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Vanishing via Lifting to Second Witt Vectors and a Proof of an Isotriviality Result

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A proof based on reduction to finite fields of Esnault and Viehweg's stronger version of the Sommese vanishing theorem for k-ample line bundles is given. This result is used to give different proofs of isotriviality results of Parshin and Migliorini. © 1999 Academic Press

INTRODUCTION

This note contains a proof of Esnault and Viehweg's improvement of the Sommese vanishing theorem for k-ample line bundles. The proof is based on Raynaud's proof of the Akizuki–Kodaira–Nakano vanishing theorem. A vector bundle version and a weak-Lefschetz-type theorem, which are easy consequences of the vanishing result, but do not seem to have appeared in the literature, are also given. The vanishing theorem allows us to give an algebraic proof of a result of Parshin and Migliorini on the isotriviality of smooth fibrations over curves of genus at most 1 with fibers either curves of genus at least 2 or minimal surfaces of general type.

This paper is organized as follows. Section 1 contains, for the convenience of the reader, basic known facts about the spreading out technique. Section 2 contains the proofs of vanishing in the line bundle case, of Theorem 2.2, and of its easy corollaries, Corollaries 2.4 and 2.5. Section 3 contains a proof of Parshin's and Migliorini's result, Theorem 3.2.

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1. PRELIMINARIES ON THE "SPREADING OUT" TECHNIQUE

Any reasonable and finite amount of geometry defined over a field of characteristic zero K can be "spread out" over an algebra of finite type over $\mathbb{Z}$, R, contained in K. We can then try to use the resulting "fibration" to say something about the original situation over K. This very vaguely presented principle is made precise in EGA [16, IV 8, 10]. A concise exposition is given by Illusie in [6, Sect. 6].

Let us exemplify the "spreading out" procedure by listing the properties we need in the sequel of the paper. Let us fix some notation. Let Z be a scheme, let $g: U \to V$ be a Z-morphism of Z-schemes, let F be an $\mathcal{O}_U$-module, and let z be a point in Z. We denote the fibers $U \times_Z \operatorname{Spec} k(z)$ and $V \times_Z \operatorname{Spec} k(z)$ simply by U_z and V_z, the restriction of F to U_z by F_z, and the induced morphism $U_z \to V_z$ by g_z.

A line bundle is said to be *semi-ample* if some positive power of it is generated by its global sections. Note that this does not imply the (false) statement that every sufficiently high power of it is generated by its global sections; e.g., a non-trivial torsion line bundle over an elliptic curve.

Let b be a non-negative integer. A semi-ample line bundle L on a projective variety X is said to be *b-ample* if given a positive integer N such that $L^{\otimes N}$ defines a morphism $\phi_{|NL|}: X \to \mathbb{P}$, then the non-empty fibers of $\phi_{|NL|}$ have dimension at most b. This notion does not depend on N. Note that the Kodaira–Iitaka dimension $\kappa(L) = \dim \phi_{|NL|}(X)$ for any N as above; see [11]. Given a ring T, $W_2(T)$ is the ring of Witt vectors of length two associated with T.

PROPOSITION 1.1. *Let $f: X \to Y$ be a projective K-morphism of projective K-varieties and let F, L, and A, respectively, be a coherent sheaf on X, a b-ample line bundle on X, and an ample line bundle on Y, respectively. Then there exist an integral $\mathbb{Z}$ algebra of finite type R contained in K, projective* $\operatorname{Spec} R =: \mathscr{R}$*-schemes $\rho: \mathscr{X} \to \mathscr{R}$ and $\sigma: \mathscr{Y} \to \mathscr{R}$, a projective $\mathscr{R}$-morphism $\mathfrak{f}: \mathscr{X} \to \mathscr{Y}$, coherent sheaves $\mathscr{F}$ and $\mathscr{L}$ on $\mathscr{X}$ and $\mathscr{A}$ on $\mathscr{Y}$, and a Zariski-dense open subset $\mathscr{U} \subset \mathscr{R}$, contained in the locus of $\mathscr{R}$ which is smooth over* $\operatorname{Spec} \mathbb{Z}$, *with the properties listed below.*

(i) *The objects X, Y, f, F, L, and A, respectively, are obtained from the corresponding objects $\mathscr{X}$, $\mathscr{Y}$, $\mathfrak{f}$, $\mathscr{F}$, $\mathscr{L}$, and $\mathscr{A}$, respectively, by means of the base change induced by $R \hookrightarrow K$.*

(ii) *The objects ρ, σ, $\mathscr{F}$, $\mathscr{L}$, $\mathscr{A}$, and $R^i\mathfrak{f}_*\mathscr{F}$ are all flat over $\mathscr{U}$. In particular the formation of the sheaves $R^i\mathfrak{f}_*\mathscr{F}$ commutes with taking the fiber over any point $u \in \mathscr{U}$.*

(iii) *The sheaves* $\mathscr{L}$ *and* $\mathscr{A}$ *are locally free on* $\rho^{-1}(\mathscr{U})$ *and* $\sigma^{-1}(\mathscr{U})$, *respectively*, $\mathscr{A}$ *is* ρ*-ample*, *and* $\mathscr{L}_u$ *is b-ample for every* $u \in \mathscr{U}$.

(iv) *If X is smooth*, *then* $\mathscr{U}$ *can be chosen so that* ρ *and* σ *are smooth and flat*, *respectively*, *over* $\mathscr{U}$.

(v) *If s is a closed point in* $\mathscr{U}$, *then* $\mathscr{X}_s$ *lifts to* $W_2(k(s))$. *Moreover*, *we can choose* $\mathscr{U}$ *so that* $\operatorname{char} k(s) > \dim X$, *for every closed point* $s \in \mathscr{U}$.

Proof. The first step of properties follows from EGA [16, IV 8]. The second one is Grothendieck's theorem on flattening stratifications, and cohomology and base change as established in EGA [16, III 6.9.10]. The third is EGA [16, IV 8]; the statement about b-ampleness stems from the fact that a line bundle L on X is b-ample and not $(b-1)$-ample iff there exist ample divisors $D_1, \dots, D_b$ such that $L_{|\bigcap_{i=1}^{b} D_i}$ is ample and b is the minimum number for which this is possible. The fourth one follows from generic smoothness and generic flatness, respectively. The last one follows after shrinking $\mathscr{U}$, if necessary, so that the conclusion on the characteristics is true, and by the fact that $\mathscr{U}$ is smooth (over $\mathbb{Z}$); see [6, pp. 152–153]. ∎

The following elementary lemma contains basic facts to be used later.

LEMMA 1.2. *Every closed point in* $\mathscr{R}$ *has a finite and a fortiori perfect residue field*. *Every Zariski-dense open subset of* $\mathscr{R}$ *contains a Zariski-dense set of closed points*.

Proof. EGA [16, IV 10.4.6, 10.4.7]. ∎

The following result contains the basic fact that we need about smooth projective varieties over a perfect field $\mathfrak{k}$ which lift to $W_2(\mathfrak{k})$. It is proven in [3] as a consequence of Théorème 2.1 (and Corollaire 2.3).

THEOREM 1.3 (Akizuki–Kodaira–Nakano vanishing theorem). *Let X be a smooth projective variety of dimension d over a perfect field* $\mathfrak{k}$ *of characteristic* $p > d$ *which admits a lifting to* $W_2(\mathfrak{k})$, *let M b a line bundle on X and let* ν *be a non-negative integer*. *Assume that*, *for some positive integer n*,

$$H^j\left(X, \Omega_X^i \otimes M^{\otimes p^n}\right) = \{0\} \qquad \forall i + j = \nu.$$

Then

$$H^j\left(X, \Omega_X^i \otimes M\right) = \{0\} \qquad \forall i + j = \nu.$$

In particular, *if* $\check{M}$ *is ample*, *then any* $\nu < d$ *will do*.

Proof. See [3, Corollaire 2.3 and Lemme 2.9]. Note that in our setting we can conclude that the relevant hypercohomology group $\mathbb{H}^\nu = \{0\}$, which is what is needed. ∎

If M is ample, then one proves the Kodaira–Akizuki–Nakano vanishing theorem as an easy consequence of Serre vanishing (Raynaud). This theorem allows one to re-prove the classical Kodaira–Akizuki–Nakano vanishing theorem in characteristic zero by spreading out $X \to K$ to $\mathscr{X} \to \mathscr{R}$ and $\Omega^{\bullet}_{X/K}$ to $\Omega^{\bullet}_{\mathscr{X}/\mathscr{R}}$ by the upper semicontinuity properties of the dimensions of cohomology groups.

2. ESNAULT AND VIEHWEG'S IMPROVEMENT OF THE SOMMESE VANISHING THEOREM FOR k-AMPLE VECTOR BUNDLES USING W_2-LIFTING

The following result is a slight improvement of the Sommese vanishing theorem for k-ample line bundles; see [13, Theorem 3.36 and Corollary 5.20]. This improvement in the line bundle case is owing to Esnault and Viehweg [4, Theorem 2.4], who used analytical methods.

We offer a new proof which is algebraic and passes through reduction to finite fields, the Deligne–Illusie decomposition theorem, and characteristic zero vanishing theorems made valid in the finite field case by "propagation." This technique already has been employed in the context of non-complete varieties in [1], where one finds, as a particular case, a proof of the Sommese vanishing theorem.

The "shifted" version for the vector bundle case follows, as is now standard, by a theorem of Le Potier simplified by Schneider; see [13, Theorem 5.16].

We need the following fact in the sequel when we use the improved Grauert–Riemenschneider theorem in the finite field context, where it does not hold in general.

LEMMA 2.1. *Let f: $X \to Y$, let $F, \mathscr{X} \xrightarrow[\mathfrak{f}]{} \mathscr{Y}$, let $\mathscr{F}$ be as in Lemma* 1.1, *and let η be the generic point of $\mathscr{R}$. Assume that $R^j f_* F = 0$ for a fixed integer j. Then $R^j \mathfrak{f}_{u*} \mathscr{F}_u = 0$ for every point u in a suitable Zariski-dense open subset $\mathscr{U}$ of $\mathscr{R}$.*

Proof. The base changes induced by $R \hookrightarrow k(\eta)$ and $k(\eta) \hookrightarrow K$ are both flat; the second one is even faithfully flat. It follows that $[R^j \mathfrak{f}_* \mathscr{F}]_\eta = R^j \mathfrak{f}_{\eta *} \mathscr{F}_\eta = 0$. Since σ is proper, we see that, after shrinking $\mathscr{U}$ if necessary, the assertion holds by Lemma 1.1(ii). ∎

We fix K, a field of characteristic zero.

THEOREM 2.2. *Let X be a smooth projective variety of dimension d over K and let L be a b-ample line bundle over X of Kodaira–Iitaka dimension $\kappa(L)$.*

Then

$$H^j\big(X, \Omega_X^i \otimes L^\vee\big) = \{0\} \qquad \forall (i,j)\ s.t.\quad i+j < \min(\kappa(L), d-b+1). \tag{1}$$

Equivalently

$$H^q(X, \Omega_X^p \otimes L) = \{0\}$$
$$\forall (p,q)\ s.t.\quad p+q > 2d - \min(\kappa(L), d-b+1). \tag{2}$$

Proof. The two statements are equivalent by virtue of Serre duality and the canonical isomorphisms $\Omega_X^{d-l} \simeq K_X \otimes \check{\Omega}_X^l$.

Step I. We first prove the assertion under the additional hypothesis that mL is generated by its global sections for every $m \gg 0$. We prove statement (2). There are a surjective and projective morphism $g\colon X \to Y$ with connected fibers onto a normal variety Y and an ample line bundle A on Y such that $L = g^*A$. By assumption, Y has dimension $\kappa(L)$ and the fibers of g are at most b dimensional.

We have the following two properties:

(a) $R^l g_*$ is the zero functor for every $l > b$.

(b) $R^l g_* K_X = 0$ for every $l > d - \kappa(L)$; this improved Grauert–Riemenschneider vanishing theorem follows from the improved Kawamata–Viehweg vanishing theorem [13, Corollary 7.50] and from [2, Proposition 8.9]. Note that everything is algebraic here and that Hironaka's resolution of singularities is needed.

We apply Proposition 1.1 to $g\colon X \to Y, L, A$ and $F := K_X \simeq \omega_{X/K}$ with the choice of $\mathscr{F} \simeq \omega_{\mathscr{X}/\mathscr{R}}$. Let $s \in \mathscr{R}$ be a closed point belonging to the open set $\mathscr{U}$ (recall Lemma 1.2) over which all the conditions of Proposition 1.1 for $\omega_{\mathscr{X}/\mathscr{R}}$ and all of its direct images via $\mathfrak{g}$ are met.

By virtue of Lemma 2.1, we may shrink $\mathscr{U}$, if necessary, so that the two conditions (a) and (b) above are met for $\mathfrak{g}_s$ as well.

By abuse of notation, we denote $\mathscr{L}_s$, $\mathscr{A}_s$, $\mathscr{X}_s$, $\mathscr{Y}_s$, and $\mathfrak{g}_s$, respectively, by L_s, A_s, X_s, Y_s, and g_s. In order to apply Theorem 1.3, we need to check that $H^q(X_s, \Omega_{X_s}^p \otimes L_s^{\otimes m}) = \{0\}$ for every $m \gg 0$ in the prescribed range for p and q. This is an immediate consequence of the Leray spectral sequence for g_s and conditions (a) and (b) for g_s, as we now show.

By virtue of the Leray spectral sequence and of Serre vanishing, there exists m_s such that for every $m \geq m_s$ and for every pair of indices p and q we have

$$H^q\big(X_s, \Omega_{X_s}^p \otimes L_s^{\otimes m}\big) \simeq H^0\big(Y_s, R^q g_{s*} \Omega_{X_s}^p \otimes A_s^{\otimes m}\big).$$

If p and q are in the prescribed range, then $R^q g_{s*}\Omega^p_{X_s}$ is zero so that the group on the right vanishes: in fact, $p+q \geq d+b$ and $p \leq d$ so that we can use (a) and (b) above.

Now the standard semicontinuity argument. By virtue of what we have proved and by virtue of Proposition 1.1 we can assert the existence of a Zariski-dense open subset $\mathscr{U} \subset \mathscr{R}$ over which the sheaves $\mathscr{L}$ and $\Omega^{\bullet}_{\mathscr{X}/\mathscr{R}}$ are locally free and such that, given any closed point $s \in \mathscr{U}$, we have that $\operatorname{char} k(s) > d$, X_s lifts to $W_2(k(s))$, and there is a certain positive integer m_s such that for every p and q in the prescribed range and for every $m \geq m_s$ we have

$$H^q\left(X_s, \Omega^p_{X_s} \otimes L_s^{\otimes m}\right) = \{0\}.$$

We choose $m := [\operatorname{char}(k(s)]^{m_s}$ and apply a straightforward descending induction coupled with Theorem 1.3 to deduce that the vanishings above hold with $m = 1$. The vanishing in characteristic zero follows from the fact that, by the upper-semicontinuity of these dimensions, we have vanishing over the generic point $\eta \in \mathscr{R}$ and, by the flat base change induced by $k(\eta) \hookrightarrow K$, we therefore have vanishing over K. The proof of Step I is now complete.

Step II. We now remove the additional assumption of Step I: we prove the theorem by induction on $d - \kappa(L)$ using Step I and by means of an easy procedure to construct, on a suitable covering of X, $d - \kappa(L)$ sections of the pullback of L with a base locus of dimension $d - \kappa(L)$. Note that we may assume that $\kappa(L) > 0$, since, if $\kappa(L) = 0$, then there is nothing to prove.

Let c be a positive integer such that cL is generated by its global sections. We get a surjective morphism onto a variety Y of dimension $\kappa(L)$. Choose a general section σ_1 of cL so that it defines a smooth divisor D'_1 on X. Consider the corresponding cyclic covering $C_1\colon X_1 \to X$ branched along D'_1 and ramified along the smooth divisor $D_1 := C_1^{-1}(D'_1)$ which is the zero set of a section of the line bundle $L_1 := C_1^* L$. We also have that Ω^l_X is a direct summand of $C_{1*}\Omega^l_{X_1}$ for every integer l such that $0 \leq l \leq d$ (see [5, p. 6]).

By iterating this procedure, we obtain a sequence of cyclic coverings $X_{i+1} \xrightarrow{C_{i+1}} X_i$ together with line bundles $L_{i+i} = C^*_{i+1} L_i$ such that $\dim Bs|L_{\kappa(L)}| \leq d - \kappa(L)$ and Ω^l_X is a direct summand of $C_*\Omega^l_{X_{\kappa(L)}}$, where $C\colon X_{\kappa(L)} \to X$ is the induced morphism. Since C is finite, $H^r(X, \Omega^l_X \otimes L)$ is a direct summand of $H^r(X_{\kappa(L)}, \Omega^l_{X_{\kappa(L)}} \otimes L_{\kappa(L)})$. It is therefore enough to prove the theorem under the additional assumption that $|L| \neq \emptyset$ and that $\dim Bs|L| \leq d - \kappa(L)$.

We work by ascending induction on $\mathfrak{l} := d - \kappa(L)$. Let $\mathfrak{l} = 0$. Then $\dim Bs|L| = 0$. By Zariski's [14] Theorem 6.2, we obtain that mL is generated by its global sections for *every* $m \gg 0$ and we conclude by virtue of Step I.

Let the contention be true for every $\mathfrak{l}' < \mathfrak{l}$. Let us prove it for $\mathfrak{l}$. We prove statement (1).

Let H be an ample hypersurface of X such that it is smooth, $L_{|H}$ is $(b-1)$-ample, and $\kappa(L_{|H}) = \kappa(L)$. Such an H exists by a result of Hironaka [13, Theorem 3.39] and by the fact that H maps onto Y by the assumption $d - \kappa(L) > 0$. Consider the exact sequences

$$0 \to \Omega_X^l \otimes L^\vee \otimes H^\vee \to \Omega_X^l \otimes L^\vee \to \Omega_X^l \otimes L^\vee \otimes \mathcal{O}_H \to 0, \tag{3}$$

$$0 \to \Omega_H^{l-1} \otimes L_{|H}^\vee \otimes H_{|H}^\vee \to \Omega_X^l \otimes L^\vee \otimes \mathcal{O}_H \to \Omega_H^l \otimes L_{|H}^\vee \to 0. \tag{4}$$

Since $L + H$ is ample, we can use the standard Akizuki–Kodaira–Nakano vanishing theorem on X and on H and get, for i and j such that $i + j < \min(\kappa(L), d - b + 1)$, the two injective maps:

$$H^j\big(X, \Omega_X^i \otimes L^\vee\big) \hookrightarrow H^j\big(H, \Omega_X^i \otimes L^\vee \otimes \mathcal{O}_H\big) \hookrightarrow H^j\big(H, \Omega_H^i \otimes L_{|H}^\vee\big).$$

Since

$$\mathfrak{l}_H := (d-1) - \kappa\big(L_{|H}\big) = (d-1) - \kappa(L) < d - \kappa(L) = \mathfrak{l}$$

and

$$i + j < \min(\kappa(L), d - b + 1) = \min\Big(\kappa\big(L_{|H}\big), \big(d - 1 - (b-1) + 1\big)\Big),$$

we can apply the induction hypothesis and conclude that the last group on the right is trivial. This gives the wanted vanishing result. ∎

Remark 2.3. Sommese's theorem is sharp, as stated. However, note that $\kappa(L) \geq d - b$ and that the strict inequality is possible. If we have equality, then f is equidimensional and we get Sommese's statement. If we have strict inequality, then Theorem 2.2 improves Sommese's by one unit.

Consider the case where K is algebraically closed and $f\colon X \to Y$ is the blowing up of $\mathbb{P}^3$ at either a (closed) point or along a line. Let L be the pullback of the hyperplane bundle. In the former case $\min(\kappa(L), d - b + 1) = \min(3, 2) = 2$; the Sommese vanishing theorem predicts vanishing for $1 + j < 1$; Theorem 2.2 predicts vanishing for $i + j < 2$; moreover, $H^1(X, \Omega_X^1 \otimes L^\vee)$ is one dimensional. In the latter case $\min(\kappa(L), d - b + 1) = \min(3, 3) = 3$; the Sommese's theorem predicts vanishing for $i + j < 2$ and Theorem 2.2 for $i + j < 3$. This example shows that Theorem 2.2 is sharp and improves upon Sommese's. Moreover it shows concretely why it is sharp: for in the case we blow up a point p, we have that $R^1 f_* \Omega^1_X$ is isomorphic to the skyscaper sheaf at p of stalk $K = k(p)$. The second case puts in evidence that, for the purpose of Akizuki–Kodaira–Nakano-type statements, a line bundle which is semi-ample, big and 1-ample is as good as ample. We use this fact in an essential way in the proof of Theorem 3.2.

The following two results follow easily from Theorem 2.2 and they do not need the proof given above. The first one admits a dual formulation which we omit for brevity. The interested reader can easily formulate and prove vanishing results analogous to the first one which involve $K_X \otimes \bigwedge^l E$ and more generally $\Omega^p_X \otimes \bigwedge^l E$; see [13, Sect. 5].

Recall that a vector bundle E is said to be b-ample if the associated tautological line bundle ξ_E is b-ample.

COROLLARY 2.4. *Let things be as in Theorem 2.2 except that we replace the line bundle L by a rank r, b-ample vector bundle E and we set $\kappa(E) := \kappa(\xi_E)$. Then*

$$H^q(X, \Omega^p_X \otimes E) = \{0\}$$

$$\forall (p,q) \ s.t.\ p + q > 2[d + (r-1)] - \min(\kappa(E), d + r - b).$$

Proof. By virtue of a result of Le Potier (cf. [13, 5.17, 5.21, and 5.28]), $H^q(X, \Omega^p_X \otimes E) \simeq H^q(\mathbb{P}(E), \Omega^p_{\mathbb{P}(E)} \otimes \xi_E)$. Apply Theorem 2.2 to the pair $(\mathbb{P}(E), \xi_E)$. ∎

COROLLARY 2.5 (Weak Lefschetz theorem). *Let X be a smooth projective variety of dimension d defined over K. Let D be an effective smooth divisor on X such that the associated line bundle L is b-ample and is of Kodaira–Iitaka dimension $\kappa(L)$. Then the canonical morphisms of de Rham cohomology $H^l_{\mathrm{DR}}(X/K) \to H^l_{\mathrm{DR}}(D/K)$ are*

(i) *isomorphisms for $l < \min(\kappa(L), d - b + 1) - 1$,*

(ii) *injective for $l < \min(\kappa(L), d - b + 1)$.*

Proof. Note that $\kappa(L_{|D}) = \kappa(L) - 1$ and that $b_D \leq b$. We conclude by means of easy diagram considerations on the long cohomology sequences associated with (3) and (4). ∎

For more statements in the vein of the corollary above, see [13, Theorem 3.40].

3. A PROOF OF A RESULT OF PARSHIN AND MIGLIORINI

We give an algebraic proof of Theorem 3.2 below. The proof hinges on Theorem 2.2 and on the following positivity result of Kollár (which holds in greater generality than the one stated below), whose original proof is Hodge theoretic and has been proven again algebraically by Kollár and Viehweg.

In what follows everything is defined over an algebraically closed field of characteristic zero K.

THEOREM 3.1 (cf. [8]). *Let $f: X \to P$ be a surjective morphism with connected fibers of nonsingular projective varieties, where P is a nonsingular curve. Assume that the fibers of f are of general type and are not all birationally isomorphic to each other. Then the vector bundle $f_* \omega_{X/P}^{\otimes m}$ is ample for infinitely many values of the positive integer m.*

THEOREM 3.2 (cf. [12, 10]). *Let X be a nonsingular projective variety of dimension d, let P be a nonsingular complete curve of genus $g(P) \leq 1$, and let $f: X \to P$ be a surjective smooth morphism such that all the fibers are connected varieties of general type with nef canonical bundle. If $d \leq 3$, then all the fibers are isomorphic to each other.*

Proof. Since, if necessary, we can take a double cover $P \to \mathbb{P}^1$, P any elliptic curve, we can assume that $g(P) = 1$. Note that, in this case, $K_X = \omega_{X/P} \simeq \Omega_{X/P}^{d-1}$.

Seeking a contradiction, we assume that the fibers of f are not all birationally isomorphic to each other. By the base-point-free theorem of Kawamata and Shokurov (cf. [2]) applied to the pluricanonical line bundles of the fibers and by Noetherian induction on P, there exists a positive integer m_0 such that for every $m \geq m_0$, the natural morphism $f^* f_* mK_X \to mK_X$ is surjective and induces a P-morphism $g_m: X \to \mathbb{P}(f_* \omega_{X/P}^{\otimes m})$ with the property that $mK_X \simeq g_m^* \xi_{E_m}$. This morphism induces the birationally isomorphic stable pluricanonical morphisms on the fibers of $f: X \to P$.

By virtue of Theorem 3.1, we can choose the integer m above so that $E_m := f_* \omega_{X/P}^{\otimes m}$ is ample. It follows that mK_X, being the pullback of an ample line bundle via g_m, is semi-ample and $(d-2)$-ample. This conclusion holds for K_X as well.

The following argument is owing to Kovács (cf. [9, p. 370]). Consider the exact sequences

$$0 \to \Omega_{X/P}^{i-1} \otimes K_X \to \Omega_X^i \otimes K_X \to \Omega_{X/P}^i \otimes K_X \to 0.$$

For every $1 \leq p \leq d-1$, we get short exact sequences

$$\begin{aligned} H^{d-p}\big(X, \Omega_{X/P}^p \otimes K_X\big) &\xrightarrow{\alpha_p} H^{d-(p-1)}\big(X, \Omega_{X/P}^{p-1} \otimes K_X\big) \\ &\longrightarrow H^{d-(p-1)}\big(X, \Omega_X^p \otimes K_X\big), \end{aligned}$$

where, when $d \leq 3$, the maps α_p are all surjective by Theorem 2.2. We compose all these surjective maps α_p and get a surjection

$$\{0\} = H^1(X, K_X \otimes K_X) \twoheadrightarrow H^d(X, K_X) \simeq K,$$

the first isomorphism on the left being the Kawamata–Viehweg vanishing theorem. This is a contradiction.

The fibers are therefore birationally isomorphic to each other. The result follows from the uniqueness of minimal models for curves and surfaces. ∎

Remark 3.3. The case $d = 2$ is owing to Parshin; see [12]. The case $d = 3$ is owing to Migliorini; see [10]; his proof uses analytic techniques. The result is false without the restriction on the genus of the base; see [7].

Since the automorphism group of the fibers is finite and the fibers are all isomorphic to each other, the fibration is isotrivial, i.e., it becomes trivial after a finite base change $P' \to P$. If we drop the nefness assumption, then a birationally isomorphic statement still holds; see [10]. A similar but weaker statement holds for any value of d and it is owing to Kovács [9], who has also proved the case $d = 4$ of Theorem 3.2. Zhang [15] has proved a similar statement in any dimension under the assumption that all fibers have ample canonical bundle.

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