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Noise Issues of Modal Identification using Eigensystem Realization Algorithm

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Abstract

The eigensystem realization algorithm (ERA) is one of the most popular methods in civil engineering applications for estimating the modal parameters, including complex-valued modal frequencies and modal vectors, of dynamic systems. In dealing with noisy measurement data, the ERA partitions the realized model into principal (signal) and perturbational (noise) portions so that the noise portion can be disregarded. During the separation of signal and noise, a critical issue is the determination for the dimensions of the block Hankel matrix which is built from noisy measurement data. We show that the signal and noise matrices can be better separated when the number of block-rows and number of block-columns of the corresponding block Hankel matrix are chosen to be close to each other. We introduce the concept of using the Frobenius norm (L_2-norm) of the signal and noise matrices to quantify the signal to noise ratio in the global sense (involving multiple signals). We also propose a verification procedure to justify that the estimated modal parameters are noise insensitive and thus indeed associated with the true system. The procedure involves artificially injecting random noise into the measured signals (which are noisy signals) to create noisy-noisy signals, then comparing the identification results obtained respectively from the measured and noisy-noisy signals. Using experimental data collected from a test plate, we demonstrate that if signal and noise portions have been properly separated while using the measured data, then the artificial noise would almost completely accumulate to the noise portion. Therefore, the modal estimation based on the signal portion only would remain the same by using either the measured or the noisy-noisy signals.

Keywords: Model identification; ERA, noise; modal parameters; model order

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1. INTRODUCTION

Modal identification involves estimating the modal parameters, such as complex-valued modal frequencies and modal vectors, of a structural system from measured input-output data. Many different modal identification methods have been developed, analyzed and tested. In the early eighties when multi-input multi-output (MIMO) testing became popular, the eigensystem realization algorithm (ERA) was developed to handle MIMO test data (Juang and Pappa 1985; Juang 1994). Today, the ERA method has become one of the most popular methods in civil engineering applications for experimental modal analysis.

The ERA is a time-domain method based on the Markov parameters (i.e., pulse response). The knowledge of Markov parameters makes it possible to construct a block Hankel matrix as the basis for realization of a discrete-time state-space model. Developed based on the minimum realization theory, the ERA identifies modal parameters from noisy measurement data. In handling the noise, the ERA partitions the realized model into principal and perturbational (noise) portions so that the noise portion can be disregarded (Juang and Pappa 1986). Although the ERA is mathematically sound, a critical issue that has not been addressed adequately is the optimum determination for the dimensions of the block Hankel matrix (Juang 1994). An inappropriate choice for the dimensions might cause the ambiguity between the signal and noise portions, and lead to inaccurate estimation for the modal parameters.

One objective of this paper is to provide insight on how to choose the dimensions of the block Hankel matrix such that the signal portion can be properly separated from the noise. Furthermore, a novel procedure is proposed to evaluate the robustness and accuracy of the modal parameters estimated from measurements with unknown degree of contamination. The numerical investigation will be based on experimental data collected from a test plate with 4 inputs and 32 outputs.

2. THEORETICAL BACKGROUND

A time-invariant system can be described by its state-space representation as:

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A_c x} + \mathbf{B_c u} \\ \mathbf{y} &= \mathbf{Cx}\end{aligned} \tag{1}$$

where $\mathbf{x} \in \Re^n, \mathbf{u} \in \Re^m$, and $\mathbf{y} \in \Re^r$ are the state, input, and output vectors, respectively; and *n*, *m* and *r* are the corresponding numbers of those vectors. The constant matrices $\mathbf{A_c}, \mathbf{B_c}, \mathbf{C}$ with appropriate dimensions represent the internal operation of the linear system.

The discrete-time version of the state-space representation is expressed as:

$$\begin{aligned}\mathbf{x}(k+1) &= \mathbf{Ax}(k) + \mathbf{Bu}(k) \\ \mathbf{y}(k) &= \mathbf{Cx}(k)\end{aligned} \tag{2}$$

where *k* is the time index, and constant matrices **A** and **B** are derivable from $\mathbf{A_c}$ and $\mathbf{B_c}$.

When the unit impulse excitation is taken for each input element, the results can be assembled into a pulse-response matrix $\mathbf{Y}_k \in \Re^{m \times r}$ as follows:

$$\mathbf{Y}_1 = \mathbf{CB}, \mathbf{Y}_2 = \mathbf{CAB}, \ldots, \mathbf{Y}_k = \mathbf{CA}^{k-1}\mathbf{B} \tag{3}$$

The constant matrices in the sequence are known as *Markov parameters*. A system realization is the computation of a triplet [**A**,**B**,**C**] from the Markov parameters shown in Eq. 3, for which the discrete-time model, Eq. 2, is satisfied.

Assume that the state matrix **A** of order n has a complete set of linearly independent eigenvectors ($\boldsymbol{\psi}_1, \boldsymbol{\psi}_2, \cdots, \boldsymbol{\psi}_n$) with corresponding eigenvalues ($\lambda_1, \lambda_2, \cdots, \lambda_n$) which are not necessarily distinct. Define $\boldsymbol{\Lambda}$ as the diagonal matrix of eigenvalues and $\boldsymbol{\Psi}$ as the matrix of eigenvectors. The realization [**A**,**B**,**C**] can then be transformed to the realization $[\boldsymbol{\Lambda}, \boldsymbol{\Psi}^{-1}\mathbf{B}, \mathbf{C}\boldsymbol{\Psi}]$. The diagonal matrix $\boldsymbol{\Lambda}$ contains the information of modal damping rates and damped natural frequencies. The matrix $\boldsymbol{\Psi}^{-1}\mathbf{B}$ defines the initial modal amplitudes and the matrix $\mathbf{C}\boldsymbol{\Psi}$ the mode shapes at the sensor points. All the modal parameters of a dynamic system can thus be identified by the triplet $[\boldsymbol{\Lambda}, \boldsymbol{\Psi}^{-1}\mathbf{B}, \mathbf{C}\boldsymbol{\Psi}]$. The desired modal damping rates and damped natural frequencies are simply the real and imaginary parts of the eigenvalues $\boldsymbol{\Lambda}_\mathbf{c}$, after transformation from the discrete-time domain to the continuous-time domain using the relation $\boldsymbol{\Lambda}_\mathbf{c} = \ln(\boldsymbol{\Lambda})/\Delta t$.

System realization begins by forming the generalized $\alpha m \times \beta r$ Hankel matrix, composed of the Markov parameters from Eq. 3:

$$\mathbf{H}(k-1) = \begin{bmatrix} \mathbf{Y}_k & \mathbf{Y}_{k+1} & \cdots & \mathbf{Y}_{k+\beta-1} \\ \mathbf{Y}_{k+1} & \mathbf{Y}_{k+2} & \cdots & \mathbf{Y}_{k+\beta} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{Y}_{k+\alpha-1} & \mathbf{Y}_{k+\alpha} & \cdots & \mathbf{Y}_{k+\alpha+\beta-1} \end{bmatrix} \tag{4}$$

For simplicity, we use the block Hankel matrix given in Eq. 4 as the ERA block data matrix. The ERA process starts with the factorization of $\mathbf{H}(0)$, which is obtained by replacing $k = 1$ in Eq. 4, using singular value decomposition,

$$\mathbf{H}(0) = \mathbf{R}\boldsymbol{\Sigma}\mathbf{S}^T \tag{5}$$

where the columns of matrices **R** and **S** are orthonormal and $\boldsymbol{\Sigma}$ is a rectangular matrix

$$\boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_n & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{6}$$

with $\boldsymbol{\Sigma}_\mathrm{n} = \mathrm{diag}[\sigma_1, \sigma_2, \cdots, \sigma_n]$ and monotonically non-increasing σ_i, i.e. $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$.

Define $\mathbf{0}_i$ as a null matrix of order i, $\mathbf{I}_i$ as an identity matrix of order i, $\mathbf{E}_\mathrm{m}^\mathrm{T} = [\mathbf{I}_m, \mathbf{0}_m, \cdots, \mathbf{0}_m]$ where m is the number of outputs, and $\mathbf{E}_\mathrm{r}^\mathrm{T} = [\mathbf{I}_r, \mathbf{0}_r, \cdots, \mathbf{0}_r]$ where r is the number of inputs. The triplet

$$\hat{\mathbf{A}} = \boldsymbol{\Sigma}_n^{-1/2}\mathbf{R}_n^T\mathbf{H}(1)\mathbf{S}_n\boldsymbol{\Sigma}_n^{-1/2}, \hat{\mathbf{B}} = \boldsymbol{\Sigma}_n^{1/2}\mathbf{S}_n^T\mathbf{E}_r, \hat{\mathbf{C}} = \mathbf{E}_\mathrm{m}^\mathrm{T}\mathbf{R}_n\boldsymbol{\Sigma}_n^{1/2} \tag{7}$$

is a minimum realization, where $\mathbf{R}_\mathrm{n}$ and $\mathbf{S}_\mathrm{n}$ are the matrices formed by the first n columns of **R** and **S**, respectively. Here the quantities with ˆ mean estimated quantities to distinguish from the true quantities. The order of the matrix $\hat{\mathbf{A}}$ is n which is the order of the system for sufficiently low-noise data. The realized discrete-time model represented by the matrices $[\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}]$ can be transformed to the continuous-time model. The system frequencies and dampings may then be computed from the eigenvalues of the estimated continuous-time state matrix.

3. MODEL ORDER DETERMINATION AND NOISE HANDLING

If **H**(0) has been built from noise-free Markov parameters, then the rank of **H**(0) should be equal to n, as long as all system modes are active and αm and βr are greater than or equal to n. One common way of knowing the rank of a matrix is based on the singular value decomposition (SVD) of the matrix. When the singular values are arranged in the descending order, for theoretical data the singular values should go to zero when the rank of the matrix is exceeded. For measured data, however, due to random errors and small inconsistencies in the data, the singular values will not become zero but will become very small.

In the traditional ERA method, a low rank approximation method has been employed to reduce the model order and to decompose **H**(0) into two parts:

$$\mathbf{H}(0) = \mathbf{S} + \mathbf{N} \tag{8}$$

where **S** and **N** represent matrices associated with the signal and noise, respectively. While approximating **H**(0) to its "nearest" matrix **S**, an often used criterion is based on minimizing the *Frobenius norm* (*L_2-norm*) of the difference between **H**(0) and **S**, denoted $\|\mathbf{H}(0)\text{-}\mathbf{S}\|_2$.

While low rank approximation problem is to approximate a data matrix **A** with another matrix $\hat{\mathbf{A}}$ that has a specific lower rank by minimizing $\| \mathbf{A} - \hat{\mathbf{A}} \|_2$, the matrix $\hat{\mathbf{A}}$ can be obtained via the truncation of the SVD of the data matrix **A**. This approach has often been referred to as based on the *truncated singular value decomposition* (TSVD) technique or Eckart-Young theorem. Although the optimal L_2-norm lower rank approximation to **H**(0) can be obtained from the TSVD of **H**(0), but the resulting matrix **S** will not maintain the block-Hankel structure of **H**(0).

For $\mathbf{H}(0) \in \Re^{\alpha m \times \beta r}$, the L_2-norm of **H**(0) can be obtained as:

$$\|\mathbf{H}(0)\|_2 = \sqrt{\sum_{i=1}^{p} \sigma_i^2} \tag{9}$$

where $p = \min\{\alpha m, \beta r\}$ and σ_i is the ith singular value of **H**(0). As **S** has been obtained from the TSVD of **H**(0) with truncated rank n and **N** is the remaining part, we thus have

$$\|\mathbf{S}\|_2 = \sqrt{\sum_{i=1}^{n} \sigma_i^2} \quad \text{and} \quad \|\mathbf{N}\|_2 = \sqrt{\sum_{i=n+1}^{p} \sigma_i^2} \tag{10}$$

Since **S** and **N** represent the signal (principal) and noise (perturbational) portions, a way to quantify the signal to noise ratio in the global sense (involves multiple measured signals) is $\|\mathbf{S}\|_2/\|\mathbf{N}\|_2$. Geometrically, $\|\mathbf{S}\|_2$, $\|\mathbf{N}\|_2$ and $\|\mathbf{H}(0)\|_2$ form a right triangle.

As **H**(0) is α- and β-dependent, the optimum determination of α and β is still a critical issue which has not been addressed adequately. In the presence of significant noise at the measured signals, the success of applying ERA to obtain accurate estimate for modal parameters does not depend on enhancing the ratio $\|\mathbf{S}\|_2/\|\mathbf{N}\|_2$, but rely on properly separating **S** from **N**. To this end, a straightforward way is to create the **H**(0) that has a large gap between σ_n and σ_{n+1}. Given that γ-point signals are employed, we have various options for α and β to meet $\gamma = \alpha+\beta-1$. Clearly, the size of **H**(0) is the largest when α and β are chosen to be closest to each other. For example, if 99-point signals are available, we can choose (α, β) to be (50, 50) to have the largest **H**(0). In turn, this **H**(0) also has the highest p, the number of singular values. When the number of components associated with **S** is the model order n, then the number of components associated with **N** is p-n. For the sake of theoretical argument, we assume that all p-n noise components are corresponding to almost identical singular values, then the value of σ_{n+1} would become smaller when p is getting larger. As a result, we expect a more dramatic drop between σ_n and σ_{n+1} when a larger size of $\mathbf{H}(0)$ is taken, or α and β are chosen to be closest to each other.

In this study, we also propose a "verification" procedure to verify the estimated true modal parameters, and to examine their accuracy. The procedure involves artificially injecting random noise into the measured signals (which are noisy signals to start with) to create noisy-noisy signals. We then compare the modal estimation from measured signals to that from noisy-noisy signals. If they agree well to each other, it indicates that the modal estimation has not been affected by the injected noise, let alone the measured noise. The hypothesis is as follows: If **S** and **N** are well separated, the artificial noise would be mainly added to the **N**, thus the modal estimation based on **S** would remain unchanged. In short, we expect that the artificial noise would be easily absorbed by **N** when a large size of $\mathbf{H}(0)$ is taken.

4. EXPERIMENT STUDIES

The experimental studies will be based on the data collected from a steel plate. Shown in Figure 1 is a sketch of the set-up for the test plate, of which the length, width and thickness are 1m, 0.8m, and 3.7mm, respectively; marked by solid circles are the locations of 32 accelerometers. With 4 input locations at the four corners of the plate, there are 128 measured acceleration signals available, recorded at sampling rate 200 Hz. Although thousands of data points have been recorded for each acceleration signal, throughout this study only 100 sample points of each measured signal (see an example in Figure 2) are utilized in ERA for estimating system parameters. To simulate noisy-noisy signals for our proposed verification procedure, we inject randomly simulated Gaussian white noise to the measured data. The level of the additive noise is quantified by a stated percentage, which is defined as the ratio of the standard deviation of the white noise to that of the measured acceleration segment. Also shown in Figure 2 is a realization of a 20% noise signal.

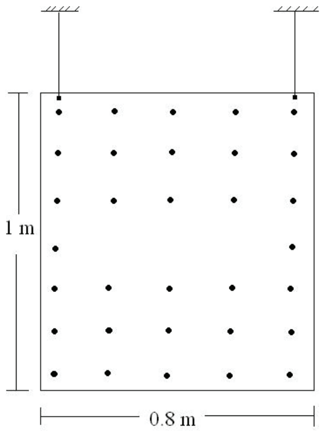
The singular values of Hankel matrices (based on α = 50 and β = 51) associated with the 100-point measured and noisy-noisy signals (see Figure 2) are shown in Figure 3, where each singular value has been normalized by the first (largest) singular value. From Figure 3, we observe that the artificially injected 20% noise has a noticeable influence to the singular values, starting at 9th singular value.

Including all 128 measured acceleration signals, we use the first 99 points of each measured signal to build the associated block Hankel matrix $\mathbf{H}(0)$. Determining the number of modes that have to the 128 measured acceleration signals is by estimating the rank of $\mathbf{H}(0)$. In theorem, the rank of $\mathbf{H}(0)$ should be twice the number of modes.

Based on the first 99 points of each measured signal to form $\mathbf{H}(0)$, two cases with different α and β are investigated: (1) α = 50, β = 50, and $\mathbf{H}(0) \in \Re^{1600\times 200}$ and (2) α = 2, β = 98, and $\mathbf{H}(0) \in \Re^{64\times 392}$. The normalized singular values associated with both block Hankel matrices are plotted in Figure 4. While there is a sharp drop starting at the 20th singular value (σ_{20}) at the curve for Case 1, no such a feature could be observed for Case 2. In other words, the model order can be determined to be 20 from Case 1 as the signal and noise portions are nicely separated, but it is not clear how to determine the model order from Case 2. However, both cases should have the same model order because they are based on the same measured data.

To test the robustness of the modal identification based on our proposed verification procedure, we inject two levels of noise, 10% and 20% respectively, into the measured data to create the cor- responding noisy-noisy signals. The normalized singular values associated with the measured and noisy-noisy signals for Case 1 and Case 2 are plotted in Figure 5 and Figure 6, respectively. For Case 1 (see Figure 5), the added 10% or 20% noise almost contributes all to the region after σ_{20}, and has a negligible effect to the first 20 singular values, which represent the region associated with the true system. For Case 2 (see Figure

6), the influence of the added noise occurs mainly at the region after σ_{20}, but the added noise also noticeably affects the singular values prior to σ_{20}.



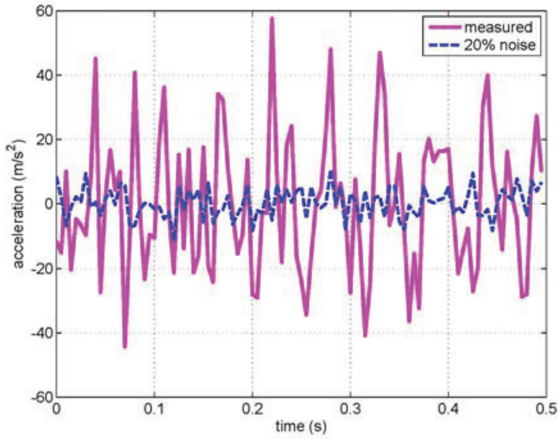
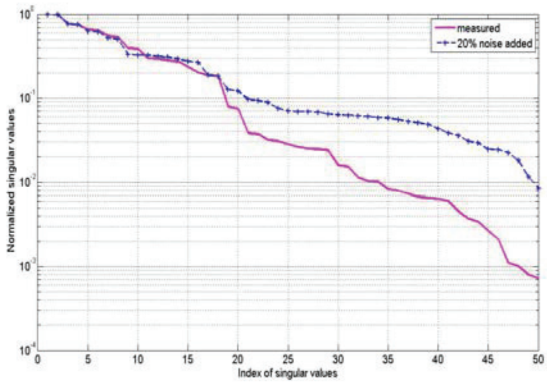


Figure 1: Sketch of the set-up for the test plate. Figure 2: An example of the measured and injected 20% noise signal.

To quantify the signal and noise contribution to $\mathbf{H}(0)$, we use Eq. 10 based on $n = 20$ to compute $||\mathbf{S}||_2$ and $||\mathbf{N}||_2$, respectively. The resulting norms of $||\mathbf{S}||_2$ and $||\mathbf{N}||_2$ associated with the measured and noisy-noisy signals for Cases 1 and 2 are shown at Table 1. As expected, the signal to noise ratio (SNR), i.e. $||\mathbf{S}||_2/||\mathbf{N}||_2$, decreases when more artificial noise has been added. We should not make a direct SNR comparison between Case 1 and Case 2, because more noise terms have been included in Case 1 and more noise at Case 2 has been counted towards $||\mathbf{S}||_2$. Precisely, the numbers of noise terms for Case 1 and Case 2 are 180 and 44, respectively. Listed at Table 2 are the changes of $||\mathbf{S}||_2/||\mathbf{N}||_2$ between the measured and noisy-noisy signals. Much higher percentage of the added noise has been absorbed by $||\mathbf{S}||_2$ in Case 1 than Case 2.



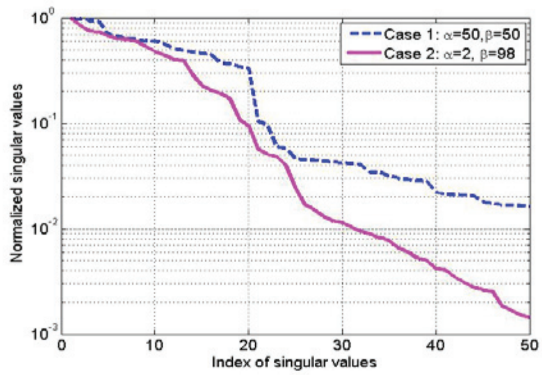
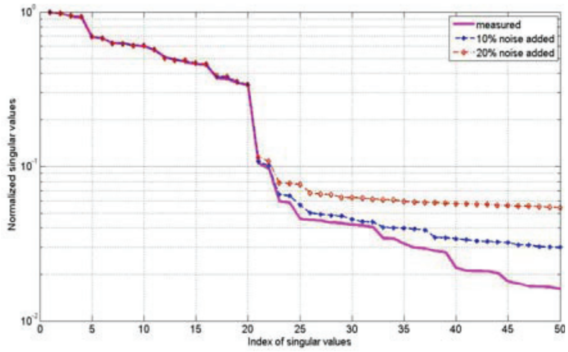


Figure 3: Singular values of the Hankel matrix from one signal; Figure 4: Singular values of the block Hankel matrices from 128 signals



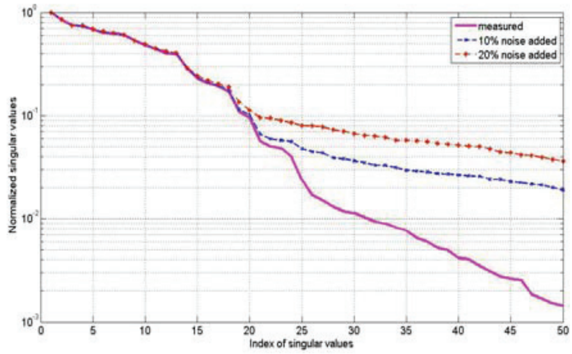


Figure 5: Singular values of the block Hankel matrices for Case 1; Figure 6: Singular values of the block Hankel matrices for Case 2

Table 1: L_2-norm of signal and noise matrices

Signal	Case 1			Case 2		
	$\|\|\mathbf{S}\|\|_2$	$\|\|\mathbf{N}\|\|_2$	$\|\|\mathbf{S}\|\|_2/\|\|\mathbf{N}\|\|_2$	$\|\|\mathbf{S}\|\|_2$	$\|\|\mathbf{N}\|\|_2$	$\|\|\mathbf{S}\|\|_2/\|\|\mathbf{N}\|\|_2$
measured	2.8463	0.2413	11.7939	2.3935	0.1089	21.9807
10% added	2.8514	0.3660	7.7903	2.4061	0.2053	11.7227
20% added	2.8585	0.5992	4.7704	2.4258	0.3614	6.7115

Table 2: The change of $||\mathbf{S}||_2$ and $||\mathbf{N}||_2$ norms between measured and noisy-noisy signals

Signal	Case 1			Case 2		
	$\triangle\|\|\mathbf{S}\|\|_2$	$\triangle\|\|\mathbf{N}\|\|_2$	$\triangle\|\|\mathbf{S}\|\|_2/\triangle\|\|\mathbf{N}\|\|_2$	$\triangle\|\|\mathbf{S}\|\|_2$	$\triangle\|\|\mathbf{N}\|\|_2$	$\triangle\|\|\mathbf{S}\|\|_2/\triangle\|\|\mathbf{N}\|\|_2$
10%	0.005	0.1247	24.94	0.0126	0.0964	7.651
20%	0.0122	0.3579	29.399	0.0323	0.2525	7.817

Tables 3 and 4 are the estimated modal frequencies and damping ratios, respectively, from the measured, 10%-noise added and 20%-noise added signals. For Case 1, the estimated frequencies and damping ratios, with or without injected noise are almost unchanged. This suggests that the modal estimation from Case 1 has been insensitive to the injected noise, thus most likely insensitive to the measured noise as well. We conclude that the modal identification in Case 1 is robust and accurate. Certainly, we cannot state the same to Case 2.

5. CONCLUDING REMARKS

A key requirement to the success of accurately estimating modal parameters from noisy measurements is a proper separation of the system information (signal) and noise from the measured data. Via the truncated singular value decomposition technique, the ERA decomposes the block Hankel matrix **H**(0) into two matrices: **S** for signal and **N** for noise. This paper demonstrated that we can best separate **S** and **N** from the corresponding **H**(0) when the number of block-rows and the number of block-columns are chosen to be closest to each other. We also proposed a procedure to ensure that the estimated modal parameters are indeed associated with the true system.

Table 3: Estimated frequencies: measured, 10%-noise and 20%-noise added signals

Component		Case 1			Case 2	
r	Measured	10% noise	20% noise	Measured	10% noise	20% noise
1	17.5098	17.5024	17.4941	17.5519	17.7563	18.3558
2	19.5113	19.5147	19.5191	19.5495	19.9038	20.9414
3	30.5248	30.5154	30.5046	30.6026	30.7629	31.2208
4	36.8163	36.8153	36.8140	36.8223	36.8576	36.9662
5	44.5120	44.5075	44.5028	44.5100	44.5218	44.5582
6	55.6433	55.6471	55.6503	55.6448	55.5933	55.5084
7	69.8036	69.8036	69.8035	69.8264	69.7330	69.4020
8	71.2253	71.2182	71.2106	71.1966	71.0622	70.6707
9	83.3762	83.3874	83.3982	83.2222	82.6586	85.5717
10	95.4995	95.5037	95.5085	98.8435	98.9223	—

Table 4: Estimated damping ratios: measured, 10%-noise and 20%-noise added signals

Component		Case 1			Case 1	
r	Measured	10% noise	20% noise	Measured	10% noise	20% noise
1	0.0036	0.0036	0.0038	0.0120	0.0434	0.1287
2	0.0034	0.0033	0.0033	0.0107	0.0532	0.1764
3	0.0038	0.0036	0.0035	0.0063	0.0165	0.0464
4	0.0014	0.0016	0.0019	0.0021	0.0039	0.0093
5	0.0027	0.0026	0.0025	0.0031	0.0037	0.0062
6	0.0015	0.0015	0.0015	0.0017	0.0030	0.0069
7	0.0030	0.0031	0.0031	0.0036	0.0065	0.0162
8	0.0031	0.0030	0.0031	0.0034	0.0074	0.0187
9	0.0029	0.0028	0.0027	0.0055	0.0346	0.1673
10	0.0025	0.0026	0.0027	0.0086	0.0128	—

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