Column ranks and their preservers over nonnegative real matrices

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Abstract

The column rank of an m by n matrix A over the nonnegative reals is the dimension over the nonnegative reals of the column space of A. We compare the column rank with the factor rank of matrices over the nonnegative reals. We also characterize the linear operators which preserve the column rank of matrices over the nonnegative reals.

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1. Introduction

There is much literature on the study of linear operators preserving the rank or column rank of matrices over several semirings [1–9]. Nonnegative matrices also have been of great interest to many authors [1,3–7,9].

In 1985, Beasley et al. [1] obtained characterizations of the linear operators preserving the rank of matrices over \( U_{+} \), the nonnegative elements of the unique factorization domain \( U \) in the reals \( \mathbb{R} \). In 1992, Beasley and Song [4] characterized
linear operators that preserve the column rank of matrices over \( \mathbb{Z}_+ \), the semiring of nonnegative integers. In 1997, Song and Hwang \cite{9} characterized those linear operators preserving the spanning column rank of matrices over \( \mathbb{U}_+ \), the nonnegative part of a unique factorization domain \( \mathbb{U} \) in \( \mathbb{R} \) which has only one unit 1.

In this paper, we study on the column rank of matrices over the nonnegative part of the reals. Consequently, we analyze the relationship between rank and column rank. We also have characterizations of the linear operators preserving the column rank of matrices over the nonnegative part of the reals.

2. Rank versus column rank of matrices over \( \mathbb{R}_+ \)

Let \( \mathbb{R}_+ \) be the nonnegative part of the reals \( \mathbb{R} \). The set of \( m \times n \) matrices with entries in \( \mathbb{R}_+ \) is denoted by \( \mathcal{M}_{m,n}(\mathbb{R}_+) \). Addition, multiplication by scalars, and the product of matrices are defined as if \( \mathbb{R}_+ \) were a field.

The rank or factor rank, \( r(A) \), of a nonzero matrix \( A \in \mathcal{M}_{m,n}(\mathbb{R}_+) \) is defined as the least integer \( k \) for which there exist \( m \times k \) and \( k \times n \) matrices \( B \) and \( C \) with \( A = BC \). The real rank of \( A \) will be denoted by \( \rho(A) \). The rank of a zero matrix is zero. Also we can easily obtain that

\[
0 \leq \rho(A) \leq r(A) \leq \min(m, n) \quad \text{and} \quad r(AB) \leq \min(r(A), r(B)). \tag{2.1}
\]

The rank of a matrix may strictly exceed its real rank. The following matrix \( A \) has rank 4, but real rank 3 (see \cite[Example 2.3.1]{1}).

\[
A = \begin{bmatrix}
1 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1
\end{bmatrix}.
\]

If \( \mathcal{V} \) is a nonempty subset of \( (\mathbb{R}_+)^n = \mathcal{M}_{n,1}(\mathbb{R}_+) \) that is closed under addition and multiplication by scalars, then \( \mathcal{V} \) is called a vector space over \( \mathbb{R}_+ \). The notions of subspace and of spanning sets are the same as if \( \mathbb{R}_+ \) were a field.

A subset \( \mathcal{S} \) of a vector space \( \mathcal{V} \) is linearly dependent if there exists \( x \in \mathcal{S} \) such that \( x \) is a linear combination of elements in \( \mathcal{S} \setminus \{x\} \). Otherwise \( \mathcal{S} \) is linearly independent. Thus an independent set cannot contain a zero vector. As with fields, a basis for a vector space \( \mathcal{V} \) is a spanning subset of least cardinality. That cardinality is the dimension, \( \dim(\mathcal{V}) \), of \( \mathcal{V} \).

The column space of an \( m \times n \) matrix \( A \) over \( \mathbb{R}_+ \) is the vector space that is spanned by its columns. The column rank, \( c(A) \), of \( A \in \mathcal{M}_{m,n}(\mathbb{R}_+) \) is the dimension of its column space. The column rank of a zero matrix is zero.

It follows that

\[
0 \leq r(A) \leq c(A) \leq n \tag{2.2}
\]

for all matrices \( A \in \mathcal{M}_{m,n}(\mathbb{R}_+) \). And Beasley and Pullman \cite{3} obtained the following relation between factor rank and column rank:
for all matrices $A \in M_{m,n}(\mathbb{R}^+)$. The column rank of a matrix may actually exceed its rank. For an example, we consider a matrix

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \in M_{3,4}(\mathbb{R}^+).$$

Then Example 2.6 (below) implies that $r(A) = 3$, but $c(A) = 4$.

**Lemma 2.1.** For any $A \in M_{m,n}(\mathbb{R}^+)$, we have that $r(A) = 1$ if and only if $c(A) = 1$.

**Proof.** If $r(A) = 1$, then $A$ can be factored as

$$A = b[c_1 \ c_2 \ \cdots \ c_n],$$

where $b$ is an $m \times 1$ matrix and $[c_1 \ c_2 \ \cdots \ c_n]$ is an $1 \times n$ matrix. Since $r(A) = 1$, $b$ is not a zero column vector. Then it is obvious that span$(\{b\})$ is the column space of $A$. Therefore $c(A) = 1$. The converse is obvious from (2.2). $\square$

Let $\mu(\mathbb{R}^+, m, n)$ be the largest integer $k \leq n$ such that for all $A \in M_{m,n}(\mathbb{R}^+)$, $r(A) = c(A)$ if $r(A) \leq k$. The matrix $A$ in (2.4) shows that $\mu(\mathbb{R}^+, 3, 4) < 3$. In general $1 \leq \mu(\mathbb{R}^+, m, n) \leq n$. Moreover, we obtain that

$$r\left( \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \right) = r(A) \quad \text{and} \quad c\left( \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \right) = c(A)$$

for all matrices $A \in M_{m,n}(\mathbb{R}^+)$. $\square$

**Lemma 2.2.** If $c(A) > r(A)$ for some $p \times q$ matrix $A$ over $\mathbb{R}^+$, then for all $m \geq p$ and $n \geq q$, $\mu(\mathbb{R}^+, m, n) < r(A)$.

**Proof.** Since $c(A) > r(A)$ for some $p \times q$ matrix $A$, we have $\mu(\mathbb{R}^+, p, q) < r(A)$ from the definition of $\mu$. Let $B = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$ be an $m \times n$ matrix containing $A$ as a submatrix. Then by (2.5), $r(B) = r(A) < c(A) = c(B)$. So, $\mu(\mathbb{R}^+, m, n) < r(A)$ for all $m \geq p$ and $n \geq q$. $\square$

**Lemma 2.3.** For any $A \in M_{2,n}(\mathbb{R}^+)$ with $n \geq 2$, $r(A) = 2$ if and only if $c(A) = 2$.

**Proof.** Let $r(A) = 2$. If $n = 2$, then (2.2) implies that $c(A) = 2$. So we can assume that $n \geq 3$. Since any zero column does not change the dimension of the column space of $A$, we may assume that there is no zero column in $A$. Then we may write
A = \begin{bmatrix}
\alpha_1 & \cdots & \alpha_i & \cdots & \alpha_n \\
\beta_1 & \cdots & \beta_i & \cdots & \beta_n
\end{bmatrix}.

Since \(c(A) \geq 2\), there exist at least two different columns in \(A\) such that they are linearly independent.

Let

\[
\frac{\beta_i}{\alpha_i} = \min_{\alpha_i \neq 0} \left\{ \frac{\beta_h}{\alpha_h} \right\}
\quad \text{and} \quad
\frac{\alpha_j}{\beta_j} = \min_{\beta_j \neq 0} \left\{ \frac{\alpha_h}{\beta_h} \right\}.
\]

Then any column \([\alpha_k \beta_k]\) of \(A\) can be written as

\[
\begin{bmatrix}
\alpha_k \\
\beta_k
\end{bmatrix} =
\begin{cases}
\begin{bmatrix}
\alpha_i \\
\beta_i
\end{bmatrix}, & \text{if } \alpha_k = 0, \\
\begin{bmatrix}
\alpha_i \\
\beta_i
\end{bmatrix} + \begin{bmatrix}
\frac{\alpha_i - \alpha_j \beta_i}{\alpha_i - \beta_i \beta_j} \\
\frac{\beta_i - \beta_j \beta_i}{\alpha_i - \beta_i \beta_j}
\end{bmatrix} \left( \begin{bmatrix}
\alpha_j \\
\beta_j
\end{bmatrix} - \begin{bmatrix}
\alpha_i \\
\beta_i
\end{bmatrix} \right), & \text{if } \alpha_k \beta_k \neq 0.
\end{cases}
\]

Thus \(c(A) = 2\) because \(\{\begin{bmatrix}
\alpha_i \\
\beta_i
\end{bmatrix}, \begin{bmatrix}
\alpha_j \\
\beta_j
\end{bmatrix}\}\) is a basis of the column space of \(A\). The converse follows from (2.2) and Lemma 2.1. □

**Theorem 2.4.** For any \(A \in \mathbb{M}_{m,n}(\mathbb{R}_+)\) with \(m \geq 2\) and \(n \geq 2\), \(r(A) = 2\) if and only if \(c(A) = 2\).

**Proof.** Let \(r(A) = 2\). Then \(A\) can be factored as \(A = BC\) for some \(m \times 2\) matrix \(B\) and \(2 \times n\) matrix \(C\), which are expressed as

\[
B = \begin{bmatrix} x & y \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix}
\alpha_1 & \cdots & \alpha_i & \cdots & \alpha_n \\
\beta_1 & \cdots & \beta_i & \cdots & \beta_n
\end{bmatrix}.
\]

Then the \(k\)th column of \(A\) is \(\alpha_k x + \beta_k y\). If \(n = 2\), then (2.2) implies that \(c(A) = 2\). So we may assume that \(n \geq 3\). Then Lemma 2.3 implies that \(c(C) = 2\). Furthermore, as in the proof of Lemma 2.3, we can write

\[
\begin{bmatrix}
\alpha_k \\
\beta_k
\end{bmatrix} = s \begin{bmatrix}
\alpha_i \\
\beta_i
\end{bmatrix} + t \begin{bmatrix}
\alpha_j \\
\beta_j
\end{bmatrix}
\]

for some \(s, t \in \mathbb{R}_+\), where \(\{\begin{bmatrix}
\alpha_i \\
\beta_i
\end{bmatrix}, \begin{bmatrix}
\alpha_j \\
\beta_j
\end{bmatrix}\}\) is a basis of the column space of \(C\). Then we have

\[
\alpha_k x + \beta_k y = (s \alpha_i + t \alpha_j)x + (s \beta_i + t \beta_j)y = s(\alpha_i x + \beta_i y) + t(\alpha_j x + \beta_j y) \in \text{span}(\{\alpha_i x + \beta_i y, \alpha_j x + \beta_j y\}).
\]
This shows that \([\alpha_i x + \beta_i y, \alpha_j x + \beta_j y]\) is a basis of the column space of \(A\), which implies that \(c(A) = 2\). The converse follows from (2.2) and Lemma 2.1.

\[\square\]

Lemma 2.5. If the columns of an \(m \times n\) matrix \(A\) over \(\mathbb{R}_+\) are linearly independent, then \(c(A) = n\).

Proof. By the property (2.3), we have \(c(A) = \min\{r(X)|AX = A, X \in \mathcal{M}_{m,n}(\mathbb{R}_+)\}\). Let \(A = [a_1, a_2, \ldots, a_n] \in \mathcal{M}_{m,n}(\mathbb{R}_+)\), where each \(a_i \in \mathcal{M}_{m,1}(\mathbb{R}_+)\) is a column of \(A\). Suppose that \(AX = A\), where \(X = [x_{ij}]\) is a matrix in \(\mathcal{M}_{n,n}(\mathbb{R}_+)\). Then the \(j\)th column of \(A\) is given by

\[a_j = x_{1j}a_1 + x_{2j}a_2 + \cdots + x_{ij}a_i + \cdots + x_{nj}a_n\]  \hspace{1cm} (2.6)

for all \(j = 1, \ldots, n\). Since the columns of \(A\) are linearly independent, there is no zero column in \(A\). It follows that \(X\) is not a zero matrix. If \(x_{jj} = 0\), (2.6) shows that the columns of \(A\) are not linearly independent, a contradiction. Thus we have \(x_{jj} > 0\) for all \(j = 1, \ldots, n\). Since there is at least one positive entry in \(a_j\), without loss of generality we may assume \(a_{kj} > 0\) for some \(k\). Then we have \(a_{kj} \geq x_{jj}a_{kj}\) from (2.6). This implies that \(0 < x_{jj} \leq 1\) for all \(j = 1, \ldots, n\). Assume that \(x_{jj} < 1\) for some \(j\). Then \(1 - x_{jj} > 0\); equivalently \(\frac{1}{1 - x_{jj}} > 0\). Therefore (2.6) becomes \(a_j = \sum_{i \neq j} x_{ij}a_i\), a contradiction to the fact that the columns of \(A\) are linearly independent. We then have \(x_{jj} = 1\) for all \(j = 1, \ldots, n\). It follows that \(\sum_{i \neq j} x_{ij}a_i = 0\), and hence \(x_{ij} = 0\) for all \(i \neq j\). Therefore \(X = I\). By (2.3), we have \(c(A) = n\). \[\square\]

Example 2.6. Consider a matrix

\[
A = \begin{bmatrix}
0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0
\end{bmatrix} \in \mathcal{M}_{3,4}(\mathbb{R}_+).
\]

Since the columns of \(A\) are linearly independent over \(\mathbb{R}_+\), we have \(c(A) = 4\) by Lemma 2.5. Also \(2 \leq r(A) \leq 3 = \min(3,4)\) by Lemma 2.1. It follows from Theorem 2.4 that \(r(A) \neq 2\). Therefore \(r(A) = 3\). \[\square\]

Theorem 2.7. For \(m \times n\) matrices over \(\mathbb{R}_+\), we have the values of \(\mu\) as follows:

\[
\mu(\mathbb{R}_+, m, n) = \begin{cases}
1 & \text{if } \min(m, n) = 1; \\
3 & \text{if } m \geq 3, \text{ and } n = 3; \\
2 & \text{otherwise}.
\end{cases}
\]

Proof. If \(\min(m, n) = 1\), then we have \(\mu(\mathbb{R}_+, m, n) = 1\) from Lemma 2.1. Consider the matrix \(A \in \mathcal{M}_{3,4}(\mathbb{R}_+)\) in Example 2.6. Then \(r(A) = 3\) and \(c(A) = 4\). Thus we have \(\mu(\mathbb{R}_+, m, n) \leq 2\) for all \(m \geq 3\) and \(n \geq 4\) by Lemma 2.2.

Suppose \(m \geq 2\) and \(n \geq 2\). Then we have \(\mu(\mathbb{R}_+, m, n) \geq 2\) for all \(m \geq 2\) and \(n \geq 2\) by Theorem 2.4.
Finally, consider the case with \( m \geq 3 \) and \( n = 3 \). Then we have \( \mu(\mathbb{R}_+, m, n) = 3 \) by Lemma 2.1 and Theorem 2.4.
Therefore we have the values of \( \mu \) as required. \( \square \)

3. Column rank preservers

In this section we have characterizations of the linear operators that preserve the column rank of matrices over \( \mathbb{R}_+ \).

A linear operator \( T \) on \( \mathcal{M}_{m,n}(\mathbb{R}_+) \) is said to preserve column rank if \( c(T(A)) = c(A) \) for all \( A \in \mathcal{M}_{m,n}(\mathbb{R}_+) \). It preserves column rank \( r \) if \( c(T(A)) = r \) whenever \( c(A) = r \). For the terms rank preserver and rank \( r \) preserver on \( \mathcal{M}_{m,n}(\mathbb{R}_+) \), they are defined similarly.

For any matrix \( A = [a_{ij}] \) in \( \mathcal{M}_{m,n}(\mathbb{R}_+) \), we define the \((0,1)\)-matrix \( A^* = [a^*_{ij}] \) by putting \( a^*_{ij} = 1 \) if and only if \( a_{ij} \neq 0 \) for all \( i \) and \( j \).

If \( S \) is any set, an \( n \times n \) square matrix \( A \) over \( S \) is called \( S \)-invertible if there exists a matrix \( B \in \mathcal{M}_{n,n}(S) \) such that \( AB = BA = I \). It is well known [1] that a square matrix \( A \) over \( \mathbb{R}_+ \) is \( \mathbb{R}_+ \)-invertible if and only if some permutation of its rows is a diagonal matrix all of whose diagonal entries are nonzero in \( \mathbb{R}_+ \). In other words, \( A \in \mathcal{M}_{n,n}(\mathbb{R}_+) \) is \( \mathbb{R}_+ \)-invertible if and only if \( A^* \) is a permutation matrix.

If \( S \) is any set, an \( n \times n \) square matrix \( A \) over \( S \) is called \( S \)-left-nonsingular if for any vector, \( v \), with entries from \( S \), \( Av = 0 \) implies that \( v = 0 \). It is called \( S \)-right-nonsingular if for any vector, \( v \), with entries from \( S \), \( v^t A = 0 \) implies that \( v = 0 \). If \( S \) is a field, nonsingularity and invertibility are equivalent. However, over antinegative semirings like \( \mathbb{R}_+ \), any matrix with no zero column is \( \mathbb{R}_+ \)-left-nonsingular and one with no zero row is \( \mathbb{R}_+ \)-right-nonsingular. We say that \( A \) is \( \mathbb{R}_+ \)-left-singular if it is not \( \mathbb{R}_+ \)-left-nonsingular, etc. Also, over fields, left- and right-singularity are equivalent for square matrices so, we drop the right- and left- and say, for example, \( A \) is \( \mathbb{R}_+ \)-nonsingular or \( \mathbb{R}_+ \)-singular.

We say that a linear operator \( T \) on \( \mathcal{M}_{m,n}(\mathbb{R}_+) \) is a \((U,V)\)-operator if there exist invertible matrices \( U \) and \( V \) in \( \mathcal{M}_{m,m}(\mathbb{R}_+) \) and \( \mathcal{M}_{n,n}(\mathbb{R}_+) \), respectively such that either \( T(A) = UAV \) or \( m = n \), \( T(A) = UA^t V \) for all \( A \in \mathcal{M}_{m,n}(\mathbb{R}_+) \).

Beasley et al. [1] obtained the following two theorems:

**Theorem 3.1.** Let \( T \) be a linear operator on \( \mathcal{M}_{m,n}(\mathbb{R}_+) \) with \( \min(m, n) \geq 2 \). Then the following are equivalent:

(a) \( T \) preserves ranks 1 and 2;
(b) \( T \) is injective, and there exists matrices \( U, V \) over \( \mathbb{R}_+ \) such that either

1. \( T(X) = U XV \) for all \( X \) in \( \mathcal{M}_{m,n}(\mathbb{R}_+) \), or
2. \( T(X) = U X^t V \) for all \( X \) in \( \mathcal{M}_{m,n}(\mathbb{R}_+) \), possibly \( m \neq n \).

[Here, \( T \) need not be a \((U,V)\)-operator because \( U \) or \( V \) need not be invertible.]
Theorem 3.2. Let $T$ be a linear operator on $\mathcal{M}_{m,n}(\mathbb{R}_+)$ with $\min(m,n) \geq 4$. Then the following are equivalent:

(a) $T$ preserves ranks 1, 2, and 4;
(b) $T$ is a $(U, V)$-operator on $\mathcal{M}_{m,n}(\mathbb{R}_+)$;
(c) $T$ preserves all ranks.

The next sequence of lemmas is needed to prove the main theorem.

Lemma 3.3. Let $A$ be a given matrix in $\mathcal{M}_{m,n}(\mathbb{R}_+)$ and define a linear operator $T$ on $\mathcal{M}_{m,n}(\mathbb{R}_+)$ by $T(X) = AX$. Then $T$ preserves column ranks if and only if $A$ is $\mathbb{R}$-nonsingular.

Proof. Assume that $A$ is $\mathbb{R}$-nonsingular. For any $X = [x_1 \ x_2 \ \cdots \ x_n] \in \mathcal{M}_{m,n}(\mathbb{R}_+)$, we have $AX = [Ax_1 \ Ax_2 \ \cdots \ Ax_n]$. Let $\{v_1, v_2, \ldots, v_r\}$ be a basis of the column space of $X$. Then we have that $\{Av_1, Av_2, \ldots, Av_r\}$ spans the column space of $AX$. Hence, $c(AX) \leq r = c(X)$.

Let $\{w_1, w_2, \ldots, w_s\}$ be a basis of the column space of $AX$. Then for each $i = 1, 2, \ldots, s$, we have

$$w_i = \sum_{j=1}^n \alpha_{ij}x_j = A \alpha_{ij}y_j, \quad \alpha_{ij} \in \mathbb{R}_+,$$

where $y_j = \sum_{i=1}^n \alpha_{ij}x_i$. Since $A$ is $\mathbb{R}$-nonsingular, (3.1) becomes $y_j = A^{-1}w_i \in \mathcal{M}_{m,1}(\mathbb{R}_+)$ for each $i = 1, 2, \ldots, s$. Let $x_j$ be the $j$th column of $X$. Then $Ax_j$ is the $j$th column of $AX$. Since $\{w_1, w_2, \ldots, w_s\}$ is a basis of the column space of $AX$, we have $Ax_j = \sum_{i=1}^s \beta_{ji}w_i$ for some scalars $\beta_{ji} \in \mathbb{R}_+$, and hence

$$x_j = A^{-1} \sum_{i=1}^s \beta_{ji}w_i = \sum_{i=1}^s \beta_{ji}A^{-1}w_i = \sum_{i=1}^s \beta_{ji}y_i, \quad \text{for } j = 1, \ldots, n.$$

This shows that $\{y_1, y_2, \ldots, y_s\}$ spans the column space of $X$. Thus, $c(X) \leq s = c(AX)$. Therefore we have $c(X) = c(AX)$, and hence $T$ preserves column ranks on $\mathcal{M}_{m,n}(\mathbb{R}_+)$. Conversely, assume that $A$ is $\mathbb{R}$-singular. If $m = 1$, then $A = 0$, and therefore, $A$ does not preserve column rank 1. Let $m \geq 2$. We show that $T$ is not injective. In view of Theorem 3.1, this will imply that $T$ does not preserve ranks 1 or 2, and therefore $T$ does not preserve column ranks 1 or 2 by Theorem 2.7. Since $A$ is $\mathbb{R}$-singular, $Ax = 0$ for some nonzero vector $x$ in $\mathcal{M}_{m,1}(\mathbb{R}_+)$. We choose a positive real $\alpha$ such that $z = j + x \in \mathcal{M}_{m,1}(\mathbb{R}_+)$, where $j$ is the vector in $\mathcal{M}_{m,1}(\mathbb{R}_+)$ with all entries $\alpha$. Then $Az = A(j + x) = Aj$. Consider the vector $e_1 = [1, 0, \ldots, 0]^t \in \mathcal{M}_{n,1}(\mathbb{R}_+)$. Then $je_1$ and $ae_1$ are distinct elements of $\mathcal{M}_{m,n}(\mathbb{R}_+)$ such that

$$T(ze_1) = Az e_1 = Aj e_1 = T(j e_1).$$

This shows that $T$ is not injective and the result follows. \qed
Lemma 3.4. Let $A$ be a given matrix in $\mathcal{M}_{n,n}(\mathbb{R}_+).$ Define a linear operator $T$ on $\mathcal{M}_{m,n}(\mathbb{R}_+)$ by $T(X) = XA.$ If $A$ is $\mathbb{R}$-singular, then $T$ does not preserve column ranks 1 or 2.

Proof. If $n = 1,$ then $A = 0,$ and the result is obvious. Let $n \geq 2.$ We show that $T$ is not injective. Since $A$ is $\mathbb{R}$-singular, $x^tA = 0$ for some nonzero vector $x$ in $\mathcal{M}_{n,1}(\mathbb{R}_+).$ We choose a positive real $\alpha$ such that $z = j + x \in \mathcal{M}_{n,1}(\mathbb{R}_+).$ Then $z^tA = (j^t + x^t)A = \alpha j^t.$ Consider the vector $e_1 = [1, 0, \ldots, 0]^t \in \mathcal{M}_{n,1}(\mathbb{R}_+).$ Then $e_1z^t$ and $e_1j^t$ are distinct elements of $\mathcal{M}_{m,1}(\mathbb{R}_+)$ such that $T(e_1z^t) = e_1z^tA = e_1j^tA = T(e_1j^t).$

This shows that $T$ is not injective and the result follows from Theorems 3.1 and 2.7. □

If $A$ and $B$ are in $\mathcal{M}_{m,n}(\mathbb{R}_+),$ we say $A$ dominates $B$ (written $B \leq A$ or $A \geq B$) if $a_{ij} = 0$ implies $b_{ij} = 0.$ Let $A$ be a fixed matrix in $\mathcal{M}_{n,n}(\mathbb{R}_+).$ Then Lemma 3.4 shows that if $T(X) = XA$ preserves column ranks 1 and 2 on $\mathcal{M}_{m,n}(\mathbb{R}_+),$ then $A$ dominates a permutation matrix $P.$

Example 3.5. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathcal{M}_{4,4}(\mathbb{R}_+).$$

Then $A$ is nonsingular but not invertible in $\mathcal{M}_{4,4}(\mathbb{R}_+).$ Let $T$ be the linear operator defined by $T(X) = XA$ on $\mathcal{M}_{4,4}(\mathbb{R}_+).$ Consider a matrix

$$X = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \in \mathcal{M}_{4,4}(\mathbb{R}_+).$$

Since the columns of $X$ are linearly independent, we have $c(X) = 4$ by Lemma 2.5. But

$$XA = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

has $c(XA) = 3$ because $[1, 1, 0, 0]^t, [1, 0, 1, 0]^t, [0, 1, 0, 1]^t$ is a basis of the column space of $XA$ over $\mathbb{R}_+.$ Therefore $T$ does not preserve column rank 4 on $\mathcal{M}_{4,4}(\mathbb{R}_+).$ □
Let $A \in \mathcal{M}_{4,4}(\mathbb{R}^+)$. Then Example 3.5 shows that $T(X) = AX$ may not preserve column rank 4 on $\mathcal{M}_{4,4}(\mathbb{R}^+)$ for some $\mathbb{R}$-invertible matrix $A$ in $\mathcal{M}_{4,4}(\mathbb{R}^+)$. 

**Lemma 3.6.** If $A \in \mathcal{M}_{n,n}(\mathbb{R}^+)$ is $\mathbb{R}$-invertible, then $T(X) = AX$ preserves column ranks on $\mathcal{M}_{m,n}(\mathbb{R}^+)$. 

**Proof.** Since $A$ is $\mathbb{R}$-invertible, we have $A^* = P$ for some permutation matrix $P$. Then we can write $A = PD$, where $D = \text{diag}(d_1, \ldots, d_n)$ is a diagonal matrix with $d_i > 0$ for all $i = 1, \ldots, n$. Let $\sigma$ be the permutation of $\{1, \ldots, n\}$ corresponding to $P$. Then for any $X = [x_1 \ x_2 \ \cdots \ x_n] \in \mathcal{M}_{m,n}(\mathbb{R}^+)$, we have 

$$T(X) = AX = XPD = [d_1 x_{\sigma(1)} d_2 x_{\sigma(2)} \ \cdots \ d_n x_{\sigma(n)}].$$ 

Therefore the column spaces of $X$ and $XA$ are the same, and the result follows. □

**Lemma 3.7.** Let $A, B \in \mathcal{M}_{n,n}(\mathbb{R}^+)$ be such that $B = PAQ$, where $P$ and $Q$ are $\mathbb{R}$-invertible matrices. Then $T(X) = AX$ preserves column ranks if and only if $T(X) = XB$ preserves column ranks on $\mathcal{M}_{m,n}(\mathbb{R}^+)$. 

**Proof.** The proof follows from Lemma 3.6. □

We say that a subspace $V$ of $\mathbb{R}^m$ is **solid** if it has a nonempty interior. Clearly the full subspace $\mathbb{R}^m$ is solid. If $A \in \mathcal{M}_{m,m}(\mathbb{R}^+)$ is $\mathbb{R}$-nonsingular, then $V \equiv \{Ax | x \in \mathbb{R}^m\}$ is a solid subspace of $\mathbb{R}^m$, since $A$ corresponds to an injective linear transformation of $\mathbb{R}^m$.

**Lemma 3.8.** Let $m \geq 3$ and $n \geq 4$ be integers. Let $V$ be a solid subspace of $\mathbb{R}^m$. Then, for any positive reals $\alpha_1, \alpha_2, \ldots, \alpha_n$, there is a set of linearly independent vectors $\{x_1, x_2, \ldots, x_n\}$ in $V$ such that

$$\alpha_1 x_1 + \alpha_2 x_2 = \alpha_3 x_3 + \cdots + \alpha_n x_n.$$ 

(3.2)

**Proof.** Choose an interior point $y = (y_1, y_2, \ldots, y_m)^t$ and a closed-ball $S$ with positive radius around $y$ contained in $V$. Let $C$ be the intersection of $S$ with the plane 

$$x_1 + x_2 + x_3 = r, \quad x_i = y_i, \quad 4 \leq i \leq m,$$

where $r = y_1 + y_2 + y_3$ and $(x_1, x_2, x_3, x_m)^t$ is a variable point in $\mathbb{R}^m$. Then $C$ is a circle with center $y$ contained in $V$. Any $n$ distinct points on $C$ will generate a convex cone with the points lying on different extremal rays, and therefore are linearly independent over $\mathbb{R}^+$. 

Now, we choose two points $z_1$ and $z_2$ which are extremities of a diameter of $C$. If $n = 4$, we choose points $z_3$ and $z_4$ in $C \setminus \{z_1, z_2\}$ which are extremities of another
diameter of $C$. If $n > 4$, then we choose $\{z_3, z_4, \ldots, z_n\} \subset C \setminus \{z_1, z_2\}$ such that $z_i, 3 \leq i \leq n$, form a regular $(n - 2)$-gon with $y$ as its centroid. Now, consider the points

$$x_i = (n - 2)a_i^{-1}z_i, \quad i = 1, 2, \quad \text{and} \quad x_i = 2a_i^{-1}z_i, \quad 3 \leq i \leq n$$

of $\mathbb{V}$. These points, being positive multiples of distinct $z_i$, are linearly independent over $\mathbb{R}_+$. Clearly, they satisfy (3.2). \qed

**Lemma 3.9.** Let $m \geq 3$, $n \geq 4$, and $A$ be a $\mathbb{R}$-nonsingular matrix in $\mathcal{M}_{n,n}(\mathbb{R}_+)$ with $A^* \neq P$ for any permutation matrix $P$. If $\mathbb{V}$ is any solid subspace of $\mathbb{R}_+^n$, then there exists $X \in \mathcal{M}_{m,n}(\mathbb{R}_+)$ with columns in $\mathbb{V}$ such that $c(X) = n$ and $c(XA) \leq n - 1$.

**Proof.** Let $A = [a_{ij}] \in \mathcal{M}_{n,n}(\mathbb{R}_+)$ be a $\mathbb{R}$-nonsingular matrix such that $A^* \neq P$ for all permutation matrix $P$. Then for any matrix $X = [x_1 \ x_2 \ \cdots \ x_n] \in \mathcal{M}_{n,n}(\mathbb{R}_+)$, we have $XA = [y_1 \ y_2 \ \cdots \ y_n]$, where for $j = 1, 2, \ldots, n$

$$y_j = a_{1j}x_1 + a_{2j}x_2 + \cdots + a_{nj}x_n.$$ 

Since $A$ is $\mathbb{R}$-nonsingular and $A^* \neq P$, one of the columns of $A$ has more than one positive entry. Without loss of generality we assume that the first column has at least two nonzero entries. We show that there is a linearly independent set $\{x_1, x_2, \ldots, x_n\}$ in $\mathbb{V}$ such that for some nonnegative reals $\beta_2, \ldots, \beta_n$

$$y_1 = \beta_2y_2 + \beta_3y_3 + \cdots + \beta_ny_n. \quad (3.3)$$

Now, (3.3) is equivalent to

$$\lambda_1x_1 + \lambda_2x_2 + \cdots + \lambda_nx_n = 0, \quad (3.4)$$

where

$$\lambda_i = a_{i1} - a_{i2}\beta_2 - a_{i3}\beta_3 - \cdots - a_{in}\beta_n, \quad 1 \leq i \leq n. \quad (3.5)$$

**Claim.** There are nonnegative reals $\beta_2, \beta_3, \ldots, \beta_n$ such that exactly two of $\lambda_1, \lambda_2, \ldots, \lambda_n$ are positive and the other $n - 2$ are negative. More precisely, there exist positive reals $\beta_2, \beta_3, \ldots, \beta_n$ and a permutation $\sigma$ of the set $\{1, 2, \ldots, n\}$ such that

$$\lambda_{\sigma(i)} \text{ is positive if } i \leq 2, \quad \text{negative if } i \geq 3. \quad (3.6)$$

Assume that the claim holds. Let $\alpha_i = \lambda_{\sigma(i)}$ for $i \leq 2$, and $\alpha_i = -\lambda_{\sigma(i)}$ for $i \geq 3$. Then the Eq. (3.4) becomes

$$\alpha_1x_{\sigma(1)} + \alpha_2x_{\sigma(2)} = \alpha_3x_{\sigma(3)} + \alpha_4x_{\sigma(4)} + \cdots + \alpha_nx_{\sigma(n)}. \quad (3.7)$$

Now, by Lemma 3.8, the Eq. (3.7) has a linearly independent solution for $x_k$ in $\mathbb{V}$. For this solution $X = [x_1 \ x_2 \ \cdots \ x_n] \in \mathcal{M}_{m,n}(\mathbb{R}_+)$, the corresponding $y_i$ satisfy (3.3). Thus we have found $X \in \mathcal{M}_{m,n}(\mathbb{R}_+)$ with columns in $\mathbb{V}$ such that $c(X) = n$ but $c(XA) \leq n - 1$. 

Proof of Claim. Consider the relations
\begin{align*}
a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= a_{11} \\
a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= a_{21} \\
& \vdots \\
a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n &= a_{n1}.
\end{align*}
(3.8)

First we consider the case when each of the relations in (3.8) is an equation in \(x_2, x_3, \ldots, x_n\). (This is the case when coefficients of \(x_2, x_3, \ldots, x_n\) in any of the relations do not vanish simultaneously.) Then (3.8) represents equations of \(n\) hyperplanes in \(\mathbb{R}^{n-1}\). Since \(A\) is \(\mathbb{R}\)-nonsingular, the hyperplanes are distinct. Moreover, each of them have nonempty intersection with \(\mathbb{R}^{n-1}_+\). For any positive vector \((b_2, b_3, \ldots, b_n) \in \mathbb{R}^{n-1}_+\), let
\[
\alpha_k = \frac{a_{k1}}{a_{k2}b_2 + a_{k3}b_3 + \cdots + a_{kn}b_n} \quad \text{for} \quad k = 1, 2, \ldots, n.
\]
Then the positive ray passing through the origin and \((b_2, b_3, \ldots, b_n)\) meets the \(k\)th plane in (3.8) at the point \(P_k = (\alpha_kb_2, \alpha_kb_3, \ldots, \alpha_kb_n)\). By our choice of the first column of the matrix \(A\), at least two of \(\alpha_k\) are nonzero. Thus any positive ray from the origin meets at least two of the hyperplanes. Now, we choose the vector \((b_2, b_3, \ldots, b_n)\) in such a way that those points \(P_k\) which are not the origin are all distinct. Let \(\sigma\) be a permutation of \([1, 2, \ldots, n]\) such that \(P_{\sigma(1)}, P_{\sigma(2)}, \ldots, P_{\sigma(n)}\) are arranged so that their distances from the origin are in descending order. Then \(P_{\sigma(2)}\) and \(P_{\sigma(3)}\) are distinct points in \(\mathbb{R}^{n-1}_+\). Let \(Q\) be the mid-point of the line joining \(P_{\sigma(2)}\) and \(P_{\sigma(3)}\). If \(Q\) has coordinate \((\beta'_2, \beta'_3, \ldots, \beta'_n)\) then for these values of \((\beta'_2, \beta'_3, \ldots, \beta'_n)\), (3.6) holds and we are done.

Next, consider the case when some of the relations in (3.8) is not an equation of \(x_j\). Then \(a_{i2} = a_{i3} = \cdots = a_{in} = 0\) for some \(i \in \{1, 2, \ldots, n\}\). Since \(A\) is \(\mathbb{R}\)-nonsingular, \(a_{i1} > 0\) and, consequently, \(\lambda_i > 0\). Moreover, there can not be two such \(i\), because in that case the rows of \(A\) would be linearly dependent. So exactly \(n - 1\) of the relations in (3.8) represent equations in variables \(x_2, x_3, \ldots, x_n\) in \(\mathbb{R}^{n-1}\). Using the argument of the previous case we can find a point \(Q\) with positive coordinates \((\beta'_2, \beta'_3, \ldots, \beta'_n)\) such that for these values of \((\beta'_2, \beta'_3, \ldots, \beta'_n)\) exactly one of \(\lambda_k\), \(k \neq i\), is positive and the other \(n - 2\) are negative. This completes the proof of the claim. □

Lemma 3.10. Let \(m \geq 3\), \(n \geq 4\), and \(A\) be a matrix in \(\mathcal{M}_{n,m}(\mathbb{R}_+)\). If \(T(X) =XA\) preserves column ranks 1, 2 and \(n\) on \(\mathcal{M}_{m,n}(\mathbb{R}_+)\), then \(A^* = P\) for some permutation matrix \(P\).

Proof. By Lemma 3.4, \(A\) is nonsingular. Suppose that \(A^* \neq P\) for any permutation matrix \(P\). Then by Lemma 3.9 (for \(\mathbb{V} = \mathbb{R}^m_+\)), there exists \(X \in \mathcal{M}_{m,n}(\mathbb{R}_+)\) such that \(c(X) = n\) and \(c(T(X)) = c(XA) \leq n - 1\), contradicting the hypothesis that \(T\) preserves column rank \(n\). Hence the result follows. □
Let $T$ be an operator on $\mathcal{M}_{n,n}(\mathbb{R}^+)$ defined by $T(X) = X^t$, the transpose of $X \in \mathcal{M}_{n,n}(\mathbb{R}^+)$. Then $T$ preserves all factor ranks since it is a $(U, V)$-operator. But the following example shows that the transposition operator does not preserve column rank.

**Example 3.11.** Let

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

be a matrix in $\mathcal{M}_{4,4}(\mathbb{R}^+)$. Then we have $c(B) = 3$ since the first three columns of $B$ constitute a basis of the column space of $B$. But the column rank of

$$B^t = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is 4 by (2.5) and Example 2.6. Thus the transposition operator does not preserve column rank 3 on $\mathcal{M}_{4,4}(\mathbb{R}^+)$. □

**Lemma 3.12.** If $T$ is the transposition operator on $\mathcal{M}_{m,m}(\mathbb{R}^+)$ with $m \geq 4$, then $T$ does not preserve column rank $r$ for $r \geq 3$.

**Proof.** Let $B$ be the matrix in Example 3.11. Consider $C = B \oplus 0_{m-4} \in \mathcal{M}_{m,m}(\mathbb{R}^+)$. Then $c(C) = 3$ by (2.5) but $T(C) = C^t$ has column rank 4 by (2.5). Let

$$D = B \oplus I_k \oplus 0_{m-k-4} \in \mathcal{M}_{m,m}(\mathbb{R}^+),$$

where $I_k$ is the identity matrix of order $k$. Then $c(D) = 3 + k$ but $T(D) = D^t$ has column rank $4 + k$. Therefore $T$ does not preserve column rank $r$ for $r \geq 3$. □

**Theorem 3.13.** Suppose $T$ is a linear operator on $\mathcal{M}_{m,n}(\mathbb{R}^+)$ with $m \geq 3$ and $n \geq 4$. If $T$ preserves column ranks 1, 2 and $n$, then there exist $U \in \mathcal{M}_{m,m}(\mathbb{R}^+)$ and $V \in \mathcal{M}_{n,n}(\mathbb{R}^+)$ such that $T(X) = U XV$ for all $X \in \mathcal{M}_{m,n}(\mathbb{R}^+)$, where $U$ is $\mathbb{R}$-nonsingular and $V^*$ is a permutation matrix.

**Proof.** Since $T$ preserves column ranks 1 and 2, it preserves factor ranks 1 and 2 by Theorem 2.7. Thus, by Theorem 3.1, $T$ is injective and has the form (1) $T(X) = U XV$ or (2) $T(X) = UX^t V$, where $U$ and $V$ are $\mathbb{R}$-matrices of appropriate sizes.

Suppose that (1) holds. We have $U \in \mathcal{M}_{m,m}(\mathbb{R}^+)$ and $V \in \mathcal{M}_{n,n}(\mathbb{R}^+)$. If $U$ is $\mathbb{R}$-singular, then by Lemma 3.3 we have $X_1, X_2 \in \mathcal{M}_{m,n}(\mathbb{R}^+)$ such that $X_1 \neq X_2$ and $UX_1 = UX_2$. We then have $T(X_1) = T(X_2)$, contradicting the fact that $T$ is injective. Thus, $U$ is $\mathbb{R}$-nonsingular and therefore, $\mathcal{W} = \{UX | x \in \mathbb{R}^m\}$ is a solid subspace of $\mathbb{R}^n$. Assume that $V^*$ is not a permutation matrix. Then, by Lemma 3.9, we...
have $X \in \mathcal{M}_{m,n}(\mathbb{R}_+)$ with columns in $\mathcal{W}$ such that $c(X) = n$ and $c(XV) \leqslant n - 1$. Since $X \in \{ U|A \in \mathcal{M}_{m,n}(\mathbb{R}_+) \}$, there exists $Y \in \mathcal{M}_{m,n}(\mathbb{R}_+)$ such that $X = UV$. Then we have $c(Y) = n$, whereas $c(T(Y)) = c(XV) \leqslant n - 1$. This contradicts the hypothesis that $T$ preserves column rank $n$. Hence $V^*$ must be a permutation matrix.

Next, suppose that (2) holds. Then $T^2(X) = WXZ$, where $W = UV^t$ and $Z = U^tV$. Since $T$ preserves column ranks 1, 2, and $n$, so does $T^2$. By (1), $W \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ is $\mathbb{R}$-invertible and $Z^*$ is a permutation matrix. If $m > n$, then $\rho(W) < m$ by the property (2.1), a contradiction. If $m < n$, we obtain a contradiction to $Z^*$ being a permutation matrix, similarly. Thus we have $m = n$, so $U^*$ and $V^*$ are permutation matrices. But, in that case, $T(X) = UXV$ does not preserve column rank $n$ by Lemma 3.7 and Lemma 3.12. This shows that the case (2) cannot hold. This completes the proof. □

**Corollary 3.14.** Suppose $T$ is a linear operator on $\mathcal{M}_{m,n}(\mathbb{R}_+)$ with $m \geqslant 3$ and $n \geqslant 4$. Then the following are equivalent:

1. $T$ preserves column ranks 1, 2, and $n$;
2. There exist $U \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ and $V \in \mathcal{M}_{n,n}(\mathbb{R}_+)$ such that $T(X) = UXV$ for all $X \in \mathcal{M}_{m,n}(\mathbb{R}_+)$, where $U$ is $\mathbb{R}$-nonsingular and $V^*$ is a permutation matrix;
3. $T$ preserves all column ranks.

**Proof.** The proof follows from Theorem 3.13. □

If $\min(m, n) \leqslant 3$, then in view of Theorem 2.7, the linear operators that preserve column ranks on $\mathcal{M}_{m,n}(\mathbb{R}_+)$ are the same as the factor rank preservers, which were characterized in [1]. Thus we have complete characterizations of the linear operators which preserve the column rank of matrices over the nonnegative reals.

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**References**