant \exp(-|z_n|^M)$ for $m \neq n$ and n large.*

Thus the zeros of a Bank–Laine function of finite order cannot be too close together. See [2] for a related result. It is clear from ElZaidi's proof that any M greater than the order $\rho(E)$ of E will suffice in Theorem 1.2. It seems natural to conjecture that Theorem 1.2 holds with the positive constant M depending only on the exponent of convergence of the zero-sequence (z_n). Although we have not been able to prove this in general, it is certainly true if the zeros are all real, as the following theorem shows.

Theorem 1.3. *Let E be a Bank–Laine function of finite order having only real zeros (a_n), and denote by λ their exponent of convergence. Let $\lambda < \mu < \infty$. Then for all large n and for all $m \neq n$ we have*

$$|a_m - a_n| \geqslant \exp\bigl(-|a_n|^{\mu}\bigr). \tag{9}$$

We remark that results on the zero distribution of Bank–Laine functions with real zeros appear in [11,21].

To motivate the next result we observe that Bank–Laine functions of exponential type with sparsely distributed zeros were constructed by Langley [21] using a variational method. It is then natural to ask whether a similar construction could be employed to produce a Bank–Laine function of finite order whose zeros occur in sparsely distributed small discs, with at least two zeros in each disc. The next theorem proves a lower bound for the radius of any such discs, depending on the exponent of convergence of the zeros, and lends some support to the conjecture stated following Theorem 1.2.

Theorem 1.4. *Let E be a Bank–Laine function of finite order, and let $\lambda = \lambda(E)$ be the exponent of convergence of the zeros of E. Let $\varepsilon > 0$ and assume that all but finitely many zeros of E lie in the union of the discs*

$$D_m = B\bigl(a_m, \exp\bigl(-|a_m|^{\lambda+\varepsilon}\bigr)\bigr), \tag{10}$$

where a_m tends to infinity with

$$|a_m - a_n| \geqslant \varepsilon |a_n| \quad \text{for } m \neq n. \tag{11}$$

Then for large m the function E has at most one zero in D_m.

Here $B(a,r)$ denotes the open disc of centre a and radius r, and $S(a,r)$ will be used to denote the corresponding boundary circle.

The hypothesis that E has finite order is not redundant in Theorems 1.3 and 1.4, because of the result of Shen [26] already mentioned.

2. Application of minimum modulus theorems

The upper logarithmic density of a set $E \subseteq [1,\infty)$ is defined [15, p. 331] using the characteristic function $\chi(t)$ of E by

$$\overline{\log \operatorname{dens}}(E) = \limsup_{r\to\infty} \frac{1}{\log r}\left(\int_1^r \frac{\chi(t)\,dt}{t}\right), \qquad \chi(t) = \begin{cases} 1 & \text{if } t \in E, \\ 0 & \text{if } t \notin E. \end{cases}$$

The lower logarithmic density is defined analogously, but with lim sup replaced by lim inf. The classical $\cos \pi\rho$ theorem [15, p. 331] states that if A is a transcendental entire function of order $\rho < \alpha < 1$ then

$$\inf_{|z|=r} \log\bigl|A(z)\bigr| \geqslant \cos \pi\alpha \log M(r, A)$$

on a set E_1 of lower logarithmic density at least $1 - \rho/\alpha$. We will also require the following modified $\cos \pi\rho$ theorem [9,10].

Theorem 2.1. *(See [9,10].) Suppose that A is a transcendental entire function with $\rho(A) = \rho < \frac{1}{2}$. If $\sigma < \rho$, then the set*

$$E_1 = \Bigl\{ r \geqslant 1\colon \inf_{|z|=r} \log\bigl|A(z)\bigr| > r^{\sigma} \Bigr\} \tag{12}$$

has positive upper logarithmic density.

The standard example $\cos\sqrt{z}$ shows that we cannot take $\rho = \frac{1}{2}$ in Theorem 2.1.

The choice $\alpha = 1/4$ in the $\cos \pi\rho$ theorem then makes it immediately clear that under the hypothesis (a) following the statement of Theorem 1.1, the conditions (2), (3) and (4) are satisfied.

Suppose next that hypothesis (b) holds. Pick τ, σ such that

$$\max\bigl\{\lambda(f), \rho(h)\bigr\} < \tau < \sigma < \rho(A) < \frac{1}{2} \tag{13}$$

and

$$\max\bigl\{\rho(A_j), \rho(B_j)\bigr\} < \tau \quad \text{for } 1 \leqslant j \leqslant k-2.$$

This time applying Theorem 2.1 gives a set E_1 as in (12) of positive upper logarithmic density and so infinite logarithmic measure, and for $r \in E_1$ conditions (2) and (4) are satisfied, using the well-known fact that $\rho(h') = \rho(h)$ [16]. Moreover, since (13) and standard inequalities yield

$$n(r, 1/f)\log r = O\big(N(2r, 1/f)\log r\big) = o\big(r^{\tau}\big),$$

condition (3) also holds for $r \in E_1$.

3. Results needed for the proof of Theorem 1.1

We need the following definition and lemma [18, pp. 84–87] for our main results.

Definition 3.1. (See [18].) Let $B(z_n, r_n)$ be open discs in the complex plane. We say that the countable union $\bigcup B(z_n, r_n)$ is an R-set if the centres z_n tend to infinity with n and the sum $\sum r_n$ of the radii r_n is finite.

Lemma 3.1. *(See [18].) Suppose that $f \not\equiv 0$ is a meromorphic function of finite order in the plane, and let $j \in \mathbb{N}$. Then there exists a positive integer N such that*

$$\frac{f^{(j)}(z)}{f(z)} = O\big(|z|^N\big)$$

holds for large z outside of an R-set.

We require next a special case of a theorem from [20], which gives a local representation for the logarithmic derivative of a solution of (1) with few zeros.

Theorem 3.1. *(See [20].) Let $k \geqslant 2$ and let $A_0, \dots, A_{k-2}$ be entire functions of finite order, with $A = A_0$ transcendental. Let E_1 be a subset of $[1, \infty)$, of infinite logarithmic measure, and with the following property. For each $r \in E_1$ there exists an arc a_r of the circle $S(0, r)$, such that*

$$\lim_{r\to\infty,\, r\in E_1} \frac{\min\{\log|A(z)|:\ z \in a_r\}}{\log r} = +\infty, \tag{14}$$

and, if $k \geqslant 3$,

$$\lim_{r\to\infty, r\in E_1} \max\left\{\frac{\log^+|A_j(z)|}{\log|A(z)|}:\ z \in a_r\right\} = 0, \tag{15}$$

for $j = 1, \dots, k-2$. Let f be a solution of (1) with $\lambda(f) < \infty$. Then there exists a set $E_2 \subseteq [1, \infty)$ of finite measure, such that for large $r \in E_0 = E_1 \setminus E_2$ the following is true. We have

$$\frac{f'(z)}{f(z)} = c_r A(z)^{1/k} - \left(\frac{k-1}{2k}\right)\frac{A'(z)}{A(z)} + O\big(r^{-2}\big) \quad \text{for all } z \in a_r. \tag{16}$$

Here the constant c_r satisfies $c_r^k = -1$ and may depend on r but not, for a given $r \in E_0$, on z. The branch of $A^{1/k}$ in (16) is analytic on a_r (including in the case where a_r is the whole circle $S(0, r)$).

We will employ the following well-known representation for higher order logarithmic derivatives [14, p. 73].

Lemma 3.2. *(See [14].) Let f be an analytic function, and let $F = f'/f$. Then for $k \in \mathbb{N}$ we have*

$$\frac{f^{(k)}}{f} = F^k + \frac{k(k-1)}{2}F^{k-2}F' + P_{k-2}(F),$$

where P_{k-2} is a differential polynomial with constant coefficients, which vanishes identically for $k \leqslant 2$ and has degree at most $k-2$ when $k > 2$.

4. Proof of Theorem 1.1

Let Eqs. (1) and (6) and their coefficients, as well as the set E_1, be as in the statement of the theorem. Suppose that (1) has a solution f with $\lambda(f) < \infty$ and satisfying (3), and that (6) has a solution g with $\lambda(g) < \infty$. We set

$$f = Pe^U, \qquad g = Qe^V, \tag{17}$$

where P, Q, U and V are entire functions which satisfy $\rho(P) = \lambda(f) < \infty$ and $\rho(Q) = \lambda(g) < \infty$. Let

$$F = \frac{f'}{f} = \frac{P'}{P} + U', \qquad G = \frac{g'}{g} = \frac{Q'}{Q} + V'. \tag{18}$$

It is clear from (2), (4) and the inequality

$$M(r,h) \leqslant rM(r,h') + |h(0)|$$

that

$$\log^+|h(z)| + \log^+|h'(z)| = o(\log|A(z)|) \quad \text{for } |z| = r \in E_1. \tag{19}$$

Let $E_2 \subseteq [1,\infty)$ be a set of finite measure so that, for some $M_1 \in \mathbb{N}$, and for $j = 1,\dots,k$,

$$\left|\frac{A'(z)}{A(z)}\right| + \left|\frac{P^{(j)}(z)}{P(z)}\right| + \left|\frac{Q^{(j)}(z)}{Q(z)}\right| \leqslant r^{M_1} \quad \text{for } |z| = r \in [1,\infty) \setminus E_2. \tag{20}$$

Such E_2 and M_1 exist by Lemma 3.1.

The next step is to estimate f'/f and g'/g in terms of A. We apply Theorem 3.1 to (1) and (6), choosing a_r to be the whole circle $|z| = r \in E_1$. This is possible since (2), (4), (5) and (19) imply that (14) and (15) hold, and also that (14) and (15) are satisfied with A, A_j replaced by B, B_j. Hence for large $r \in E_0 = E_1 \setminus E_3$, where $E_2 \subset E_3$ and E_3 has finite measure, the following is true. We have, by (16),

$$\frac{f'(z)}{f(z)} = cA(z)^{1/k} - \left(\frac{k-1}{2k}\right)\frac{A'(z)}{A(z)} + O(r^{-2}), \quad |z| = r,\ c^k = -1, \tag{21}$$

and

$$\frac{g'(z)}{g(z)} = d\,B(z)^{1/k} - \left(\frac{k-1}{2k}\right)\frac{B'(z)}{B(z)} + O(r^{-2}), \quad |z| = r,\ d^k = -1. \tag{22}$$

Next, we apply the binomial theorem to expand $B^{1/k}$ and B'/B in terms of $A^{1/k}$ and A'/A. Using (5) and (19), we get for $|z| = r \in E_0$, on suppressing the variable z for brevity,

$$B^{1/k} = (A+h)^{1/k} = A^{1/k}\left(1 + \frac{h}{A}\right)^{1/k} = A^{1/k}\left(1 + O\left(\frac{|h|}{|A|}\right)\right) \tag{23}$$

and

$$\frac{B'}{B} = \frac{A'+h'}{A+h} = \frac{A'+h'}{A(1+h/A)} = \frac{A'}{A}\left(1 + O\left(\frac{|h|}{|A|}\right)\right) + O\left(\frac{|h'|}{|A|}\right). \tag{24}$$

Using (19), (20), (22), (23) and (24), we deduce that, for $|z| = r \in E_0$,

$$\frac{g'(z)}{g(z)} = dA(z)^{1/k} - \left(\frac{k-1}{2k}\right)\frac{A'(z)}{A(z)} + O(r^{-2}), \quad d^k = -1. \tag{25}$$

We recall from Theorem 3.1 that c and d may depend on r but, for a given r, do not depend on z. The following two lemmas are then the key to the proof of Theorem 1.1.

Lemma 4.1. *Suppose that c, d are as in (21) and (25) respectively. Then $c = d$ for all large $r \in E_0$.*

Proof. We may write $d = \omega c$ where $\omega^k = 1$. Multiplying (21) by ω and subtracting (25) we get

$$\omega\left(\frac{f'(z)}{f(z)} + \left(\frac{k-1}{2k}\right)\frac{A'(z)}{A(z)}\right) = \frac{g'(z)}{g(z)} + \left(\frac{k-1}{2k}\right)\frac{A'(z)}{A(z)} + O(r^{-2}).$$

Integrating around $|z| = r_n$, where $r_n \to \infty$ with $r_n \in E_0$, we then find that

$$\omega\left[n\left(r_n, \frac{1}{f}\right) + \left(\frac{k-1}{2k}\right)n\left(r_n, \frac{1}{A}\right)\right] + o(1) = n\left(r_n, \frac{1}{g}\right) + \left(\frac{k-1}{2k}\right)n\left(r_n, \frac{1}{A}\right). \tag{26}$$

But the right-hand side of (26) must be positive since

$$n\left(r_n, \frac{1}{g}\right) \geqslant 0 \quad \text{and} \quad n\left(r_n, \frac{1}{A}\right) > 0,$$

using the fact that (2) gives

$$\log r = o\big(T(r, A)\big) = o\big(T(r, 1/A)\big) = o\big(N(r, 1/A)\big) = o\big(n(r, 1/A)\log r\big)$$

for $r \in E_0$. Applying the same considerations to the left-hand side of (26) and recalling that $\omega^k = 1$, we deduce that $\omega = 1$ and $c = d$. □

Lemma 4.2. *The quotient f/g is non-constant.*

Proof. Assume the contrary. Then f solves both equations (1) and (6), so that

$$\sum_{1 \leqslant j \leqslant k-2} (B_j - A_j) f^{(j)} + hf = 0. \tag{27}$$

Hence we must have $k \geqslant 3$ and $B_j - A_j \not\equiv 0$ for at least one $j \geqslant 1$, since $h \not\equiv 0$. Let q be the largest integer such that $B_q - A_q \not\equiv 0$. Then Lemma 3.2, (18) and (27) give

$$(B_q - A_q)\left(F^q + \frac{q(q-1)}{2}F^{q-2}F' + P_{q-2}(F)\right) + \cdots + h = 0,$$

from which we deduce, using (2), (4), (18), (21), the fact that F has finite order [8], and the notation

$$S(r) = \log r + T(r, h) + \sum_{1 \leqslant j \leqslant k-2} \big(T(r, A_j) + T(r, B_j)\big), \tag{28}$$

that

$$m(r, A) \leqslant O\big(m(r, F) + \log r\big) = O\big(S(r)\big) = o\Big(\inf_{|z|=r} \log\big|A(z)\big|\Big) = o\big(T(r, A)\big)$$

as $r \to \infty$ with $r \in E_0$, which is an obvious contradiction. □

To complete the proof of Theorem 1.1, we can now use (21), (25) and Lemma 4.1 to get, as $r \to \infty$ with $r \in E_0$,

$$\frac{f'(z)}{f(z)} = \frac{g'(z)}{g(z)} + O\big(r^{-2}\big), \quad |z| = r,$$

and hence

$$n\left(r, \frac{1}{f}\right) = n\left(r, \frac{1}{g}\right) \quad \text{for large } r \in E_0. \tag{29}$$

Using (18) and (20) we obtain

$$\big|U'(z) - V'(z)\big| \leqslant 3r^{M_1}$$

for $|z| = r$ and large $r \in E_0$, and since U and V are entire we deduce that $Q_0 = U' - V'$ is a polynomial. Thus (18) becomes

$$F = G + M, \quad M = \frac{P'}{P} - \frac{Q'}{Q} + Q_0, \tag{30}$$

in which M does not vanish identically, since f/g is non-constant by Lemma 4.2. Combining (6) and (18) with Lemma 3.2 and its notation, we obtain

$$-A - h = G^k + \frac{k(k-1)}{2}G^{k-2}G' + P_{k-2}(G) + \sum_{1 \leqslant j \leqslant k-2} B_j\left[G^j + \frac{j(j-1)}{2}G^{j-2}G' + P_{j-2}(G)\right]. \tag{31}$$

Similarly, using (1), (30) and Lemma 3.2 leads to

$$\begin{aligned}
-A &= \frac{f^{(k)}}{f} + \sum_{1 \leqslant j \leqslant k-2} A_j \frac{f^{(j)}}{f} \\
&= F^k + \frac{k(k-1)}{2}F^{k-2}F' + P_{k-2}(F) + \sum_{1 \leqslant j \leqslant k-2} A_j\left[F^j + \frac{j(j-1)}{2}F^{j-2}F' + P_{j-2}(F)\right] \\
&= (G+M)^k + \frac{k(k-1)}{2}(G+M)^{k-2}\big(G' + M'\big) + P_{k-2}(G+M) \\
&\quad + \sum_{1 \leqslant j \leqslant k-2} A_j\left[(G+M)^j + \frac{j(j-1)}{2}(G+M)^{j-2}\big(G' + M'\big) + P_{j-2}(G+M)\right].
\end{aligned}$$

On subtracting (31) we then get

$$h = kMG^{k-1} + S_{k-2}(G, M)$$

where $S_{k-2}(G, M)$ is a differential polynomial in G and M, of total degree at most $k-2$ in G and its derivatives, and with coefficients which are linear combinations of those A_j and B_j with $1 \leqslant j \leqslant k-2$. This gives

$$kG = \frac{1}{MG^{k-2}}\big[h - S_{k-2}(G, M)\big].$$

Since G and M have finite order, we deduce using (2), (3), (4), (18), (20), (25), (28), (29) and (30) that

$$\begin{aligned}
m(r, A) &\leqslant C_0\big(m(r, G) + \log r\big)\\
&\leqslant C_1\left(m(r, M) + m\left(r, \frac{1}{M}\right)\right) + C_1 S(r)\\
&\leqslant C_2 T(r, M) + C_1 S(r) + O(1)\\
&\leqslant C_2 N(r, M) + C_3 S(r)\\
&\leqslant C_2\left(N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{g}\right)\right) + C_3 S(r)\\
&\leqslant C_2\left(n\left(r, \frac{1}{f}\right) + n\left(r, \frac{1}{g}\right)\right)\log r + C_3 S(r)\\
&\leqslant 2C_2 n\left(r, \frac{1}{f}\right)\log r + C_3 S(r)\\
&= o\big(T(r, A)\big)\\
&= o\big(m(r, A)\big)
\end{aligned}$$

for large $r \in E_0$, where the C_j denote positive constants. This is evidently a contradiction, and Theorem 1.1 is proved.

5. Proof of Theorem 1.3

We require the following result from [21].

Theorem 5.1. *(See [21].) Let $F = We^Q$ be a Bank–Laine function, with Q a polynomial of positive degree N and W an entire function of order $\rho(W) < N$. Let $\theta_1 < \theta_2$ and $c > 0$ and suppose that $|\mathrm{Re}(Q(z))| > c|z|^N$ as $z \to \infty$ in the sector S given by $\theta_1 \leqslant \arg z \leqslant \theta_2$. Then F has finitely many zeros in S.*

We now prove Theorem 1.3. We may assume that the sequence (a_n) is infinite, since otherwise there is nothing to prove. By hypothesis there is a Bank–Laine function E of finite order with zero sequence (a_n). Thus we can write

$$E = We^P, \quad P = Q + iR, \tag{32}$$

where W is a real entire function of order λ and Q and R are polynomials with real coefficients. Since E is a Bank–Laine function and $R(a_n)$ is real, we must have

$$E'(a_n) = W'(a_n)e^{Q(a_n)+iR(a_n)} = \pm 1 \quad \text{and} \quad e^{iR(a_n)} = \pm 1.$$

It follows that $F = We^Q$ is a Bank–Laine function.

Lemma 5.1. *Let $\deg Q = N$. Then we have $N \leqslant \lambda$.*

Proof. Suppose that this is not the case. Since Q is a real polynomial there exists $c > 0$ such that $|\mathrm{Re}(Q(z))| > c|z|^N$ for large z near the real axis. Hence we may apply Theorem 5.1 twice, first in a sector $|\arg z| < \varepsilon$, and subsequently in a sector $|\arg z - \pi| < \varepsilon$, to conclude that F has finitely many zeros on the real axis, and so has E. This contradiction proves Lemma 5.1. □

Hence we have $\log M(r, F) = o(r^\mu)$ as $r \to \infty$. Now let a_n, a_m be zeros of E with a_n large and $|a_n - a_m| \leqslant 1$. Let

$$g(z) = \frac{F(z)}{(z - a_m)(z - a_n)}.$$

Then g is entire and for $|z| = 2|a_n|$ we have

$$|z - a_m| \geqslant |z - a_n| - 1 \geqslant |a_n| - 1 \geqslant 1.$$

It follows that

$$\left|g(z)\right| \leqslant M\big(2|a_n|, F\big) \leqslant \exp\big(|a_n|^{\mu}\big)$$

for $|z| \leqslant 2|a_n|$. Now Cauchy's integral formula gives

$$1 = \left|F'(a_n)\right| = \left|(a_n - a_m)g(a_n)\right| = \left|\frac{1}{2\pi i}\int_{|z-a_n|=1} g(z)\left(\frac{a_n - a_m}{z - a_n}\right)dz\right| \leqslant \exp\big(|a_n|^{\mu}\big)|a_n - a_m|,$$

which implies (9) and completes the proof of Theorem 1.3.

6. Nehari's univalence criterion

For the proof of Theorem 1.4, we require the following disconjugacy result, which is a standard consequence of a well-known univalence criterion of Nehari [17,22,23].

Lemma 6.1. *Let $a \in \mathbb{C}$ and $r \in (0, \infty)$ and let the function A be analytic on $B(a, r)$ with $|A(z)| \leqslant r^{-2}$ there. Then each non-trivial solution of (7) has at most one zero in $B(a, r)$.*

We include the standard proof for completeness. Let f be a non-trivial solution of (7) in $B(a, r)$, and choose a second solution g such that f and g are linearly independent. Then $F = f/g$ is locally univalent in $B(a, r)$. Let $G(z) = F(a + rz)$ on $B(0, 1)$. Then the Schwarzian derivatives of F and G satisfy, for $z \in B(0, 1)$,

$$S_G = \frac{G'''}{G'} - \frac{3}{2}\left(\frac{G''}{G'}\right)^2, \qquad \big(1 - |z|^2\big)^2\left|S_G(z)\right| \leqslant \left|S_G(z)\right| = r^2\left|S_F(a + rz)\right| = 2r^2\left|A(a + rz)\right| \leqslant 2.$$

Hence G is univalent on $B(0, 1)$ by Nehari's criterion [17,22,23], and f has at most one zero in $B(a, r)$.

7. Proof of Theorem 1.4

Let the Bank–Laine function E and the sequence (a_m) be as in the statement of Theorem 1.4. Then there exists an entire function A such that E is the product of linearly independent solutions of (7). Since the zeros of E have exponent of convergence λ we may write $E = \Pi e^P$, where Π is an entire function of order $\rho(\Pi) = \lambda$ and P is a polynomial of degree N. Let δ be small and positive, and assume that m is large and that E has at least two zeros in D_m.

Suppose first that there exists $\zeta \in D'_m = B(a_m, 2|a_m|^{-N})$ such that

$$\left|e^{P(\zeta)}\right| < \exp\big(-|a_m|^{\lambda+\delta}\big).$$

Then for all $z \in D'_m$ we have

$$\left|e^{P(z)}\right| \leqslant \left|e^{P(\zeta)}\right| \exp\left(\int_{\zeta}^{z} \left|P'(t)\right| |dt|\right) \leqslant 2\exp\big(-|a_m|^{\lambda+\delta}\big)$$

and, since Π has order λ,

$$\left|E(z)\right| \leqslant \left|e^{P(z)}\right| \exp\big(|a_m|^{\lambda+o(1)}\big) \leqslant \exp\left(-\frac{1}{2}|a_m|^{\lambda+\delta}\right).$$

Since m is large we have $D_m \subseteq B(a_m, |a_m|^{-N})$, and Cauchy's estimate for derivatives then gives $E'(z) = o(1)$ for all $z \in D_m$. But this contradicts the assumption that the Bank–Laine function E has zeros in D_m, at each of which we have $E'(z) = \pm 1$.

Hence we have

$$\left|e^{P(z)}\right| \geqslant \exp\big(-|a_m|^{\lambda+\delta}\big) \quad \text{for all } z \in D'_m. \tag{33}$$

We use (33) and the Bank–Laine formula (8) to estimate A on D_m. Since E has no zeros in $D'_m \setminus D_m$, the Poisson–Jensen formula and standard estimates for logarithmic derivatives (see Lemma 3.1) give a positive real number M_1, independent of m, such that

$$\left|\frac{E'(z)}{E(z)}\right| + \left|\frac{E''(z)}{E(z)}\right| \leqslant |a_m|^{M_1} \quad \text{and} \quad \left|\log\left|\Pi(z)\right|\right| \leqslant |a_m|^{\lambda+\delta}$$

for all $z \in S(a_m, |a_m|^{-N})$. Combining these estimates with (8) and (33) then gives

$$\left|A(z)\right| \leqslant \exp\left(|a_m|^{\lambda+2\delta}\right)$$

for all $z \in S(a_m, |a_m|^{-N})$ and hence for all $z \in D_m$, by the maximum principle. Since δ may be chosen arbitrarily small, it follows from Lemma 6.1 that no non-trivial solution of (7) has more than one zero in D_m.

Since we are assuming that E has at least two zeros in D_m we conclude that E has precisely two zeros a and b in D_m, with $E'(a) = 1$ and $E'(b) = -1$. Hence we must have $N > \lambda$, because if this is not the case then E has order λ and we obtain

$$2 = \left|E'(a) - E'(b)\right| = \left|\int_b^a E''(t)\,dt\right| \leqslant |a-b| \exp\left(|a_m|^{\lambda+o(1)}\right) = o(1),$$

a contradiction. Set $E(z) = (z-a)(z-b)h(z)$. Then

$$1 = E'(a) = (a-b)h(a), \qquad -1 = E'(b) = (b-a)h(b), \qquad h(a) = h(b). \tag{34}$$

Since a and b are the only zeros of E in $B(a_m, \varepsilon|a_m|/2)$ by (11), and since $N > \lambda$ and δ is small, simple estimates give

$$\frac{\Pi'(z)}{\Pi(z)} = O\left(|a_m|^{\lambda+\delta-1}\right) \quad \text{and} \quad \frac{E'(z)}{E(z)} = \frac{\Pi'(z)}{\Pi(z)} + P'(z) \sim P'(z)$$

for $z \in S(a_m, \delta|a_m|)$. This yields

$$\frac{h'(z)}{h(z)} = \frac{E'(z)}{E(z)} - \frac{1}{z-a} - \frac{1}{z-b} \sim P'(z)$$

for $z \in S(a_m, \delta|a_m|)$ and hence for $z \in D_m$, by the maximum principle. Combining this with (34) then gives

$$1 = \frac{h(a)}{h(b)} = \exp\left(\int_b^a \frac{h'(t)}{h(t)}\,dt\right) = \exp\left(\int_b^a P'(t)\big(1+o(1)\big)\,dt\right) = \exp\left(\int_b^a P'(a)\big(1+o(1)\big)\,dt\right)$$
$$= \exp\big((a-b)P'(a)\big(1+o(1)\big)\big) \neq 1,$$

and this contradiction proves the theorem.

Acknowledgments

The authors thank the referee for several helpful suggestions to improve the readability of the paper.

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