ELSEVIER
Available online at www.sciencedirect.com
SCIENCE DIRECT
J. Math. Anal. Appl. 303 (2005) 61–80
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

# Coexistence states for systems of mutualist species

Zongming Guo [1]

*Department of Mathematics, Dong Hua University, Shanghai, 200051, PR China*

Received 27 March 2003
Available online 21 December 2004
Submitted by P. Broadbridge

**Abstract**

Coexistence states for a class of systems of mutualist species are obtained via bifurcation theory and monotone techniques.


*Keywords:* Coexistence states; Multiplicity; Positive solutions; Semilinear elliptic systems

## 1. Introduction

We investigate the coexistence states for the systems of mutualist species

$$
\begin{aligned}
&-\Delta u = \lambda u + af(u) + buv, \quad x \in \Omega,\\
&-\Delta v = \mu v + dg(v) + cuv, \quad x \in \Omega,\\
&u = v = 0, \quad x \in \partial\Omega,
\end{aligned}
\tag{1.1}
$$

where $\lambda, \mu \in \mathbb{R}$ are bifurcation parameters, $a > 0$, $b > 0$, $c > 0$, $d > 0$ are constants, $f, g \in C^1([0,\infty))$ satisfy the following conditions:

*E-mail address:* guozm@public.xxptt.ha.cn.
[1] Present address: Department of Mathematics, Henan Normal University, Xinxiang, 453002, PR China.

0022-247X/$ – see front matter 
doi:10.1016/j.jmaa.2004.06.055

$(G_1)$ $f(s), g(s) > 0$ for $s > 0$, $f(0) = g(0) = 0$, $f'(0) > 0$, $g'(0) > 0$,

$$\lim_{s\to\infty} \frac{f(s)}{s} = \lim_{s\to\infty} \frac{g(s)}{s} = 0;$$

$(G_2)$ $(f(s)/s)' < 0$, $(g(s)/s)' < 0$ for $s \in (0, \infty)$.

In the rest of this paper we always assume that $f$ and $g$ satisfy $(G_1)$ and $(G_2)$.

A typical example of $f$ and $g$ is

$$f(s) = se^{-s}, \qquad g(s) = se^{-2s}.$$

We say $(u, v)$ a positive solution of (1.1) if $(u, v) \in C_0^1(\bar{\Omega}) \times C_0^1(\bar{\Omega})$ satisfies (1.1) in the weak sense with $u > 0$, $v > 0$ in $\Omega$.

(1.1) models the stationary case of the situation of two species co-existing in $\Omega$, where $\Omega$ is the inhabiting region, $u(x)$ and $v(x)$ are the densities of each of the species, $a$ and $d$ describe the limiting effects of crowding in each population, $b$ and $c$ are the supporting rates between the species. In this model we are assuming that $\Omega$ is fully surrounded by inhospitable areas, because both population densities are subject to homogeneous Dirichlet boundary conditions. Such kind of systems was studied extensively by many authors, see, for example, [2,3,5–14,17] and the references therein. They were interested in the existence and multiplicity of positive solutions, i.e., $(u, v) \in C_0^1(\bar{\Omega}) \times C_0^1(\bar{\Omega})$ with $u > 0$ and $v > 0$ in $\Omega$.

Without loss of generality we assume

$$a = d = 1$$

and then (1.1) changes to the form

$$\begin{aligned} &-\Delta u = \lambda u + f(u) + buv, \quad x \in \Omega, \\ &-\Delta v = \mu v + g(v) + cuv, \quad x \in \Omega, \\ &u = v = 0, \quad x \in \partial\Omega. \end{aligned} \tag{1.2}$$

We will study the existence, stability, and multiplicity of non-negative solutions $(u, v)$ of (1.2). Thanks to the strong maximum principle, if $(u, v) \in C_0^1(\bar{\Omega}) \times C_0^1(\bar{\Omega})$ is a non-negative solution of (1.2) with $u \neq 0$ (respectively $v \neq 0$), then $u$ (respectively $v$) is strongly positive in the sense of Section 2. Therefore, (1.2) admits three types of non-negative component-wise solutions: the *trivial* one, $(0, 0)$; those with one component positive and the other zero, $(u, 0)$ or $(0, v)$, the *semi-trivial* positive solutions; and those with both components positive, the *coexistence states*.

## 2. Preliminaries

In this section we obtain some results which will be useful in the following.

Let $E = L^\infty(\Omega)$ and $F = C_0^1(\bar{\Omega})$. We consider the spaces $E$ and $F$ as being ordered by the usual cones of non-negative functions $P_E$ and $P_F$. Clearly $u \in \operatorname{int} P_F$ if $u > 0$ in $\Omega$ and $\partial u/\partial n < 0$ on $\partial\Omega$, where $n$ is the outward normal vector of $\partial\Omega$; we will write $u \gg 0$ in this case and call $u$ strongly positive.

Let $q \in L^\infty(\Omega)$. The linear eigenvalue problem

$$-\Delta u + qu = \lambda u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega \tag{2.1}$$

has an infinite sequence of eigenvalues, which are bounded below. We denote the lowest eigenvalue of (2.1) by $\lambda_1^\Omega(q)$. It is well known that $\lambda_1^\Omega(q)$ is a simple eigenvalue and that the corresponding eigenfunction $\phi(q)$ does not change sign on $\Omega$. In the following, we assume $\phi(q) > 0$ in $\Omega$ and $\|\phi(q)\|_\infty = 1$. The following lemma is well-known.

**Lemma 2.1.**

(i) *Monotonicity with respect to the potential*: *let $q_1, q_2 \in L^\infty(\Omega)$ such that $q_1 \leqslant q_2$ and $q_1 < q_2$ on a set of positive measure. Then*

$$\lambda_1^\Omega(q_1) < \lambda_1^\Omega(q_2). \tag{2.2}$$

(ii) *Continuity with respect to the potential*: *if $q_n \in L^\infty(\Omega)$, $n \geqslant 1$, is a sequence of potentials such that*

$$\lim_{n\to\infty} \|q_n - q\|_\infty = 0,$$

*then*

$$\lim_{n\to\infty} \lambda_1^\Omega(q_n) = \lambda_1^\Omega(q).$$

(iii) *If $\Omega_1$ is a proper subdomain of $\Omega$ with $\partial\Omega_1$ of class $C^2$, then*

$$\lambda_1^{\Omega_1}(q) > \lambda_1^\Omega(q). \tag{2.3}$$

Consider now the nonlinear eigenvalue problem

$$-\Delta w + qw = \gamma w + f(w) \quad \text{in } \Omega, \qquad w = 0 \quad \text{on } \partial\Omega. \tag{2.4$_\gamma$}$$

**Theorem 2.2.** *Problem* $(2.4)_\gamma$ *has a positive solution in $C_0^1(\bar\Omega)$ if and only if $\Gamma_{[q,f]} < \gamma < \lambda_1^\Omega(q)$, where*

$$\Gamma_{[q,f]} = \lambda_1^\Omega(q) - f'(0). \tag{2.5}$$

*Moreover, for each $\Gamma_{[q,f]} < \gamma < \lambda_1^\Omega(q)$,* $(2.4)_\gamma$ *has a unique positive solution $\theta_{[\gamma,q,f]}$, the map $\gamma \mapsto \theta_{[\gamma,q,f]}$ from $(\Gamma_{[q,f]}, \lambda_1^\Omega(q))$ to $C_0^1(\bar\Omega)$ is strongly increasing* (*i.e., $\theta_{[\gamma_1,q,f]} \gg \theta_{[\gamma_2,q,f]}$ if $\gamma_1 > \gamma_2$*) *and continuous. Furthermore, we have*

$$\lim_{\gamma\downarrow\Gamma_{[q,f]}} \theta_{[\gamma,q,f]} = 0 \quad \textit{uniformly in } \Omega. \tag{2.6}$$

**Proof.** Writing $f(u)$ in the form $uf(u)/u$ and using the condition $(G_2)$, we obtain the proof of the first part of this theorem from [16, Theorem 1.1] or [15, Lemma 2].

The fact that

$$\lim_{\gamma\downarrow\Gamma_{[q,f]}} \theta_{[\gamma,q,f]} = 0 \quad \text{uniformly in } \bar\Omega$$

can be easily seen from the non-existence of positive solution of $(2.4)_\gamma$ with $\gamma = \Gamma_{[q,f]}$. The fact that $\gamma \mapsto \theta_{[\gamma,q,f]}$ is strongly increasing and continuous follows from the sub- and supersolution argument together with the strong maximum principle. $\square$

The following result provides us with the behavior of $\theta_{[\gamma,q,f]}$ as $\gamma \uparrow \lambda_1^\Omega(q)$.

**Theorem 2.3.** *Let $\mathcal{F}(s) = f(s)/s$. The following inequality holds*:

$$\theta_{[\gamma,q,f]} \geqslant \mathcal{F}^{-1}\big(\lambda_1^\Omega(q) - \gamma\big)\phi(q) \quad \text{in } \Omega, \tag{2.7}$$

*for $\Gamma_{[q,f]} < \gamma < \lambda_1^\Omega(q)$.*

**Proof.** We know from $(G_2)$ that $\mathcal{F}$ is a strictly decreasing function on $(0,\infty)$. For $\gamma \in (\Gamma_{[q,f]}, \lambda_1^\Omega(q))$, we show that $\mathcal{F}^{-1}(\lambda_1^\Omega(q) - \gamma)\phi(q)$ is a subsolution of $(2.4)_\gamma$. Indeed, for any $\alpha > 0$, if

$$\lambda_1^\Omega(q) - \gamma \leqslant \mathcal{F}\big(\alpha\phi(q)\big), \tag{2.8}$$

then $\alpha\phi(q)$ is a subsolution of $(2.4)_\gamma$. Now we choose

$$\alpha = \mathcal{F}^{-1}\big(\lambda_1^\Omega(q) - \gamma\big).$$

It follows from the monotonicity of the function $\mathcal{F}$ that (2.8) holds. Thus, the uniqueness of $\theta_{[\gamma,q,f]}$ implies

$$\theta_{[\gamma,q,f]} \geqslant \mathcal{F}^{-1}\big(\lambda_1^\Omega(q) - \gamma\big)\phi(q) \quad \text{in } \Omega.$$

Since $\lim_{s\to 0^+} \mathcal{F}^{-1}(s) = \infty$, we have

$$\lim_{\gamma\uparrow\lambda_1^\Omega(q)} \mathcal{F}^{-1}\big(\lambda_1^\Omega(q) - \gamma\big) = \infty. \qquad \square$$

**Lemma 2.4.**

(i) *If $\gamma \leqslant \Gamma_{[q,f]}$, then $(2.4)_\gamma$ does not admit a positive subsolution and, if $\gamma \geqslant \lambda_1^\Omega(q)$, then $(2.4)_\gamma$ does not admit a positive supersolution.*
(ii) *If $\gamma \in (\Gamma_{[q,f]}, \lambda_1^\Omega(q))$ and $\bar{w} \in C^1(\Omega)$ is a positive strict supersolution of $(2.4)_\gamma$, then $\bar{w} \gg \theta_{[\gamma,q,f]}$.*
(iii) *Similarly, if $\gamma \in (\Gamma_{[q,f]}, \lambda_1^\Omega(q))$ and $\underline{w} \in C^1(\Omega)$ is a positive strict subsolution of $(2.4)_\gamma$, then $\theta_{[\gamma,q,f]} \gg \underline{w}$.*

**Proof.** (i) Suppose that $\gamma \leqslant \Gamma_{[q,f]}$ and that $(2.4)_\gamma$ possesses a positive subsolution $\zeta^0$. Then, $\theta_{[\gamma^0,q,f]}$ with $\Gamma_{[q,f]} < \gamma^0 < \lambda_1^\Omega(q)$ and near $\lambda_1^\Omega(q)$ is a positive supersolution of $(2.4)_\gamma$. By (2.7), we see $\zeta^0 < \theta_{[\gamma^0,q,f]}$ in $\Omega$. Therefore, $(2.4)_\gamma$ has a positive solution. This contradicts Theorem 2.2. Suppose that $\gamma \geqslant \lambda_1^\Omega(q)$ and that $(2.4)_\gamma$ possesses a positive supersolution $\zeta^1$. Then $\theta_{[\gamma^1,q,f]}$ with $\Gamma_{[q,f]} < \gamma^1 < \lambda_1^\Omega(q)$ and near $\Gamma_{[q,f]}$ is a positive subsolution of $(2.4)_\gamma$ and $\theta_{[\gamma^1,q,f]} < \zeta^1$ in $\Omega$. Therefore, $(2.4)_\gamma$ has a positive solution. This contradicts Theorem 2.2.

(ii) Assume $\Gamma_{[q,f]} < \gamma < \lambda_1^{\Omega}(q)$. Then there exists $\varepsilon_\gamma > 0$ such that for $0 < \varepsilon < \varepsilon_\gamma$,

$$f'(0) \leqslant \frac{f(\varepsilon\phi(q))}{\varepsilon\phi(q)} + \frac{1}{2}(\gamma - \Gamma_{[q,f]}).$$

Thus,

$$\begin{aligned}-\Delta\big(\varepsilon\phi(q)\big) + q\big(\varepsilon\phi(q)\big) &= \big(\Gamma_{[q,f]} + f'(0)\big)\big(\varepsilon\phi(q)\big)\\ &\leqslant \frac{1}{2}(\gamma + \Gamma_{[q,f]})\big(\varepsilon\phi(q)\big) + f\big(\varepsilon\phi(q)\big)\\ &< \gamma\big(\varepsilon\phi(q)\big) + f\big(\varepsilon\phi(q)\big).\end{aligned}$$

This implies that $\varepsilon\phi(q)$ with $0 < \varepsilon < \varepsilon_\gamma$ is a subsolution to $(2.4)_\gamma$. Thus $(2.4)_\gamma$ has a positive solution $w$ between $\varepsilon\phi(q)$ and $\bar{w}$. The uniqueness of $\theta_{[\gamma,q,f]}$ implies that $w \equiv \theta_{[\gamma,q,f]}$ in $\Omega$. The fact that $\bar{w} \gg \theta_{[\gamma,q,f]}$ can be obtained from the strong maximum principle.

(iii) We first construct a supersolution of $(2.4)_\gamma$. Since $\lim_{s\to\infty} f(s)/s = 0$, for a fixed $0 < \delta < \frac{1}{2}(\lambda_1^{\Omega}(q) - \gamma)$ there exists $S = S(\delta) > 0$ such that $f(s) \leqslant \delta s$ for $s \geqslant S$. Setting $M = \max_{0\leqslant s\leqslant S} f(s)$ (note that $M$ depends on $\delta$) and considering the problem

$$-\Delta y + qy = \gamma y + M \quad \text{in } \Omega, \qquad y = 0 \quad \text{on } \partial\Omega, \tag{2.9}$$

we easily know that (2.9) has a unique solution $y_M$ which is the global minimizer of the functional

$$J(y) = \frac{1}{2}\int_{\Omega} |\nabla y|^2\,dx + \frac{1}{2}\int_{\Omega} (q-\gamma)y^2\,dx - \int_{\Omega} My\,dx$$

in $H_0^1(\Omega)$. Since $|y_M|$ is also a global minimizer, then $y_M \geqslant 0$. The regularity of $-\Delta$ implies $y_M \in C^2(\Omega)$ and thus the maximum principle implies that $y_M > 0$ in $\Omega$. We claim that $W := C\phi(q) + y_M$ with $C > 2S$ sufficiently large is a supersolution of $(2.4)_\gamma$. Indeed, for $C > 2S$, there is a subset $\Omega_C \subset\subset \Omega$ such that $W(x) \geqslant S$ for $x \in \Omega_C$. Therefore, for $x \in \Omega_C$,

$$\begin{aligned}-\Delta W + qW &= \lambda_1^{\Omega}(q)\big(C\phi(q)\big) + \gamma y_M + M\\ &= \gamma W + \big(\lambda_1^{\Omega}(q) - \gamma\big)\big(C\phi(q)\big) + M\\ &> \gamma W + f(W),\end{aligned}$$

where we are using the fact that we can choose $C$ sufficiently large such that $(\lambda_1^{\Omega}(q) - \gamma - \delta)(C\phi(q)) > \delta y_M$ in $\Omega_C$. For $x \in \Omega\backslash\Omega_C$, we have

$$-\Delta W + qW = \lambda_1^{\Omega}(q)\big(C\phi(q)\big) + \gamma y_M + M > \gamma W + f(W).$$

Therefore $W$ is a supersolution of $(2.4)_\gamma$. Choosing $C$ in $W$ sufficiently large, we see $\underline{w} < W$ in $\Omega$. Therefore, there is a positive solution $w$ of $(2.4)_\gamma$ in $(\underline{w}, W)$. The uniqueness of $\theta_{[\gamma,q,f]}$ implies $\theta_{[\gamma,q,f]} \geqslant \underline{w}$. The fact that $\theta_{[\gamma,q,f]} \gg \underline{w}$ can be obtained from the strong maximum principle. $\square$

## 3. Change of stability of semi-trivial positive solutions

By Theorem 2.2, (1.2) possesses a semi-trivial positive solution of the form $(u, 0)$ if, and only if, $\Gamma_{[0,f]} < \lambda < \lambda_1^\Omega(0)$. Moreover, in this case the semi-trivial state is $(\theta_{[\lambda,0,f]}, 0)$. Similarly, (1.2) possesses a semi-trivial positive solution of the form $(0, v)$ if, and only if, $\Gamma_{[0,g]} < \mu < \lambda_1^\Omega(0)$ and if this is the case, then it is given by $(0, \theta_{[\mu,0,g]})$. The following result characterizes the linearized stability of each of these semi-trivial states.

**Proposition 3.1.** *Assume $\Gamma_{[0,f]} < \lambda < \lambda_1^\Omega(0)$. Then, $(\theta_{[\lambda,0,f]}, 0)$ is linearly asymptotically stable if, and only if,*

$$\mu < \lambda_1^\Omega(-c\theta_{[\lambda,0,f]}) - g'(0); \tag{3.1}$$

*linearly unstable if, and only if,*

$$\mu > \lambda_1^\Omega(-c\theta_{[\lambda,0,f]}) - g'(0); \tag{3.2}$$

*and linearly neutrally stable if*

$$\mu = \lambda_1^\Omega(-c\theta_{[\lambda,0,f]}) - g'(0). \tag{3.3}$$

*Similarly, if we assume $\Gamma_{[0,g]} < \mu < \lambda_1^\Omega(0)$, then $(0, \theta_{[\mu,0,g]})$ is linearly asymptotically stable if, and only if, $\lambda < \lambda_1^\Omega(-b\theta_{[\mu,0,g]}) - f'(0)$; linearly unstable if, and only if, $\lambda > \lambda_1^\Omega(-b\theta_{[\mu,0,g]}) - f'(0)$ and linearly neutrally stable if*

$$\lambda = \lambda_1^\Omega(-b\theta_{[\mu,0,g]}) - f'(0). \tag{3.4}$$

**Proof.** The linearized stability of $(\theta_{[\lambda,0,f]}, 0)$ is given by the sign of the real parts of the eigenvalues of the linearization of (1.2) at $(\theta_{[\lambda,0,f]}, 0)$, i.e., by the real parts of the $\tau$'s for which the following linear problem admits a solution $(h, k) \in (W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega))^2 \backslash \{(0, 0)\}$:

$$\begin{aligned} -\Delta h &= \lambda h + f'(\theta_{[\lambda,0,f]})h + b\theta_{[\lambda,0,f]}k + \tau h, \\ -\Delta k &= \mu k + g'(0)k + c\theta_{[\lambda,0,f]}k + \tau k. \end{aligned} \tag{3.5}$$

If $k = 0$, then (3.5) becomes

$$-\Delta h = \big(\lambda + f'(\theta_{[\lambda,0,f]})\big)h + \tau h. \tag{3.6}$$

On the other hand, from the definition of $\theta_{[\lambda,0,f]}$ we find from Theorem 2.2 that

$$\lambda_1^\Omega\left(-\frac{f(\theta_{[\lambda,0,f]})}{\theta_{[\lambda,0,f]}}\right) - \lambda = \lambda_1^\Omega\left(-\frac{f(\theta_{[\lambda,0,f]})}{\theta_{[\lambda,0,f]}} - \lambda\right) = 0.$$

The condition $(G_2)$ on $f$ implies

$$f'(\theta_{[\lambda,0,f]}) < \frac{f(\theta_{[\lambda,0,f]})}{\theta_{[\lambda,0,f]}}.$$

Thus, Lemma 2.1 implies

$$\lambda_1^\Omega\big(-f'(\theta_{[\lambda,0,f]})\big) - \lambda > 0, \tag{3.7}$$

and hence, any eigenvalue $\tau$ of (3.5) satisfies

$$\mathrm{Re}\tau \geqslant \lambda_1^\Omega\big(-f'(\theta_{[\lambda,0,f]}) - \lambda\big) = \lambda_1^\Omega\big(-f'(\theta_{[\lambda,0,f]})\big) - \lambda > 0.$$

Thus, the eigenvalue with associated eigenfunctions of the form $(u,0)$ has a positive real part. If $k \neq 0$, then $\tau$ is an eigenvalue of $-\Delta - g'(0) - c\theta_{[\lambda,0,f]} - \mu$. Assuming (3.1) holds, we see that

$$\lambda_1^\Omega\big(-g'(0) - c\theta_{[\lambda,0,f]} - \mu\big) > 0$$

and the real part of any eigenvalue of $-\Delta - g'(0) - c\theta_{[\lambda,0,f]} - \mu$ must be positive. Hence, under condition (3.1) the real part of any eigenvalue $\tau$ of (3.5) is positive and therefore, the state $(\theta_{[\lambda,0,f]}, 0)$ is linearly asymptotically stable. Assuming (3.2) holds, we see that

$$\tau_1 := \lambda_1^\Omega\big(-g'(0) - c\theta_{[\lambda,0,f]} - \mu\big) = \lambda_1^\Omega(-c\theta_{[\lambda,0,f]}) - g'(0) - \mu < 0$$

is an eigenvalue corresponding to a positive eigenfunction, say $\xi$, of the second equation of (3.5). Since $\tau_1 < 0$, (3.7) implies

$$\lambda_1^\Omega\big(-f'(\theta_{[\lambda,0,f]}) - \lambda - \tau_1\big) = \lambda_1^\Omega\big(-f'(\theta_{[\lambda,0,f]})\big) - \lambda - \tau_1 > 0,$$

and therefore, thanks to the strong maximum principle, the first equation of (3.5) with $\tau = \tau_1$ possesses a unique solution:

$$h = \big(-\Delta - f'(\theta_{[\lambda,0,f]}) - \lambda - \tau_1\big)^{-1}(b\theta_{[\lambda,0,f]}\xi).$$

Therefore, under the condition (3.2), $\tau_1 < 0$ is an eigenvalue of (3.5) and hence the state $(\theta_{[\lambda,0,f]}, 0)$ is linearly unstable. Finally, if we assume (3.3) holds, it is easily seen that $\tau_1 = 0$ is an eigenvalue of (3.5) and that any other eigenvalue has positive real part. Therefore, under the condition (3.3) the state $(\theta_{[\lambda,0,f]}, 0)$ is linearly neutrally stable.

The results concerning with the semi-trivial state $(0, \theta_{[\mu,0,g]})$ can be obtained similarly. (Note that $\lambda$, $b$, and $f$ are changed by $\mu$, $c$, and $g$, respectively.) $\square$

Proposition 3.1 implies that the curve (3.3) in the $(\lambda,\mu)$-plane is the curve of change of stability of the semi-trivial positive solution $(\theta_{[\lambda,0,f]}, 0)$. Similarly, the curve (3.4) is the curve of change of stability of $(0, \theta_{[\mu,0,g]})$. The following result provides us with the global behavior of these curves.

**Proposition 3.2.** *The mapping $F(\lambda)$ defined by*

$$F(\lambda) := \lambda_1^\Omega(-c\theta_{[\lambda,0,f]}) - g'(0), \quad \Gamma_{[0,f]} < \lambda < \lambda_1^\Omega(0), \tag{3.8}$$

*is continuous strictly decreasing and satisfies*

$$\lim_{\lambda \downarrow \Gamma_{[0,f]}} F(\lambda) = \Gamma_{[0,g]}, \qquad \lim_{\lambda \uparrow \lambda_1^\Omega(0)} F(\lambda) = -\infty. \tag{3.9}$$

*Similarly, the mapping $G(\mu)$ defined by*

$$G(\mu) := \lambda_1^\Omega(-b\theta_{[\mu,0,g]}) - f'(0), \quad \Gamma_{[0,g]} < \mu < \lambda_1^\Omega(0), \tag{3.10}$$

*is continuous strictly decreasing and satisfies*

$$\lim_{\mu \downarrow \Gamma_{[0,g]}} G(\mu) = \Gamma_{[0,f]}, \qquad \lim_{\mu \uparrow \lambda_1^\Omega(0)} G(\mu) = -\infty. \tag{3.11}$$

**Proof.** The continuity and monotonicity of $F(\lambda)$ can be obtained from Theorem 2.2. The first relation of (3.9) follows from (2.6). Now we prove the second relation of (3.9). Since $\phi(0) > 0$ in $\Omega$, there exists a ball $B$ with $\bar{B} \subset \Omega$ such that

$$\phi_L := \min_{\bar{B}} \phi(0) > 0.$$

On the other hand, by Theorem 2.3, for each $\Gamma_{[0,f]} < \lambda < \lambda_1^{\Omega}(0)$,

$$\theta_{[\lambda,0,f]} \geqslant \phi_L \mathcal{F}^{-1}\big(\lambda_1^{\Omega}(0) - \lambda\big) \quad \text{uniformly in } \bar{B},$$

and hence, Lemma 2.1 implies

$$F(\lambda) = \lambda_1^{\Omega}(-c\theta_{[\lambda,0,f]}) - g'(0) < \lambda_1^{B}(0) - g'(0) - c\phi_L \mathcal{F}^{-1}\big(\lambda_1^{\Omega}(0) - \lambda\big).$$

Our conclusion follows from the fact

$$\lim_{\lambda \uparrow \lambda_1^{\Omega}(0)} \mathcal{F}^{-1}\big(\lambda_1^{\Omega}(0) - \lambda\big) = \infty.$$

The same argument shows the corresponding properties of $G(\mu)$. $\square$

By Proposition 3.2, the curves of change of stability of the semi-trivial positive solutions meet at $(\Gamma_{[f,0]}, \Gamma_{[g,0]})$.

## 4. The existence of unbounded continua of coexistence states

In this section we provide a bifurcation result for the coexistence.

**Theorem 4.1.** *Fix $\Gamma_{[0,f]} < \lambda < \lambda_1^{\Omega}(0)$ and treat $\mu$ as the bifurcation parameter. Then, the point*

$$(\mu, u, v) = \big(\lambda_1^{\Omega}\big(-g'(0) - c\theta_{[\lambda,0,f]}\big), \theta_{[\lambda,0,f]}, 0\big)$$

*is the only bifurcation point to coexistence states from the semi-trivial state $(\theta_{[\lambda,0,f]}, 0)$. Moreover, the maximal component (closed and connected) of coexistence states emanating from $(\theta_{[\lambda,0,f]}, 0)$ at $\mu = F(\lambda)$, say $\mathcal{E}^{+}_{(\mu,u,0)} \subset \mathbb{R} \times (C_0^1(\bar{\Omega}))^2$, is unbounded.*

*Now, fix $\mu < \Gamma_{[0,g]}$ and treat $\lambda \in \mathbb{R}$ as the bifurcation parameter. By Proposition 3.2, there exists a unique $\Gamma_{[0,f]} < \lambda_\mu < \lambda_1^{\Omega}(0)$ such that $\mu = F(\lambda_\mu)$. Then, the point*

$$(\lambda, u, v) = (\lambda_\mu, \theta_{[\lambda_\mu,0,f]}, 0)$$

*is the only bifurcation point to coexistence states from the curve $(\theta_{[\lambda,0,f]}, 0)$. Moreover, the maximal component (closed and connected) of coexistence states emanating from $(\theta_{[\lambda,0,f]}, 0)$ at $\lambda = \lambda_\mu$, say $\mathcal{E}_{(\lambda,u,0)} \subset \mathbb{R} \times C_0^1(\bar{\Omega}) \times C_0^1(\bar{\Omega})$, is unbounded.*

*Similarly, if we fix $\Gamma_{[0,g]} < \mu < \lambda_1^{\Omega}(0)$ and treat $\lambda \in \mathbb{R}$ as the bifurcation parameter, then the point*

$$(\lambda, u, v) = \big(\lambda_1^{\Omega}\big(-f'(0) - b\theta_{[\mu,0,g]}\big), 0, \theta_{[\mu,0,g]}\big)$$

*is the only bifurcation point to coexistence states from the semitrivial state $(0, \theta_{[\mu,0,g]})$ and the maximal component (closed and connected) of coexistence states emanating from $(0, \theta_{[\mu,0,g]})$ at $\lambda = G(\mu)$, say $\mathcal{E}^{+}_{(\lambda,0,v)} \subset \mathbb{R} \times C_0^1(\bar{\Omega}) \times C_0^1(\bar{\Omega})$, is unbounded.*

*Finally, fix $\lambda < \Gamma_{[0,f]}$ and treat $\mu \in \mathbb{R}$ as the bifurcation parameter. By Proposition* 3.2, *there exists a unique $\Gamma_{[0,g]} < \mu_\lambda < \lambda_1^\Omega(0)$ such that $\lambda = G(\mu_\lambda)$. In this case, the point*

$$(\mu, u, v) = (\mu_\lambda, 0, \theta_{[\mu_\lambda,0,g]})$$

*is the only bifurcation point to coexistence states from the curve $(0, \theta_{[\mu,0,g]})$ and the maximal component* (*closed and connected*) *of coexistence states emanating from $(0, \theta_{[\mu,0,g]})$ at $\mu = \mu_\lambda$, say $\mathcal{E}^+_{(\mu,0,v)} \subset \mathbb{R} \times C_0^1(\bar{\Omega}) \times C_0^1(\bar{\Omega})$, is unbounded.*

**Proof.** Similar to the proof of [5, Theorem 5.1 ]. □

## 5. Coexistence regions of (1.2)

We first show the following lemma.

**Lemma 5.1.** *Assume that* (1.2) *possesses a coexistence state, say $(u, v)$. Then*

$$u_M < \frac{\lambda_1^\Omega(0) - \mu}{c}, \qquad v_M < \frac{\lambda_1^\Omega - \lambda}{b}, \tag{5.1}$$

*where $u_M = \max_\Omega u$ and $v_M = \max_\Omega v$.*

**Proof.** From (1.2) it is easily seen that

$$u = \theta_{[\lambda,-bv,f]}, \qquad v = \theta_{[\mu,-cu,g]}.$$

It follows from Lemma 2.1 that

$$\theta_{[\lambda,-bv,f]} \leqslant \theta_{[\lambda,-bv_M,f]} = \theta_{[\lambda+bv_M,0,f]}.$$

Similarly,

$$\theta_{[\mu,-cu,g]} \leqslant \theta_{[\mu,-cu_M,g]} = \theta_{[\mu+cu_M,0,g]}.$$

Moreover, since $\theta_{[\lambda+bv_M,0,f]} \geqslant u > 0$, we find from Theorem 2.2 that

$$\Gamma_{[0,f]} < \lambda + bv_M < \lambda_1^\Omega(0).$$

Therefore,

$$v_M < \frac{\lambda_1^\Omega(0) - \lambda}{b}.$$

Similarly,

$$u_M < \frac{\lambda_1^\Omega - \mu}{c}.$$

This completes the proof. □

**Remark 5.2.** It follows from Lemma 5.1 that if (1.2) possesses a coexistence state, then $\lambda < \lambda_1^\Omega(0)$ and $\mu < \lambda_1^\Omega(0)$.

Now we provide the following non-existence result.

**Theorem 5.3.** *The following assertions are true*:

(i) *If* $\Gamma_{[0,g]} < \mu < \lambda_1^{\Omega}(0)$, *then there exist* $-\infty < \lambda_* < \lambda_1^{\Omega}(0)$ *such that* (1.2) *does not admit a coexistence state provided* $\lambda > \lambda_*$.
(ii) *If* $\Gamma_{[0,f]} < \lambda < \lambda_1^{\Omega}(0)$, *then there exist* $-\infty < \mu_* < \lambda_1^{\Omega}(0)$ *such that* (1.2) *does not admit a coexistence state provided* $\mu > \mu_*$.
(iii) *For each* $\lambda < \Gamma_{[0,f]}$, *there exists* $\Gamma_{[0,g]} < \mu = \mu(\lambda) < \lambda_1^{\Omega}(0)$ *such that* $\lambda > \lambda_1^{\Omega}(-b\theta_{[\mu(\lambda),0,g]}) - f'(0)$ *and* (1.2) *does not admit a coexistence state if* $\mu(\lambda) < \mu < \lambda_1^{\Omega}(0)$. *Moreover,* $\mu(\lambda)$ *can be chosen continuous in* $\lambda$.
(iv) *For each* $\mu < \Gamma_{[0,g]}$, *there exists* $\Gamma_{[0,f]} < \lambda = \lambda(\mu) < \lambda_1^{\Omega}(\Omega)$ *such that* $\mu > \lambda_1^{\Omega}(-c\theta_{[\lambda(\mu),0,f]}) - g'(0)$ *and* (1.2) *does not admit a coexistence state if* $\lambda(\mu) < \lambda < \lambda_1^{\Omega}(0)$. *Moreover,* $\lambda(\mu)$ *can be chosen to be continuous in* $\mu$.

**Proof.** (i) Assume (1.2) possesses a coexistence state $(u, v)$, then

$$-\Delta u = \lambda u + f(u) + buv > \lambda u + f(u).$$

Thanks to Lemma 2.4,

$$u \gg \theta_{[\lambda,0,f]}.$$

We also know from Lemma 5.1 that

$$u_M < \frac{\lambda_1^{\Omega}(0) - \mu}{c}.$$

Thus

$$\max_{\Omega} \theta_{[\lambda,0,f]} < \frac{\lambda_1^{\Omega}(0) - \mu}{c}.$$

Theorem 2.3 implies that $\lambda_*$ as required in this theorem do exist. This completes the proof.

Part (ii) follows by symmetry.

(iii) Pick up $\lambda < \Gamma_{[0,f]}$. It follows from Proposition 3.2 that there exists $\Gamma_{[0,g]} < \mu_0(\lambda) < \lambda_1^{\Omega}(0)$ such that

$$\lambda > \lambda_1^{\Omega}\big(-f'(0) - b\theta_{[\mu,0,g]}\big) \quad \text{for each } \mu_0(\lambda) < \mu < \lambda_1^{\Omega}(0). \tag{5.2}$$

We argue by contradiction assuming that there exists a sequence of coexistence states of (1.2), say $(\mu_n, u_n, v_n)$, $n \geqslant 1$, such that $\mu_n > \mu_0(\lambda)$, $n \geqslant 1$, and $\lim_{n\uparrow\infty} \mu_n = \lambda_1^{\Omega}(0)$. Then the second equation of (1.2) gives

$$-\Delta v_n = \mu_n v_n + g(v_n) + cu_n v_n > \mu_n v_n + g(v_n)$$

and hence $v_n$ is a strict positive supersolution of

$$-\Delta w = \mu_n w + g(w) \quad \text{in } \Omega, \qquad w = 0 \quad \text{on } \partial\Omega.$$

Thus, thanks to Lemma 2.4 and Theorem 2.3,

$$v_n \geqslant \theta_{[\mu_n,0,g]} \geqslant \mathcal{F}^{-1}\big(\lambda_1^{\Omega}(0) - \mu_n\big)\phi(0). \tag{5.3}$$

Let $\Omega_1 \subset\subset \Omega$ and $\phi_L = \min_{x\in\Omega_1} \phi(0)(x)$. Then $\phi_L > 0$ and

$$v_n \geqslant \mathcal{F}^{-1}\big(\lambda_1^{\Omega}(0) - \mu_n\big)\phi_L \quad \text{in } \Omega_1.$$

On the other hand, we find from the first equation of (1.2) that

$$\lambda = \lambda_1^{\Omega}\left(-\frac{f(u_n)}{u_n} - bv_n\right) \leqslant \lambda_1^{\Omega}(-bv_n) \leqslant \lambda_1^{\Omega_1}(-bv_n)$$
$$\leqslant \lambda_1^{\Omega_1}(0) - b\mathcal{F}^{-1}\big(\lambda_1^{\Omega}(0) - \mu_n\big)\phi_L \downarrow -\infty \quad \text{as } n \to \infty.$$

This contradiction shows that (1.2) does not admit a coexistence state for $\mu$ large and completes the proof.

Part (iv) follows by symmetry. □

Now we provide the following existence result.

**Theorem 5.4.** *Assume*

$$\Gamma_{[0,f]} < \lambda < \lambda_1^{\Omega}(0), \quad \mu < \lambda_1^{\Omega}(-c\theta_{[\lambda,0,f]}) - g'(0) \tag{5.4}$$

*or*

$$\Gamma_{[0,g]} < \mu < \lambda_1^{\Omega}(0), \quad \lambda < \lambda_1^{\Omega}(-b\theta_{[\mu,0,g]}) - f'(0). \tag{5.5}$$

*Then* (1.2) *possesses a coexistence state.*

**Proof.** We only show the first part of this theorem. The second part can be obtained similarly. Fix $\lambda \in (\Gamma_{[0,f]}, \lambda_1^{\Omega}(0))$ and consider $\mu$ as the main bifurcation parameter. By Theorem 5.3, problem (1.2) does not admit a coexistence state if $\mu \geqslant \mu_*$. Moreover, by Theorem 4.1 the continuum $\mathcal{E}^+_{(\mu,u,0)}$ of coexistence states emanating from $(\theta_{[\lambda,0,f]}, 0)$ at the value of the parameter $\mu = \lambda_1^{\Omega}(-g'(0) - c\theta_{[\lambda,0,f]})$ is unbounded and thanks to Lemma 5.1 these coexistence states are bounded in $C_0^1(\bar{\Omega}) \times C_0^1(\bar{\Omega})$ uniformly on compact subintervals of $\mu$. Therefore, (1.2) possesses a coexistence state for each $\mu < \lambda_1^{\Omega}(-g'(0) - c\theta_{[\lambda,0,f]}) = \lambda_1^{\Omega}(-c\theta_{[\lambda,0,f]}) - g'(0)$. This completes the proof. □

**Theorem 5.5.**

(i) *Assume $\lambda < \Gamma_{[0,f]}$ and $\mu < \mu_\lambda$, where $\mu_\lambda$ is the unique value of $\mu$ satisfying $\lambda = \lambda_1^{\Omega}(-b\theta_{[\mu,0,g]}) - f'(0)$. Then* (1.2) *possesses a coexistence state.*
(ii) *Assume $\mu < \Gamma_{[0,g]}$ and $\lambda < \lambda_\mu$, where $\lambda_\mu$ is the unique value of $\lambda$ satisfying $\mu = \lambda_1^{\Omega}(-c\theta_{[\lambda,0,f]}) - g'(0)$. Then* (1.2) *possesses a coexistence state.*

**Proof.** We only show the first case. The second case follows by symmetry. Fix $\lambda < \Gamma_{[0,f]}$ and consider $\mu$ as the main bifurcation parameter. By Theorem 5.3(iii), there exists $\mu = \mu(\lambda) \in (\Gamma_{[0,g]}, \lambda_1^{\Omega}(0))$ such that $\lambda > \lambda_1^{\Omega}(-f'(0) - b\theta_{[\mu(\lambda),0,g]}) = \lambda_1^{\Omega}(-b\theta_{[\mu(\lambda),0,g]}) - f'(0)$ and (1.2) does not admit a coexistence state for $\mu(\lambda) < \mu < \lambda_1^{\Omega}(0)$.

Moreover, by Theorem 4.1 the continuum $\mathcal{E}^+_{(\mu,0,v)}$ of coexistence states emanating from $(0, \theta_{[\mu,0,g]})$ at $\mu_\lambda$ is unbounded, where $\mu_\lambda$ is the unique value of $\mu > \Gamma_{[0,g]}$ for which $\lambda = \lambda_1^{\Omega}(-f'(0) - b\theta_{[\mu,0,g]})$. Hence, we conclude from Lemma 5.1 that (1.2) possesses a coexistence state for each $\mu < \mu_\lambda$. This completes the proof of part (i) and completes the proof of this theorem. □

## 6. Structure of the coexistence states

In this section we analyze the structure of the set of $\lambda$'s (respectively $\mu$'s) for which (1.2) possesses a coexistence state, denoted by $\Lambda$ (respectively $M$).

**Theorem 6.1.** *The following assertions are true*:

(i) *Assume $\Gamma_{[0,g]} < \mu < \lambda_1^\Omega(0)$. Then, either $\Lambda = (-\infty, \lambda_1^\Omega(-b\theta_{[\mu,0,g]}) - f'(0))$, or there exists $\lambda_1^\Omega(-b\theta_{[\mu,0,g]}) - f'(0) \leqslant \lambda^* < \lambda_1^\Omega(\Omega)$ such that $\Lambda = (-\infty, \lambda^*]$.*
(ii) *Assume $\Gamma_{[0,f]} < \lambda < \lambda_1^\Omega(0)$. Then, either $M = (-\infty, \lambda_1^\Omega(-c\theta_{[\lambda,0,f]}) - g'(0))$, or there exists $\lambda_1^\Omega(-c\theta_{[\lambda,0,f]}) - g'(0) \leqslant \mu^* < \lambda_1^\Omega(0)$ such that $M = (-\infty, \mu^*]$.*

**Proof.** We will only prove the first case. The second case follows by symmetry. Assume $\Gamma_{[0,g]} < \mu < \lambda_1^\Omega(0)$. Then, thanks to Theorem 5.4,

$$\big(-\infty, \lambda_1^\Omega(-b\theta_{[\mu,0,g]}) - f'(0)\big) \subset \Lambda. \tag{6.1}$$

Now, suppose that (1.2) possesses a coexistence state $(u_0, v_0)$ for some

$$\lambda_0 > \lambda_1^\Omega(-b\theta_{[\mu,0,g]}) - f'(0).$$

Then it follows from Theorem 5.3 that $\lambda_0 \leqslant \lambda_*$. Moreover, $(u_0, v_0)$ is a supersolution of (1.2) for each

$$\lambda \in \big(\lambda_1^\Omega(-b\theta_{[\mu,0,g]}) - f'(0), \lambda_0\big]. \tag{6.2}$$

On the other hand, if we define $\zeta(x) > 0$ with $\|\zeta\|_\infty = 1$ is the eigenfunction corresponding to $\lambda_1^\Omega(-f'(0) - b\theta_{[\mu,0,g]})$, by the condition on $f$, we can choose an $\varepsilon^0 > 0$ such that for $0 < \varepsilon < \varepsilon^0$,

$$0 < f'(0) - \frac{f(\varepsilon\zeta)}{\varepsilon\zeta} < \frac{1}{2}\big(\lambda - \lambda_1^\Omega\big(-f'(0) - b\theta_{[\mu,0,g]}\big)\big). \tag{6.3}$$

(Note that $\lambda_1^\Omega(-f'(0) - b\theta_{[\mu,0,g]}) = \lambda_1^\Omega(-b\theta_{[\mu,0,g]}) - f'(0)$.) Thus,

$$\begin{aligned} -\Delta(\varepsilon\zeta) &= \lambda_1^\Omega\big(-f'(0) - b\theta_{[\mu,0,g]}\big)(\varepsilon\zeta) + f'(0)(\varepsilon\zeta) + b\theta_{[\mu,0,g]}(\varepsilon\zeta) \\ &< \lambda(\varepsilon\zeta) + f(\varepsilon\zeta) + b\theta_{[\mu,0,g]}(\varepsilon\zeta). \end{aligned}$$

If we choose $\mu$ near $\Gamma_{[0,g]}$ and $\varepsilon$ sufficiently small, we easily know that

$$\varepsilon\zeta < u_0, \qquad \theta_{[\mu,0,g]} < v_0 \quad \text{in } \Omega.$$

Such couple provides us with a subsolution of (1.2). Thanks to [5, Theorem 8.7], for each $\lambda$ satisfying (6.3) problem (1.2) possesses a coexistence state. Therefore, we see that (1.2) possesses a coexistence state for $\lambda$ satisfying (6.2).

Now we show that (1.2) possesses a coexistence state for $\lambda = \lambda_1^\Omega(-f'(0) - b\theta_{[\mu,0,g]})$. We fix $\mu$ and treat $\lambda$ as the main bifurcation parameter. By Theorem 4.1,

$$(\lambda, u, v) = \big(\lambda_1^\Omega\big(-f'(0) - b\theta_{[\mu,0,g]}\big), 0, \theta_{[\mu,0,g]}\big)$$

is the only bifurcation point to coexistence states from the semi-trivial curve $(u, v) = (0, \theta_{[\mu,0,g]})$ and the maximal component (closed and connected) of coexistence states emanating from $(0, \theta_{[\mu,0,g]})$ at $\lambda = \lambda_1^{\Omega}(-f'(0) - b\theta_{[\mu,0,g]})$, denoted by $\mathcal{E}^+_{(\lambda,0,v)}$, is unbounded in $\mathbb{R} \times (C_0^1(\bar{\Omega}))^2$. Moreover, by the local bifurcation theorem of [4], there exist a neighborhood $\mathcal{N} := \mathcal{N}(\lambda_1^{\Omega}(-f'(0) - b\theta_{[\mu,0,g]}), 0, \theta_{[\mu,0,g]})$ of $(\lambda_1^{\Omega}(-f'(0) - b\theta_{[\mu,0,g]}), 0, \theta_{[\mu,0,g]})$ in $\mathbb{R} \times (C_0^1(\bar{\Omega}))^2$, a real number $s_0 > 0$, and an analytic mapping

$$(\lambda, u, v) : (-s_0, s_0) \to \mathbb{R} \times C_0^1(\bar{\Omega}) \times C_0^1(\bar{\Omega})$$

such that

$$\big(\lambda(0), u(0), v(0)\big) = \big(\lambda_1^{\Omega}\big(-f'(0) - b\theta_{[\mu,0,g]}\big), 0, \theta_{[\mu,0,g]}\big)$$

and

$$\mathcal{N} \cap \mathcal{E}^+_{(\lambda,0,v)} = \big\{\big(\lambda(s), u(s), v(s)\big) \colon s > 0\big\}.$$

Indeed, the unique coexistence states of (1.2) close to the bifurcation point are those lying on the curve $(\lambda(s), u(s), v(s))$. Since $\lambda(s)$ is analytic, $s_0$ can be reduced, if necessary, so that either $\lambda(s) < \lambda_1^{\Omega}(-f'(0) - b\theta_{[\mu,0,g]})$ for each $s \in (0, s_0)$, or $\lambda(s) = \lambda_1^{\Omega}(-f'(0) - b\theta_{[\mu,0,g]})$ for each $s \in (0, s_0)$, or $\lambda(s) > \lambda_1^{\Omega}(-f'(0) - b\theta_{[\mu,0,g]})$ for each $s \in (0, s_0)$. If $\lambda(s) = \lambda_1^{\Omega}(-f'(0) - b\theta_{[\mu,0,g]})$ for each $s \in (0, s_0)$ the proof is completed.

Assume that $\lambda(s) < \lambda_1^{\Omega}(-f'(0) - b\theta_{[\mu,0,g]})$ for each $s \in (0, s_0)$. Since (1.2) possesses a coexistence state for each $\lambda \in (\lambda_1^{\Omega}(-f'(0) - b\theta_{[\mu,0,g]}), \lambda_0]$ and thanks to Lemma 5.1 uniform a priori bounds for the coexistence states of (1.2) are available in the range $\lambda \in (\lambda_1^{\Omega}(-f'(0) - b\theta_{[\mu,0,g]}), \lambda_0]$, for any sequence of coexistence states of (1.2), say $(\lambda_n, u_n, v_n)$, with $\lambda_n > \lambda_1^{\Omega}(-f'(0) - b\theta_{[\mu,0,g]})$ and $\lambda_n \downarrow \lambda_1^{\Omega}(-f'(0) - b\theta_{[\mu,0,g]})$, we can choose a convergent subsequence, relabeled by $n$, such that

$$\lim_{n\to\infty} (u_n, v_n) = (u^*, v^*)$$

for some non-negative solution couple $(u^*, v^*)$ of (1.2) with $\lambda = \lambda_1^{\Omega}(-f'(0) - b\theta_{[\mu,0,g]})$. By the uniqueness obtained from the application of [4],

$$(\lambda_n, u_n, v_n) \notin \mathcal{N}$$

for $n$ sufficiently large. Hence,

$$(u^*, v^*) \neq (0, \theta_{[\mu,0,g]}).$$

Now we show that $(u^*, v^*) \neq (0, 0)$. To show this we argue by contradiction. Indeed, if $u^* = v^* = 0$, then the new sequences $\tilde{u}_n = u_n/\|u_n\|_\infty$ and $\tilde{v}_n = v_n/\|v_n\|_\infty$ satisfy

$$\begin{aligned}
&-\Delta \tilde{u}_n = \lambda_n \tilde{u}_n + \frac{f(u_n)}{u_n}\tilde{u}_n + b\tilde{u}_n v_n, \quad x \in \Omega,\\
&-\Delta \tilde{v}_n = \mu \tilde{v}_n + \frac{g(v_n)}{v_n}\tilde{v}_n + c u_n \tilde{v}_n, \quad x \in \Omega,\\
&\tilde{u}_n = \tilde{v}_n = 0, \quad x \in \partial\Omega
\end{aligned} \tag{6.4}$$

and, since $(\tilde{u}_n, \tilde{v}_n)$ is uniformly bounded in $L^\infty(\Omega) \times L^\infty(\Omega)$, the regularity of $-\Delta$ implies that there exists a subsequence (still denoted by $(\tilde{u}_n, \tilde{v}_n)$), such that $\tilde{u}_n \to w$ and

$\tilde{v}_n \to z$, as $n \to \infty$, for some $w, z \in C_0^1(\bar{\Omega})$. Necessarily $w > 0$, $z > 0$ and passing to the limit in (6.4), we find that

$$-\Delta w = \lambda_1^{\Omega}\big(-f'(0) - b\theta_{[\mu,0,g]}\big)w + f'(0)w, \quad x \in \Omega$$
$$-\Delta z = \mu z + g'(0)z, \quad x \in \Omega,$$
$$w = z = 0, \quad x \in \partial\Omega. \tag{6.5}$$

By the uniqueness of the principal eigenvalue,

$$\mu = \Gamma_{[0,g]}$$

and this is impossible, since we are assuming $\mu > \Gamma_{[0,g]}$.

If $u^* > 0$ and $v^* \equiv 0$, then we take the sequence $(u_n, \tilde{v}_n)$ and the same compactness argument as above shows that

$$\mu = \lambda_1^{\Omega}\big(-g'(0) - cu^*\big) < \Gamma_{[0,g]},$$

which is impossible either. Therefore, $(u^*, v^*)$ must be a coexistence state.

Finally, assume that $\lambda(s) > \lambda_1^{\Omega}(-f'(0) - b\theta_{[\mu,0,g]})$ for each $s \in (0, s_0)$ and let $\mathcal{E}_1^+$ denote the maximal subcontinuum of $\mathcal{E}^+_{(\lambda,0,v)}$ outside $\mathcal{N}$. It is clear that $\mathcal{E}_1^+$ is unbounded. Thanks to Lemma 5.1 uniform a priori bounds on compact intervals of $\lambda$ are available. Moreover, thanks to Theorem 5.3(i), (1.2) does not admit a coexistence state if $\lambda > \lambda_*$. Therefore, $\mathcal{E}_1^+$ must go backwards and (1.2) possesses a coexistence state for $\lambda = \lambda_1^{\Omega}(-f'(0) - b\theta_{[\mu,0,g]})$ as well.

The analysis above implies that

$$(-\infty, \lambda_0] \subset \Lambda.$$

Let

$$\lambda^* = \sup\big\{\lambda_0 > \lambda_1^{\Omega}\big(-f'(0) - b\theta_{[\mu,0,g]}\big) \text{ for which (1.2) has a coexistence state}\big\}.$$

We have that $(-\infty, \lambda^*) \subset \Lambda$ and that

$$\lambda_1^{\Omega}\big(-f'(0) - b\theta_{[\mu,0,g]}\big) < \lambda^* \leqslant \lambda_*. \tag{6.6}$$

Due to the existence of a priori bounds, there exists a sequence of positive solutions of (1.2), say $(\lambda_n, u_n, v_n)$, $n \geqslant 1$, such that

$$\lim_{n\to\infty}(\lambda_n, u_n, v_n) = (\lambda^*, \hat{u}, \hat{v}),$$

for some non-negative solution $(\hat{u}, \hat{v})$ of (1.2) with $\lambda = \lambda^*$. Necessarily $\hat{u} > 0$ and $\hat{v} > 0$. To show this we argue by contradiction. Indeed, if $\hat{u} = \hat{v} = 0$, then the new sequences $\hat{u}_n = u_n/\|u_n\|_\infty$ and $\hat{v}_n = v_n/\|v_n\|_\infty$ satisfy

$$-\Delta\hat{u}_n = \lambda_n\hat{u}_n + \frac{f(u_n)}{u_n}\hat{u}_n + b\hat{u}_n v_n, \quad x \in \Omega,$$
$$-\Delta\hat{v}_n = \mu\hat{v}_n + \frac{g(v_n)}{v_n}\hat{v}_n + cu_n\hat{v}_n, \quad x \in \Omega,$$
$$\hat{u}_n = \hat{v}_n = 0, \quad x \in \partial\Omega \tag{6.7}$$

and, since $(\hat{u}_n, \hat{v}_n)$ is uniformly bounded in $L^\infty(\Omega) \times L^\infty(\Omega)$, the regularity of $-\Delta$ implies that there exists a subsequence (still denoted by $(\hat{u}_n, \hat{v}_n)$), such that $\hat{u}_n \to \hat{w}$ and $\hat{v}_n \to \hat{z}$, as $n \to \infty$, for some $\hat{w}, \hat{z} \in C_0^1(\bar{\Omega})$. Necessarily $\hat{w} > 0$, $\hat{z} > 0$ and passing to the limit in (6.7), we find that

$$\begin{aligned} &-\Delta \hat{w} = \lambda^* \hat{w} + f'(0)\hat{w}, \quad x \in \Omega, \\ &-\Delta \hat{z} = \mu \hat{z} + g'(0)\hat{z}, \quad x \in \Omega, \\ &\hat{w} = \hat{z} = 0, \quad x \in \partial\Omega. \end{aligned} \tag{6.8}$$

By the uniqueness of the principal eigenvalue,

$$\lambda^* = \Gamma_{[0,f]}, \qquad \mu = \Gamma_{[0,g]},$$

and this is impossible, since we are assuming that $\mu > \Gamma_{[0,g]}$.

If $\hat{u} > 0$ and $\hat{v} \equiv 0$, then we take the sequence $(u_n, \hat{v}_n)$ and the same compactness argument as above shows that $\hat{u} = \theta_{[\lambda^*,0,f]}$ and that

$$\mu = \lambda_1^\Omega\big(-g'(0) - c\theta_{[\lambda^*,0,f]}\big) \leqslant \Gamma_{[0,g]},$$

which is impossible either. Finally, if $\hat{u} \equiv 0$ and $\hat{v} > 0$, then $\hat{v} = \theta_{[\mu,0,g]}$ and

$$\lambda^* = \lambda_1^\Omega\big(-f'(0) - b\theta_{[\mu,0,g]}\big),$$

which contradicts (6.6). Therefore, $\hat{u} > 0$, $\hat{v} > 0$ and

$$\Lambda = (-\infty, \lambda^*].$$

This completes the proof. □

**Theorem 6.2.** *The following assertions are true*:

(i) *Assume $\lambda < \Gamma_{[0,f]}$. Then, either $M = (-\infty, \mu_\lambda)$ or $M = (-\infty, \mu^*]$ for some $\mu^* \geqslant \mu_\lambda$, where $\Gamma_{[0,g]} < \mu_\lambda < \lambda_1^\Omega(0)$ is the unique value of $\mu$ satisfying $\lambda = \lambda_1^\Omega(-b\theta_{[\mu,0,g]}) - f'(0)$.*

(ii) *Assume $\mu < \Gamma_{[0,g]}$. Then, either $\Lambda = (-\infty, \lambda_\mu)$ or $\Lambda = (-\infty, \lambda^*]$ for some $\lambda^* \geqslant \lambda_\mu$, where $\Gamma_{[0,f]} < \lambda_\mu < \lambda_1^\Omega(0)$ is the unique value of $\lambda$ satisfying $\mu = \lambda_1^\Omega(-c\theta_{[\lambda,0,f]}) - g'(0)$.*

**Proof.** We only prove the first case. The second case follows by symmetry. Assume $\lambda < \Gamma_{[0,f]}$. By Theorem 5.5,

$$(-\infty, \mu_\lambda) \subset M. \tag{6.9}$$

Now, suppose that (1.2) possesses a coexistence state $(u_0, v_0)$ for some $\mu_0 > \mu_\lambda$. Then, we easily know from Remark 5.2 that $\mu_0 < \lambda_1^\Omega(0)$. We now show that (1.2) possesses a coexistence state for each $\mu \in (\mu_\lambda, \mu_0]$. Assume that

$$\mu_\lambda < \mu \leqslant \mu_0.$$

Then,

$$\lambda_1^\Omega(0) > \lambda > \lambda_1^\Omega\big(-f'(0) - b\theta_{[\mu,0,g]}\big),$$

and hence,

$$\theta_{[\lambda,-b\theta_{[\mu,0,g]},f]} > 0.$$

(Note that $\lambda_1^\Omega(-f'(0) - b\theta_{[\mu,0,g]}) = \Gamma_{[-b\theta_{[\mu,0,g]},f]}$.) Moreover, since $\lambda < \Gamma_{[0,f]}$, we have $\mu_\lambda > \Gamma_{[0,g]}$ and hence, for each $\mu \in (\mu_\lambda, \mu_0]$ we find $\theta_{[\mu,0,g]} > 0$. Now, observe that the couple $(\theta_{[\lambda,-b\theta_{[\mu,0,g]},f]}, \theta_{[\mu,0,g]})$ provides us with a subsolution of (1.2), and that, thanks to Lemma 2.4, for any coexistence state $(u, v)$ of (1.2) we have

$$\left(\theta_{[\lambda,-b\theta_{[\mu,0,g]},f]}, \theta_{[\mu,0,g]}\right) < (u, v).$$

In particular,

$$\left(\theta_{[\lambda,-b\theta_{[\mu_0,0,g]},f]}, \theta_{[\mu_0,0,g]}\right) < (u_0, v_0).$$

Thus, thanks again to Lemma 2.4, for each $\mu \in (\mu_\lambda, \mu_0)$ we find that

$$\left(\theta_{[\lambda,-b\theta_{[\mu,0,g]},f]}, \theta_{[\mu,0,g]}\right) < \left(\theta_{[\lambda,-b\theta_{[\mu_0,0,g]},f]}, \theta_{[\mu_0,0,g]}\right) < (u_0, v_0)$$

and therefore, it follows from [5, Theorem 8.7] that (1.2) possesses a coexistence state for each $\mu \in (\mu_\lambda, \mu_0]$.

To complete the proof it suffices to show that (1.2) possesses a coexistence state for $\mu = \mu_\lambda$. We can show this fact by arguments similar to those in the proof of the fact that (1.2) possesses a coexistence state for $\lambda = \lambda_1^\Omega(-f'(0) - b\theta_{[\mu,0,g]})$ in Theorem 6.1. We omit the details here. Thus

$$(-\infty, \mu_0] \subset M. \tag{6.10}$$

Let $\mu^*$ denote the supremum of the set of $\mu_0 > \mu_\lambda$ for which (1.2) possesses a coexistence state for each $\mu \in (-\infty, \mu_0]$. By Remark 5.2, $\mu^* < \lambda_1^\Omega(0)$. Moreover, $\mu^* > \mu_\lambda$ and due to the existence of a priori bounds, there exists a sequence of positive solutions of (1.2), say $(\mu_n, u_n, v_n)$, $n \geqslant 1$, such that

$$\lim_{n\to\infty} (\mu_n, u_n, v_n) = (\mu^*, u^*, v^*),$$

for some non-negative solution $(u^*, v^*)$ of (1.2) with $\mu = \mu^*$. The same argument as in the proof of Theorem 6.1 shows that $u^* > 0$ and $v^* > 0$. Therefore,

$$M = (-\infty, \mu^*].$$

This completes the proof. □

Now we obtain the following multiplicity result.

**Theorem 6.3.** *The following assertions are true*:

(i) *Assume* $\lambda < \Gamma_{[0,f]}$ *and* $M = (-\infty, \mu^*]$ *with* $\mu^* > \mu_\lambda$. *Then* (1.2) *possesses at least two coexistence states for each* $\mu \in (\mu_\lambda, \mu^*)$.
(ii) *Assume* $\mu < \Gamma_{[0,g]}$ *and* $\Lambda = (-\infty, \lambda^*]$ *with* $\lambda^* > \lambda_\mu$. *Then* (1.2) *possesses at least two coexistence states for each* $\lambda \in (\lambda_\mu, \lambda^*)$.

**Proof.** To prove this result we use the fixed point index in cones. It suffices to prove Theorem 6.3(i), since (ii) follows by symmetry. The proof of Theorem 6.2, under the assumptions of Theorem 6.3, [5, Theorem 8.7] guarantees the existence of a minimal coexistence state, which will be denoted by $(u_\mu, v_\mu)$. Suppose not, there are a sequence of coexistence states which bifurcate from (0,0) or some of the semitrivial positive solutions. This is impossible by the proof of Theorem 6.2. We now show that (1.2) fits into the abstract setting of [1]. Fix $\alpha < \mu_\lambda$, $\beta > 0$ and consider $I := [\alpha, \mu^* + \beta]$. Since we have uniform a priori bounds for the non-negative solutions of (1.2), there exists $K > 0$ such that

$$-\frac{f(u)}{u} - bv < \lambda + K, \qquad -\frac{g(v)}{v} - cu < \mu + K,$$

for each $\mu \in I$ and any non-negative solution $(u, v)$ of (1.2). Let $e$ denote the unique solution of

$$-\Delta e + Ke = 1 \quad \text{in } \Omega, \qquad e = 0 \quad \text{on } \partial\Omega.$$

We have $e(x) > 0$ for each $x \in \Omega$ and $\partial_n e(x) < 0$ for each $x \in \partial\Omega$, where $n$ stands for the outward unit normal vector on $\partial\Omega$. Let $C_e(\bar{\Omega})$ denote the ordered Banach space consisting of all functions $u \in C(\bar{\Omega})$ for which there exists a positive constant $\kappa > 0$ such that $-\kappa e \leqslant u \leqslant \kappa e$, endowed with the norm

$$\|u\|_e := \inf\{\kappa > 0\colon -\kappa e \leqslant u \leqslant \kappa e\}$$

and ordered by its cone of positive functions, $P$. Then, the operators

$$\mathcal{K}_\mu : C_e(\bar{\Omega}) \times C_e(\bar{\Omega}) \to C_e(\bar{\Omega}) \times C_e(\bar{\Omega})$$

defined by

$$\mathcal{K}_\mu(u, v) = \begin{pmatrix} (-\Delta + K)^{-1}[(\lambda + K)u + f(u) + buv] \\ (-\Delta + K)^{-1}[(\mu + K)v + g(v) + cuv] \end{pmatrix}$$

for each $\mu \in I$, are compact and strongly order preserving. Moreover, the solutions of (1.2) are the fixed points of $\mathcal{K}_\mu$. Let $B_e$ denote the unit ball of $C_e(\bar{\Omega}) \times C_e(\bar{\Omega})$ and, for each $\rho > 0$, $P_\rho$ the positive part of $\rho B_e$. Since by Lemma 5.1 we have uniform a priori bounds for the non-negative solutions of (1.2), the fixed point index of $\mathcal{K}_\mu$ in $P_\rho$ makes sense for sufficiently large $\rho$. Moreover, we have the following result.

**Lemma 6.4.** *Assume $\mu \in (\mu_\lambda, \mu^* + \beta]$. Then $(0,0)$ and $(0, \theta_{[\mu,0,g]})$ are isolated fixed points of $\mathcal{K}_\mu$ in $P^2$ and*

$$i\big(\mathcal{K}_\mu, (0,0)\big) = i\big(\mathcal{K}_\mu, (0, \theta_{[\mu,0,g]})\big) = 0. \tag{6.11}$$

*Moreover,*

$$i(\mathcal{K}_\mu, P_\rho) = 0, \tag{6.12}$$

*provided that $\rho$ is sufficiently large.*

**Proof.** Since $\mu > \mu_\lambda$, $(0, \theta_{[\mu,0,g]})$ is linearly unstable by Proposition 3.1 in Section 3, and so $i(\mathcal{K}_\mu, (0, \theta_{[\mu,0,g]})) = 0$ (see [7]). On the other hand, it follows from [1, Lemma 13.1(ii)] that $i(\mathcal{K}_\mu, (0,0)) = 0$ and therefore (6.11) holds. Relation (6.12) follows by homotopy invariance, taking into account that (0,0) and $(0, \theta_{[\mu,0,g]})$ are the only non-negative solutions of (1.2) for $\mu \in (\mu^*, \mu^* + \beta]$. This completes the proof. □

To complete the proof of Theorem 6.3, we need to compute the fixed point index of the minimal solution $(u_\mu, v_\mu)$ of (1.2). The proof is similar to that of [5, Theorem 8.10]. We are only to sketch it. Thanks to [1, Proposition 20.4], $(u_\mu, v_\mu)$ is weakly stable and so

$$\lambda_1^\Omega(\mathcal{L}_\mu) \geqslant 0,$$

where

$$\mathcal{L}_\mu := \begin{pmatrix} -\Delta & 0 \\ 0 & -\Delta \end{pmatrix} - A_\mu \tag{6.13}$$

and

$$A_\mu = \begin{pmatrix} \lambda + f'(u_\mu) + bv_\mu & bu_\mu \\ cv_\mu & \mu + g'(v_\mu) + cu_\mu \end{pmatrix}. \tag{6.14}$$

If $\lambda_1^\Omega(\mathcal{L}_\mu) > 0$, the same argument as in the proof of [5, Theorem 8.10] completes the proof of Theorem 6.3.

If $\lambda_1^\Omega(\mathcal{L}_\mu) = 0$, it follows from of [5, Lemma 8.13] that there exists $\varepsilon > 0$ and a differentiable mapping $(\mu, u, v) : (-\varepsilon, \varepsilon) \to \mathbb{R} \times P^2$ which is strictly increasing in $s$ such that $(\mu(0), u(0), v(0)) = (\mu, u_\mu, v_\mu)$ and for each $s \in (-\varepsilon, \varepsilon)$, $(\mu(s), u(s), v(s))$ is a coexistence state of (1.2). Moreover,

$$\operatorname{sgn} \mu'(s) = \operatorname{sgn} \lambda_1^\Omega(\mathcal{L}_s),$$

where

$$\mathcal{L}_s = \begin{pmatrix} -\Delta & 0 \\ 0 & -\Delta \end{pmatrix} - A_s,$$

where

$$A_s = \begin{pmatrix} \lambda + f'(u(s)) + bv(s) & bu(s) \\ cv(s) & \mu(s) + g'(v(s)) + cu(s) \end{pmatrix}.$$

Arguing as in the proof of [5, Theorem 8.10], we find that

$$\mu(s) < \mu \quad \forall s \in (-\varepsilon, 0).$$

Thus, two different situations may occur for $s \in (0, \varepsilon)$:

**Case (a).** If $\mu(s) < \mu$ for all $s \in (0, \varepsilon)$, then the same argument as in the proof of [5, Theorem 8.10] applies to complete the proof of this case.

**Case (b).** If $\mu(s) > \mu$ for all $s \in (0, \varepsilon)$, then there exists $s_1 > 0$ such that $\mu'(s_1) > 0$ and hence,

$$i\big(\mathcal{K}_{\mu(s_1)}, \big(u(s_1), v(s_1)\big)\big) = 1.$$

Now, setting

$$\rho_1 := \big\|\big(u(s_1), v(s_1)\big)\big\|_e - \delta, \qquad \rho_2 := \big\|(u_\mu, v_\mu)\big\|_e - \delta,$$

we know that (1.2) does not admit a coexistence state in

$$\big[\mu(s_1), \mu(s_1)+\delta\big] \times \partial\big(P_{\rho_1}\backslash\overline{P_{\rho_2}}\big).$$

Moreover, by the uniqueness of [5, Lemma 8.13(ii)], $\delta > 0$ can be chosen so that (1.2) does not have a coexistence state in $P_{\rho_1}\backslash\overline{P_{\rho_2}}$ for $\mu = \mu(s_1)+\delta$ either. Thus, the homotopy invariance implies

$$i\big(\mathcal{K}_{\mu(s_1)}, P_{\rho_1}\backslash\overline{P_{\rho_2}}\big) = 0.$$

Now, for $\delta > 0$ sufficiently small, setting

$$\rho := \big\|\big(u(s_1), v(s_1)\big)\big\|_e + \delta$$

we also know that

$$i\big(\mathcal{K}_{\mu(s_1)}, P_{\rho}\backslash\overline{P_{\rho_2}}\big) = 1.$$

Since the monotonicity of $(u(s), v(s))$ and the uniqueness given by [5, Lemma 8.13(ii)] imply that (1.2) does not admit a coexistence state on

$$\big[\mu, \mu(s_1)\big] \times \partial\big(P_{\rho}\backslash\overline{P_{\rho_2}}\big),$$

then

$$i\big(\mathcal{K}_{\mu}, (u_\mu, v_\mu)\big) = 1.$$

Therefore, our conclusion is obtained by using Lemma 6.4. This completes the proof of Theorem 6.3. □

**Theorem 6.5.** *The following assertions are true*:

(i) *Assume $\Gamma_{[0,g]} < \mu < \lambda_1^{\Omega}(0)$ and $\Lambda = (-\infty, \lambda^*]$ with $\lambda_1^{\Omega}(-b\theta_{[\mu,0,g]}) - f'(0) < \lambda^* < \Gamma_{[0,f]}$. Then* (1.2) *possesses at least two coexistence states for each $\lambda \in (\lambda_1^{\Omega}(-b\theta_{[\mu,0,g]}) - f'(0), \lambda^*)$.*

(ii) *Assume $\Gamma_{[0,f]} < \lambda < \lambda_1^{\Omega}(0)$ and $M = (-\infty, \mu^*]$ with $\lambda_1^{\Omega}(-c\theta_{[\lambda,0,f]}) - g'(0) < \mu^* < \Gamma_{[0,g]}$. Then* (1.2) *possesses at least two coexistence states for each $\mu \in (\lambda_1^{\Omega}(-c\theta_{[\lambda,0,f]}), \mu^*) - g'(0)$.*

**Proof.** The proof of this theorem is similar to that of Theorem 6.3. Define $\mathcal{K}_\lambda$ similar to $\mathcal{K}_\mu$ as above. For case (i), we need the condition $\lambda^* < \Gamma_{[0,f]}$ to guarantee that $(0,0)$ and $(0, \theta_{[\mu,0,g]})$ are the only non-negative solutions of (1.2) for $\lambda \in (\lambda^*, \Gamma_{[0,f]})$. Moreover, $(0, \theta_{[\mu,0,g]})$ is linearly unstable for $\lambda_1^{\Omega}(-b\theta_{[\mu,0,f]}) - f'(0) < \lambda < \lambda^*$ by Proposition 3.1. Thus, a similar lemma to Lemma 6.4 can be obtained. The rest of the proof of Theorem 6.5 is a little variant of the proof of [5, Theorem 8.14]. □

**Remark 6.6.** The assumption $\lambda^* < \Gamma_{[0,f]}$ in Theorem 6.5 is reasonable since we can see that there exists $\tilde{\mu} < \lambda_1^{\Omega}(0)$, which depends upon $\Gamma_{[0,f]}$, such that this assumption holds for $\mu > \tilde{\mu}$. Indeed, it follows from the proof of Theorem 5.3(i) that if (1.2) possesses a coexistence state, then $\max_{\Omega} \theta_{[\mu,0,g]} < (\lambda_1^{\Omega}(0) - \lambda)/c$. Choosing $\tilde{\mu}$ such that

$\max_\Omega \theta_{[\tilde{\mu},0,g]} = (\lambda_1^\Omega(0) - \Gamma_{[0,f]})/c$ (the existence of such $\tilde{\mu}$ can be known from Theorem 2.3), we can show that $\lambda^* < \Gamma_{[0,f]}$ provided $\mu > \tilde{\mu}$. On the contrary, there is $(\lambda, \mu)$ with $\lambda \geqslant \Gamma_{[0,f]}$, $\mu > \tilde{\mu}$ such that (1.2) has a coexistence state $(u, v)$. Then

$$\max_\Omega v > \max_\Omega \theta_{[\tilde{\mu},0,g]} = \big(\lambda_1^\Omega(0) - \Gamma_{[0,f]}\big)/c.$$

On the other hand,

$$\max_\Omega v < \big(\lambda_1^\Omega(0) - \lambda\big)/c \leqslant \big(\lambda_1^\Omega(0) - \Gamma_{[0,f]}\big)/c.$$

This is impossible. Similarly, the assumption $\mu^* < \Gamma_{[0,g]}$ in Theorem 6.5 is also reasonable.

## Acknowledgment

The author thanks one of the referees for his valuable suggestions.

## References

[1] H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, SIAM Rev. 18 (1976) 620–709.
[2] H. Amann, J.L. Gómez, A priori bounds and multiple solutions for superlinear indefinite elliptic problems, J. Differential Equations 146 (1998) 336–374.
[3] A. Cañada, J.L. Gámez, Positive solutions of nonlinear elliptic systems, Math. Models Methods Appl. Sci. 3 (1993) 823–837.
[4] M.G. Crandall, P.H. Rabinowitz, Bifurcation from simple eigenvalues, J. Funct. Anal. 8 (1971) 321–340.
[5] M. Delgado, J.L. Gómez, A. Suárez, On the symbiotic Lotka–Volterra model with diffusion and transport effects, J. Differential Equations 160 (2000) 175–262.
[6] J.L. Gómez, Nonlinear eigenvalues and global bifurcation: Application to the search of positive solutions for general Lotka–Volterra reaction–diffusion systems with two species, Differential Integral Equations 7 (1994) 1427–1452.
[7] J.L. Gómez, Positive periodic solutions of Lotka–Volterra reaction–diffusion systems, Differential Integral Equations 5 (1992) 55–72.
[8] J.L. Gómez, M.M. Meyer, The maximum principle for cooperative weakly coupled elliptic systems and some applications, Differential Integral Equations 7 (1994) 383–398.
[9] J.L. Gómez, R. Pardo, Coexistence regions in Lotka–Volterra models with diffusion, Nonlinear Anal. 19 (1992) 11–28.
[10] J.L. Gómez, J.C. Sabina, Coexistence states and global attractivity for some convective diffusive competing species models, Trans. Amer. Math. Soc. 347 (1995) 3797–3833.
[11] P. Korman, Dynamics of the Lotka–Volterra systems with diffusion, Appl. Anal. 44 (1992) 191–207.
[12] P. Korman, A. Leung, On the existence and uniqueness of positive steady-states in the Volterra–Lotka ecological models with diffusion, Appl. Anal. 26 (1987) 145–160.
[13] A. Leung, A study of three species prey–predator reaction–diffusions by monotone schemes, J. Math. Anal. Appl. 100 (1984) 583–604.
[14] L. Li, A. Ghoreishi, On positive solutions of general nonlinear elliptic symbiotic interacting systems, Appl. Anal. 40 (1991) 281–295.
[15] L. Li, Y. Liu, Spectral and nonlinear effects in certain elliptic systems of three variables, SIAM J. Math. Anal. 24 (1993) 480–498.
[16] P.L. Lions, On the existence of positive solutions of semilinear elliptic equations, SIAM Rev. 24 (1982) 441–467.
[17] Y. Lou, Necessary and sufficient condition for the existence of positive solutions of certain cooperative system, Nonlinear Anal. 26 (1996) 1079–1095.