Existence and asymptotic behavior of blow-up solutions to weighted quasilinear equations

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Abstract
Given a bounded domain Ω we consider local weak blow-up solutions to the equation Δ_p u = g(x) f(u) on Ω. The non-linearity f is a non-negative non-decreasing function and the weight g is a non-negative continuous function on Ω which is allowed to be unbounded on Ω. We show that if Δ_p w = −g(x) in the weak sense for some w ∈ W^{1,p}_0(Ω) and f satisfies a generalized Keller–Osserman condition, then the equation Δ_p u = g(x) f(u) admits a non-negative local weak solution u ∈ W^{1,p}_loc(Ω) ∩ C(Ω) such that u(x) → ∞ as x → ∂Ω. Asymptotic boundary estimates of such blow-up solutions will also be investigated.

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1. Introduction

Let Ω ⊆ ℝ^n be a bounded domain, and 1 < p < ∞. In this paper, we will be concerned with local weak solutions to quasilinear equations of the form

Δ_p u = H(x, u), \quad x ∈ Ω. \hspace{1cm} (1.1)

Here Δ_p stands for the p-Laplacian

Δ_p w := \text{div}(|∇ w|^{p-2} ∇ w),

and H : Ω × ℝ → ℝ is a continuous function to be specified later.

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We say that \( u \in W^{1,p}(\Omega) \) is a weak solution to Eq. (1.1) in the domain \( \Omega \) if and only if
\[
\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx = - \int_{\Omega} H(x,u) \varphi \, dx
\]
for all \( \varphi \in W^{1,p}_0(\Omega) \).

A function \( u \in W^{1,p}_{\text{loc}}(\Omega) \) is said to be a local weak solution to Eq. (1.1) in the domain \( \Omega \) if and only if it is a weak solution of (1.1) on \( D \) for every sub-domain \( D \subset \Omega \).

A local weak solution \( u \) of (1.1) is said to be a (local weak) blow-up solution if
\[
u(x) \to \infty \quad \text{as} \quad d(x,\partial \Omega) \to 0.
\]

Our purpose in this paper is to investigate blow-up solutions of Eq. (1.1) with \( H(x,t) = g(x)f(t) \). More specifically, we will consider solutions \( u \in W^{1,p}_{\text{loc}}(\Omega) \cap C(\Omega) \) to the problem
\[
\begin{cases}
\Delta_p u = g(x)f(u) & \text{for } x \in \Omega, \\
u(x) \to \infty & \text{as } d(x,\partial \Omega) \to 0.
\end{cases}
\]

Throughout this paper we will always assume that \( g \) is a non-negative function and that the non-linearity \( f \) satisfies
\[
f : [0,\infty) \to [0,\infty) \text{ is a non-decreasing } C^1 \text{ function such that } f(0) = 0, \text{ and } f(s) > 0 \text{ for } s > 0.
\]

The following growth condition on \( f \) at infinity, first introduced by Keller [11] and Osserman [17], is crucial in the investigation of existence of blow-up solutions:
\[
\int_1^\infty \frac{1}{(F(t))^{1/p}} \, dt < \infty, \quad \text{where } F(t) := \int_0^t f(s) \, ds.
\]

We will refer to the condition (1.4) as the generalized Keller–Osserman, or simply the Keller–Osserman condition.

In the papers [11,17] the condition (1.4) was shown to be necessary and sufficient for the equation \( \Delta u = f(u) \) to admit a blow-up solution on a bounded domain \( \Omega \). The investigation in these papers led to several papers where important contributions were made to the question of existence, uniqueness, asymptotic boundary behavior, symmetry and convexity of blow-up solutions. We refer to the papers [1–3,5,12,13,16] and references therein for such results.

When \( p = 2 \) and \( g \in C(\bar{\Omega}) \) is positive, the question of existence as well as asymptotic boundary estimates were investigated in [1,3]. In [12], Lair investigated the existence of blow-up solutions to (1.3) (when \( p = 2 \)) under the following assumption on \( g \in C(\bar{\Omega}) \).

For any \( x_0 \in \Omega \) satisfying \( g(x_0) = 0 \), there exists a sub-domain
\[
\mathcal{O} \subset \Omega \text{ containing } x_0 \text{ such that } g(x) > 0 \text{ for all } x \in \partial \mathcal{O}.
\]

In [12] it was shown that if \( g \in C(\bar{\Omega}) \) satisfies condition (1.5) above, then the Keller–Osserman condition on \( f \) remains a necessary and sufficient condition for (1.3) to admit
a blow-up solution. However, when $g$ is allowed to be unbounded near the boundary $\partial \Omega$, the situation is drastically different. In this case the rate of growth of the weight $g$ near the boundary of the domain $\Omega$ plays a decisive role in the question of whether (1.3) has a blow-up solution or not. When $\Omega$ is a ball, and $g$ is a non-decreasing, non-negative radial weight, this was investigated in the papers [4], when $p = 2$, and [18,19] for any $1 < p < \infty$. Loosely speaking, in these papers it was shown that the growth rate of the weight $g$ near the boundary of $\Omega$ and the growth of the non-linearity $f$ at infinity must be inversely related for the problem (1.3) to admit a blow-up solution.

Our objective in this paper is to investigate existence of blow-up solutions for (1.3) when the weight is not necessarily radial. In particular $g$ will be allowed, as in [12], to vanish on large portion of $\Omega$, but unlike in [12], $g$ may be unbounded on $\Omega$.

We will need the following comparison principle for weak solutions to quasi-linear equations, which is due to Tolksdorf (see [21] for a proof). We state it in a slightly more general form that would include our equations. The proof given in [21] extends trivially to cover the following

**Theorem 1.1** (Weak comparison principle). Let $G : \Omega \times \mathbb{R} \to \mathbb{R}$ be continuous and further assume that it is non-increasing in the second variable. Let $u, w \in W^{1,p}(\Omega)$ satisfy the respective inequalities
\[
\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \leq \int_{\Omega} G(x,u) \varphi \quad \text{and} \quad \int_{\Omega} |\nabla w|^{p-2} \nabla w \cdot \nabla \varphi \geq \int_{\Omega} G(x,w) \varphi
\]
for all non-negative $\varphi \in W^{1,p}_0(\Omega)$. Then the inequality $u \leq w$ on $\partial \Omega$ implies $u \leq w$ in $\Omega$.

The other fact we need is an interior regularity result for weak solutions to quasi-linear equations. It is due to DiBenedetto [7] and Tolksdorf [20].

**Theorem 1.2** (DiBenedetto–Tolksdorf $C^{1,\alpha}$ interior regularity). Suppose $h(x,t)$ is measurable in $x \in \Omega$ and continuous in $t \in \mathbb{R}$ such that $|h(x,t)| \leq \Gamma$ on $\Omega \times \mathbb{R}$. Let $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ be a weak solution of $\Delta_p u = h(x,u)$. Given a sub-domain $D \Subset \Omega$, there is $\alpha > 0$ and a positive constant $C$, depending on $n$, $p$, $\Gamma$, $\|u\|_{\infty}$ and $D$ such that
\[
|\nabla u(x)| \leq C \quad \text{and} \quad |\nabla u(x) - \nabla u(y)| \leq C|x - y|^{\alpha}, \quad x, y \in D. \quad (1.6)
\]

The paper is organized as follows. In Section 2 we give a sufficient condition on the weight $g$ for problem (1.3) to admit a local weak blow-up solution. In Section 3 some a priori asymptotic boundary estimates for blow-up solutions are given. In Section 4 we give a class of weights $g$ for which (1.3) admits a local weak blow-up solution and a class of weights $g$ for which (1.3) fails to admit a blow-up solution in $W^{1,p}(\Omega)$.

2. Existence of blow-up solutions

In this section we will make use of the DiBenedetto–Tolksdorf regularity theorem to show that the estimates (1.6) hold for bounded solutions of (1.2) as follows. Suppose
$H(x,t)$ is measurable in $x$ and continuous in $t$ such that $H(x,\cdot)$ is non-decreasing for each $x \in \Omega$. Furthermore, assume that $H(x,t) \leqslant \Gamma(t)$ for all $x \in \Omega$. If $u \in W^{1,p}(\Omega)$ is a weak solution of (1.2) such that $\|u\|_{\infty} \leqslant N$ on $\Omega$, then there is $\alpha > 0$ and there is a constant $C$, depending on $n$, $p$, $\Gamma(N)$ and $N$ such that (1.6) holds. This is the case because $u$ is a solution of $\Delta_p u = H_N(x,u)$ in $\Omega$, where

$$H_N(x,t) = \begin{cases} H(x,t) & \text{for } (x,t) \in \Omega \times [0,N], \\ H(x,N) & \text{for } (x,t) \in \Omega \times (N,\infty). \end{cases}$$

We start with the following Lemma that extends a result of Lair (see Theorem 1 of [12]) to the $p$-Laplacian case.

**Lemma 2.1.** Let $D \subseteq \mathbb{R}^n$ be a bounded domain, and suppose $g \in C(\bar{D})$ satisfies (1.5) on $D$. Let $f$ satisfy the Keller–Osserman condition. Then the problem

$$\begin{cases} \Delta_p u = g(x)f(u) & \text{for } x \in D, \\ u(x) \to \infty & \text{as } d(x,\partial D) \to 0, \end{cases}$$

admits a non-negative solution $u \in W^{1,p}_{\loc}(D) \cap C^{1,\alpha}(D)$, $0 < \alpha < 1$.

**Proof.** For each $k = 1, 2, \ldots$, let $u_k \in W^{1,p}(D)$ be a weak solution of

$$\begin{cases} \Delta_p u = g(x)f(u), & x \in D, \\ u(x) = k, & x \in \partial D. \end{cases}$$

(See [5, Theorem 4.2].) Notice that $u \equiv 0$ is a solution of the above Dirichlet problem with $k = 0$. By the comparison principle we see that

$$0 \leqslant u_k(x) \leqslant u_{k+1}(x), \quad x \in D,$$

for all $k = 1, 2, \ldots$. By proceeding as in [12] we now show that $\{u_k\}$ is uniformly bounded on sub-domains that are compactly contained in $D$. Thus let $\mathcal{U} \subseteq D$ be a sub-domain and take $x_0 \in \mathcal{U}$. If $g(x_0) > 0$ then there is a ball $B$ containing $x_0$ such that $g > 0$ on $2B$. Let $m > 0$ be the minimum of $g$ on $2B$, and let $w$ be a blow-up solution of $\Delta_p u = m f(u)$, $u = \infty$ on $\partial (2B)$. The existence of such a blow-up solution follows from [8,15,16]. Again by the comparison principle we conclude that $u_k \leqslant w$ on $2B$. Since $w$ is locally bounded we see that $u_k \leqslant C$ on $B$ for all $k = 1, 2, \ldots$, and some $C > 0$. Suppose now $g(x_0) = 0$. Then by condition (1.5) there is a sub-domain $O \subseteq D$ such that $g(x) > 0$ for all $x \in \partial O$. Then arguing as in [12] we conclude that $u_k \leqslant C$ on $\partial O$ for some $C$ and all $k = 1, 2, \ldots$. Again by the comparison principle it follows that $u_k \leqslant C$ on $\partial O$ for all $k = 1, 2, \ldots$. Thus in any case we see that given $x_0 \in \mathcal{U}$ there is a ball $B \subseteq \mathcal{U}$ containing $x_0$ and a positive constant $C_B$ such that $0 \leqslant u_k \leqslant C_B$ on $B$ for all $k = 1, \ldots$. By covering $\mathcal{U}$ by such balls we see that $\{u_k\}$ is indeed uniformly bounded on $\mathcal{U}$.

The DiBenedetto–Tolksdorf $C^{1,\alpha}$ interior regularity result implies that the sequences $\{u_k\}$ and $\{\nabla u_k\}$ are equicontinuous in subdomains compactly contained in $\Omega$, and hence we can choose a subsequence, still denoted by $\{u_k\}$, such that $u_k \to u$ and $\nabla u_k \to \nabla u$ uniformly on compact subsets of $D$ for some $u \in C(D)$ and $v \in (C(D))^n$. In fact, $v = \nabla u$ on $D$, and from the interior $C^{1,\alpha}$ estimate (1.6) we conclude that $\nabla u \in C^{\alpha}(D)$ for some $0 < \alpha < 1$. Thus $u \in W^{1,p}_{\loc}(D) \cap C^{1,\alpha}(D)$. Let $\mathcal{U} \subseteq D$ and $\varphi \in W^{1,p}_{0}(\mathcal{U})$. From (1.6) again
we see that $|\nabla u_k|^{p-1} \nabla \varphi | \leq C |\nabla \varphi |$ on $\mathcal{U}$ and since the function $\xi \to |\xi|^{p-2} \xi$ is continuous on $\mathbb{R}^n$, it follows that $|\nabla u_k(x)|^{p-2} \nabla u_k(x) \cdot \nabla \varphi(x) \to |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla \varphi(x)$ for $x \in \mathcal{U}$. Thus by the dominated convergence theorem we see that

$$\int_{\mathcal{U}} |\nabla u_k(x)|^{p-2} \nabla u_k(x) \cdot \nabla \varphi(x) \to \int_{\mathcal{U}} |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla \varphi(x).$$

Furthermore, since $0 \leq f(u_k) \leq f(u_{k+1})$, and $f(u_k(x)) \to f(u(x))$ for each $x \in \mathcal{U}$, by the monotone convergence theorem we obtain

$$\int_{\mathcal{U}} g f(u_k) \varphi \to \int_{\mathcal{U}} g f(u) \varphi.$$

Therefore it follows that

$$\int_{\mathcal{U}} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi = -\int_{\mathcal{U}} g f(u) \varphi, \quad \varphi \in W^{1,p}_0(\mathcal{U}),$$

and hence $u$ is a local weak solution of $\Delta_p u = g f(u)$ on $D$. Since $u_k = k$ on $\partial D$ we see that $u(x) \to \infty$ as $x \to \partial D$. □

Let us now introduce the following notation. Assume that $f$ satisfies the Keller–Osserman condition (1.4). Then it is known (see Lemma 2.1 of [10]) that

$$\lim_{t \to \infty} \frac{(F(t))^{(p-1)/p}}{f(t)} = 0. \quad (2.2)$$

Thus it follows that for $t > 0,$

$$\int_{t}^{\infty} \frac{1}{f(s)^{1/(p-1)}} ds < \infty.$$

In view of this we define $\gamma : (0, \infty) \to (0, \gamma(0+))$ to be the decreasing function given by

$$\gamma(t) := \int_{t}^{\infty} \frac{1}{f(s)^{1/(p-1)}} ds.$$

We will need the following condition on $g \in C(\Omega)$, which we will refer to as the $G$-condition.

There is a sequence $\{D_k\}$ of domains such that

1. $D_k \subseteq D_{k+1}$, $k = 1, 2, \ldots$.
2. $\Omega = \bigcup_{k=1}^{\infty} D_k$.
3. $g$ satisfies condition (1.5) on each $D_k$.

This is the case for instance when $g$ is positive on $\Omega \setminus \overline{\mathcal{U}}$ for some sub-domain $\mathcal{U} \subseteq D$. 
In showing the existence of local weak blow-up solutions of (1.3) the solvability of the following Dirichlet problem will be useful:

\[
\begin{cases}
\Delta_p w = -g(x), & x \in \Omega, \\
w(x) = 0, & x \in \partial \Omega.
\end{cases}
\]  

(2.3)

More precisely, we have the following

**Theorem 2.2.** Suppose \( f \) satisfies the Keller–Osserman condition, and let \( g \in C(\Omega) \) satisfy the \( \mathcal{G} \)-condition. If the Dirichlet problem (2.3) has a weak solution, then Eq. (1.3) admits a non-negative blow-up solution.

**Proof.** By the \( \mathcal{G} \)-condition we can find domains \( D_j \) with \( \bar{D}_j \subseteq D_{j+1} \subseteq \Omega \) such that \( \bigcup_{j=1}^{\infty} D_j = \Omega \), and \( g \) satisfies condition (1.5) on each \( D_j \). For each \( j \), let \( u_j \) be a local weak blow-up solution of (1.3) with \( D_j \) replacing \( \Omega \). Such a solution exists by the above lemma, since \( g \in C(\bar{D}_j) \) and \( g \) verifies condition (1.5) on \( D_j \). An application of the comparison principle shows that \( u_j \) exists by the above lemma, since \( g \in C(\bar{D}_j) \) and \( g \) verifies condition (1.5) on \( D_j \).

Let \( \epsilon > 0 \) be fixed, and let \( v_j(x) := \gamma(u_j(x) + \epsilon), \ x \in D_j \). Let us first observe that

\[
\langle \nabla v_j \rangle^{p-2} \nabla v_j = \langle \gamma'(u_j + \epsilon) \rangle^{p-2} \gamma'(u_j + \epsilon) \langle \nabla u_j \rangle^{p-2} \nabla u_j
\]

and

\[
\nabla \left( \langle \gamma'(u_j + \epsilon) \rangle^{p-2} \gamma'(u_j + \epsilon) \right) = (p-1) \langle \gamma'(u_j + \epsilon) \rangle^{p-2} \gamma''(u_j + \epsilon) \nabla u_j.
\]

Let \( \varphi \in C_0^\infty(D_j) \) be a non-negative test function. Then

\[
\int_{D_j} \langle \nabla v_j \rangle^{p-2} \nabla v_j \cdot \nabla \varphi = \int_{D_j} \langle \nabla u_j \rangle^{p-2} \nabla u_j \cdot \nabla \left( \langle \gamma'(u_j + \epsilon) \rangle^{p-2} \gamma'(u_j + \epsilon) \varphi \right)
\]

\[
- \int_{D_j} \langle \nabla u_j \rangle^{p-2} \nabla u_j \cdot \nabla \left( \langle \gamma'(u_j + \epsilon) \rangle^{p-2} \gamma'(u_j + \epsilon) \right) \varphi
\]

\[
= - \int_{D_j} g f(u_j) \langle \gamma'(u_j + \epsilon) \rangle^{p-2} \gamma'(u_j + \epsilon) \varphi
\]

\[
- (p-1) \int_{D_j} \langle \nabla u_j \rangle^{p} \langle \gamma'(u_j + \epsilon) \rangle^{p-2} \gamma''(u_j + \epsilon) \varphi.
\]

Noting that

\[
\langle \gamma'(t) \rangle^{p-2} \gamma'(t) = -\frac{1}{f(t)} \quad \text{and} \quad \gamma''(t) = \frac{1}{p-1} \frac{f'(t)}{f(t)^{p/(p-1)}},
\]

we obtain the equation

\[
\int_{D_j} \langle \nabla v_j \rangle^{p-2} \nabla v_j \cdot \nabla \varphi = \int_{D_j} g f(u_j) \langle \gamma'(u_j + \epsilon) \rangle^{p-2} \gamma'(u_j + \epsilon) \varphi - \int_{D_j} \langle \nabla u_j \rangle^{p} \frac{f'(u_j + \epsilon)}{f(u_j + \epsilon)^{p/(p-1)}} \varphi.
\]
Thus it follows that
\[
\int_{D_j} |\nabla v_j|^{p-2} \nabla v_j \cdot \nabla \varphi \leq \int_{D_j} g \varphi, \quad 0 \leq \varphi \in C_0^\infty(D_j).
\]

By density argument we see that the last inequality is still valid for all \(0 \leq \varphi \in W^{1,p}_0(D_j)\). Thus by the comparison principle, we conclude that
\[
v_j(x) \leq w(x) \quad \text{for all} \quad x \in D_j, \quad (2.4)
\]
where \(w\) is a local weak solution to the Dirichlet problem (2.3). Let \(D \Subset \Omega\), and take \(\varphi \in C_0^\infty(D)\). Let \(m \) be chosen such that \(D \subseteq D_m\). Then given \(x \in D_m\) the sequence \([u_j(x)]_{j=m+1}^\infty\) is a monotone non-increasing sequence bounded below by \(\gamma^{-1}(w)\). By the regularity theorem it also follows that \([\nabla u_j]_{j=m+1}^\infty\) is equicontinuous on \(D_k\). Thus by diagonal extraction we obtain a subsequence \([u_j]\) such that \(u_j(x) \rightarrow u(x)\) and \(\nabla u_j(x) \rightarrow \nabla u(x)\) for \(x \in D\). Also, note that for all \(k \geq m+1\) we have the inequalities
\[
|\nabla u_k|^{p-1} |\nabla \varphi| \leq C_m |\nabla \varphi|, \quad f(u_k) \leq f(u_{m+1}) \quad \text{on} \quad D.
\]
These inequalities together with the pointwise convergence
\[
|\nabla u_k|^{p-2} \nabla u_k \cdot \nabla \varphi \rightarrow |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi, \quad f(u_k) \rightarrow f(u) \quad \text{on} \quad D,
\]
allow us to conclude, via the Lebesgue convergence theorem, that
\[
\int_D |\nabla u_k(x)|^{p-2} \nabla u_k(x) \cdot \nabla \varphi(x) \rightarrow \int_D |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla \varphi(x)
\]
and
\[
\int_D gf(u_k) \varphi \rightarrow \int_D gf(u) \varphi.
\]

From this, we conclude that
\[
\int_D |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi = -\int_D gf(u) \varphi, \quad \varphi \in C_0^\infty(D).
\]

By the usual density argument, it follows that the above equation continues to hold for any \(\varphi \in W^{1,p}_0(D)\), thus showing that \(u\) is a local weak solution of \(\Delta_p u = gf(u)\) on \(\Omega\). From (2.4) we conclude, since \(\epsilon > 0\) is arbitrary, that \(\gamma(u) \leq w\) on \(\Omega\). Now let \(U \subseteq \Omega\) be a neighborhood of the boundary \(\partial \Omega\) such that \(0 \leq w \leq \gamma(0+)\). Then \(u(x) \geq \gamma^{-1}(w(x))\) on \(U\) and hence it follows that \(u(x) \rightarrow \infty\) as \(x \rightarrow \partial \Omega\). \(\square\)

**Corollary 2.3.** Suppose that \(g \in C(\Omega)\) for which the Dirichlet problem (2.3) admits a weak solution \(w\). If \(f\) satisfies the Keller–Osserman condition, then for any non-negative blow-up solution \(u\) of (1.3) we have
\[
u(x) \geq \gamma^{-1}(w(x))
\]
for \(x\) near \(\partial \Omega\).
Proof. Let \( u \) be a non-negative blow-up solution of (1.3). As in the proof of the above theorem the function
\[
v(x) = \gamma(u + \epsilon), \quad \epsilon > 0,
\]
satisfies the inequality
\[
\int_{\Omega} |\nabla v|^{p-2} \nabla v \cdot \nabla \varphi \leq \int_{\Omega} g \varphi, \quad 0 \leq \varphi \in W^{1,p}_{0}(\Omega).
\]
Thus, by the comparison principle again we conclude
\[
\gamma(u + \epsilon)(x) \leq w(x), \quad x \in \Omega,
\]
that is
\[
u(x) + \epsilon \geq \gamma^{-1}(w(x)) \quad \text{for } x \text{ near } \partial \Omega.
\]
Since \( \epsilon > 0 \) is arbitrary, we obtain the desired inequality.

3. Asymptotic boundary behavior

In this section we will give an estimate on blow-up solutions of (1.3) near the boundary \( \partial \Omega \) in terms of the distance function to the boundary. The distance of \( x \in \Omega \) to the boundary \( \partial \Omega \) will be denoted by \( \delta(x) \). To avoid ambiguities we will use \( d(x, \partial D) \) for the distance of \( x \in D \) to \( \partial D \) for other domains \( D \). To obtain asymptotic boundary estimates of blow-up solutions on \( \Omega \), we will need some regularity assumptions on the boundary \( \partial \Omega \) of \( \Omega \).

For our first result we consider a bounded domain \( \Omega \) with \( C^2 \) boundary \( \partial \Omega \). Given a bounded domain \( \Omega \) in \( \mathbb{R}^n \) whose boundary \( \partial \Omega \) is of class \( C^2 \), it is known (see [9]) that there is a positive number \( \mu = \mu(\Omega) \), depending only on \( \Omega \), such that the distance function \( \delta(x), x \in \Omega \), belongs to \( C^2(\bar{\Gamma}_\mu) \), where \( \Gamma_\mu: \{ x \in \Omega: \delta(x) < \mu \} \).

It is also known that
\[
|\nabla \delta(x)| = 1, \quad x \in \bar{\Gamma}_\mu.
\]
We use these facts to prove the next result.

**Theorem 3.1.** Suppose \( \partial \Omega \) is of class \( C^2 \) and that \( f \) satisfies the Keller–Osserman condition. Let \( g \in C(\Omega) \) be such that the Dirichlet problem (2.3) has a solution. Suppose \( \sup_{\Omega} g(x) \delta(x)^{(1-\beta)(p-1)+1} \leq N < \infty \) for some \( 0 < \beta < 1 \). Then there is a neighborhood \( \mathcal{O} \) of the boundary \( \partial \Omega \) and a positive constant \( \alpha \), depending only on \( \Omega \) and the weight \( g \), such that for any non-negative blow-up solution \( u \) of (1.3),
\[
u(x) \geq \gamma^{-1}(\alpha \delta^\beta(x)), \quad x \in \mathcal{O}.
\]
Proof. Given $0 < \beta < 1$, let $v(x) = \alpha \delta(x)^\beta$ with $\alpha > 0$ to be determined later. We note that $v \in C^2(\Gamma_\mu)$, and that $|\nabla \delta(x)| = 1$ on $\Gamma_\mu$. Then

$$\nabla v = \alpha \beta \delta(x)^{\beta-1} \nabla \delta(x),$$

and thus

$$|\nabla v|^{p-2} \nabla v = (\alpha \beta)^{p-1} \delta(x)^{(p-1)(\beta-1)} \nabla \delta(x).$$

Let $M := \max\{|\Delta \delta(x)|: x \in \Gamma_\mu\}$. On $\Gamma_\eta = \{x \in \Omega: \delta(x) < \eta\}$, where $0 < \eta \leq \mu$, we make the following estimations:

$$- \text{div}(|\nabla v|^{p-2} \nabla v) - g(x) = \delta(x)^{(p-1)(\beta-1)-1}[\alpha \beta]^{p-1}((p-1)(1-\beta) - \delta(x) \Delta \delta(x)) - g(x) \delta(x)^{(p-1)(1-\beta)+1} \geq \delta(x)^{(p-1)(\beta-1)-1}[\alpha \beta]^{p-1}((p-1)(1-\beta) - \eta M) - N].$$

We now pick $\eta > 0$ small enough such that $(p-1)(1-\beta) - \eta M > 0$, and then we choose $\alpha > 0$ big enough such that $\alpha \eta \beta \geq w$ on the set $\{x \in \Omega: \delta(x) = \eta\}$ so that $v \geq w$ on the boundary $\partial \Gamma_\eta$. Then by the weak comparison principle we conclude that $w \leq v$ on $\Gamma_\eta$. Thus we have $u \geq \gamma^{-1}(v)$ on $\Gamma_\eta$. From this follows the conclusion of the theorem. \hfill \Box

Suppose that $f$ satisfies the Keller–Osserman condition, and let

$$\psi(t) = \int_t^\infty \frac{1}{(qF(s))^{1/p}} \, ds, \quad t > 0, \quad (3.1)$$

Also let $\phi$ be the inverse of the decreasing function $\psi$ given above. For subsequent asymptotic boundary estimates we will need the following condition, introduced in [3]. We will refer to the condition as the Bandle–Marcus condition on $f$. Let the function $\psi$ defined in (3.1) satisfy

$$\liminf_{t \to \infty} \frac{\psi(\beta t)}{\psi(t)} > 1 \quad \text{for any } 0 < \beta < 1. \quad (3.2)$$

This condition implies the following lemma given in [3].

Lemma 3.2. Let $\psi \in C[t_0, \infty)$. Suppose that $\psi$ is strictly monotone decreasing and satisfies (3.2). Let $\phi := \psi^{-1}$. Given a positive number $\gamma$ there exist positive numbers $\eta_\gamma$, $\rho_\gamma$ such that the following hold:
If $\gamma > 1$, then $\phi((1 - \eta)\rho) \leq \gamma \phi(\rho)$ for all $\eta \in [0, \eta_\gamma]$, $\rho \in [0, \rho_\gamma]$; (3.3)

If $\gamma < 1$, then $\phi((1 + \eta)\rho) \geq \gamma \phi(\rho)$ for all $\eta \in [0, \eta_\gamma]$, $\rho \in [0, \rho_\gamma]$. (3.4)

Suppose $f$ satisfies the Keller–Osserman condition, and let $D$ be a bounded domain with $C^2$ boundary $\partial D$. If $u \in W^{1,p}_{\text{loc}}(D) \cap C(D)$ is a solution of
\[ \Delta_p u = f(u), \quad u(x) \to \infty, \quad x \to \partial D, \]
then it is known (see [16] for a proof) that
\[ \lim_{d(x, \partial D) \to 0} \frac{u(x)}{\phi(d(x, \partial D))} = 1. \] (3.5)

The next theorem establishes a similar result for weak blow-up solutions of (1.3) when $g \in C(\overline{\Omega})$.

A bounded domain $\Omega$ in $\mathbb{R}^n$ is said to satisfy a uniform interior (exterior) sphere condition if there is $R > 0$ such that for any $z \in \partial \Omega$ and any $0 < r < R$ there is a ball $B := B_r(x)$ contained in $\Omega$ (contained in the complement $\Omega^c$) such that $\partial B \cap \partial \Omega = \{z\}$.

**Theorem 3.3.** Let $\Omega$ be a bounded domain that satisfies both the uniform interior and exterior sphere conditions. Suppose $f$ satisfies both the Keller–Osserman and the Bandle–Marcus conditions. Let $g \in C(\overline{\Omega})$ such that $g > 0$ on the boundary $\partial \Omega$. Then for any non-negative solution $u$ of (1.3) we have
\[ \lim_{\delta(x) \to 0} \frac{u(x)}{\phi(g(x)^{1/p}(\delta(x))} = 1. \] (3.6)

**Proof.** We follow the method introduced by Bandle and Marcus in [3] (see also [16]). Let $z \in \partial \Omega$. First we show that
\[ \lim_{x \to z} \frac{u(x)}{\phi(g^{1/p}(\delta(x))} \leq 1. \]

To this end let $\epsilon > 0$ be given. Corresponding to $\gamma = 1 + \epsilon$ in Lemma 3.2 we choose $1 > \eta_\epsilon > 0$ and $\rho_\epsilon > 0$ such that (3.3) holds for all $\eta \in [0, \eta_\epsilon]$, $\rho \in [0, \rho_\epsilon]$. Let us fix $\eta > 0$ such that $2\eta - \eta^2 < \eta_\epsilon$. Since $g^{1/p}(x)\delta(x) \to 0$ as $x \to z$ and since $g^{1/p}$ is continuous on $\overline{\Omega}$ there is $r_1 = r_1(z, \eta)$ such that $|g^{1/p}(x) - g^{1/p}(z)| < g^{1/p}(z)\eta$ and $g^{1/p}(x)\delta(x, \partial B) < \rho_\epsilon$ when $|x - z| < r_1$, $x \in \Omega$.

Therefore it follows that $g^{1/p}(z) > g^{1/p}(x)(1 - \eta)$ for any $x \in \Omega$, $|x - z| < r_1$. In particular for such $x$ we also have
\[ \alpha^{1/p} > g^{1/p}(z)(1 - \eta) > g^{1/p}(x)(1 - \eta^2). \]

Since $\phi$ is decreasing, from the above inequality, we obtain
\[ \phi(\alpha^{1/p}\delta(x)) \leq \phi(g^{1/p}(x)\delta(x))(1 - (2\eta - \eta^2)), \quad x \in \Omega, \ |x - z| < r_1. \]

By (3.3) of Lemma 3.2, applied to the right-hand side of the above inequality, we get
\[ \phi(\alpha^{1/p}\delta(x)) \leq (1 + \epsilon)\phi(g^{1/p}(x)\delta(x)), \quad x \in \Omega, \ |x - z| < r_1. \] (3.7)
Since $\Omega$ satisfies the interior sphere condition we take a ball $B := B(x_0, r) \subseteq \Omega$ such that $\partial B \cap \partial \Omega = \{z\}$. We will suppose that the radius $r$ is small enough such that $r < r_1/2$ and $g > 0$ on $B$.

Now let $w \in W^{1,p}_{\text{loc}}(B) \cap C(B)$ be a blow-up solution of

$$\Delta_p w = \alpha f(w),$$

where $\alpha = \inf\{g(x) : x \in B\} > 0$. First we note, by the comparison principle, that

$$u(x) \leq w(x), \quad x \in B. \quad (3.8)$$

Moreover, using $\alpha f$ instead of $f$ in (3.1) and letting $\phi^\alpha$ stand for the corresponding inverse, we note that

$$\phi^\alpha(s) = \phi(\alpha^{1/p}s).$$

Thus, on applying (3.5), we find

$$\lim_{d(x, \partial B) \to 0} w(x) \phi(\alpha^{1/p}d(x, \partial B)) = 1.$$ 

Therefore corresponding to the given $\epsilon > 0$ there is $\rho > 0$ such that

$$w(x) \leq (1 + \epsilon)\phi(\alpha^{1/p}d(x, \partial B)), \quad \text{when } x \in B \text{ and } d(x, \partial B) < \rho. \quad (3.9)$$

Now let $x \in B$ be on the segment $\overline{x_0z}$ so that $d(x, \partial B) = \delta(x)$. Then for such $x$ we obtain from (3.7) and (3.9) that

$$\phi(\alpha^{1/p}d(x, \partial B)) \leq (1 + \epsilon)^2 \phi\left(g^{1/p}(x)\delta(x)\right).$$

This together with (3.8) shows that

$$u(x) \leq (1 + \epsilon)^2 \phi\left(g^{1/p}(x)\delta(x)\right)$$

for $x \in B$ which lies on the segment $\overline{x_0z}$. Thus

$$\limsup_{\delta(x) \to 0} \frac{u(x)}{\phi\left(g^{1/p}(x)\delta(x)\right)} \leq 1. \quad (3.10)$$

Using the exterior sphere condition on $\Omega$ we apply a similar argument as in the above to show that

$$\liminf_{\delta(x) \to 0} \frac{u(x)}{\phi\left(g^{1/p}(x)\delta(x)\right)} \geq 1.$$ 

So for $z \in \partial \Omega$ let $B = B(x_0, r) \subseteq \Omega^c$ such that $\partial B \cap \partial \Omega = \{z\}$. Consider the annulus $D(r) = B(x_0, 2r) \setminus B(x_0, r)$ centered at $x_0$. Let $w \in W^{1,p}_{\text{loc}}(D(r))$ be a solution of

$$\Delta_p w = \beta f(w), \quad w(x) \to \infty \text{ on } \partial B(x_0, r) \text{ and } w(x) = 0 \text{ on } \partial B(x_0, 2r),$$

where $\beta := \sup\{g(x) : x \in \Omega \cap D(r)\}$. (See [16] for the existence of such a solution.) Then by the comparison principle again it follows that

$$w(x) \leq u(x), \quad x \in \Omega \cap D(r). \quad (3.11)$$
It is also known (see [16]) that \( w \) satisfies
\[
\liminf_{D(r) \ni x \to \partial B(x_0, r)} \frac{w(x)}{\phi(\beta^{1/p}d(x, \partial B(x_0, r)))} \geq 1.
\]

Arguing as before, but using (3.4) of Lemma 3.2 instead, it follows that
\[
\liminf_{x \to \partial \Omega} \frac{u(x)}{\phi(g(x)^{1/p}d(x, \partial \Omega))} \geq 1. \tag{3.12}
\]

This together with (3.10) gives the desired limit. \( \square \)

Without further conditions on the weight \( g \), a solution \( u \) of (1.3) does not, in general, satisfy the limit in (3.6). In fact the limit superior in (3.10) could be zero, while the limit inferior in (3.12) could be infinity. (See [4] for an example.) However with mild conditions on \( g \), the estimate in (3.10) still holds even when \( g \) is unbounded on \( \Omega \). The conditions on \( g \) needed to obtain the estimate in (3.10) are the following:

\[
\inf_{\Gamma \eta} g(x) > 0 \quad \text{for some} \quad \eta > 0,
\]

\[
g(x)^{1/p}d(x, \partial \Omega) \to 0 \quad \text{as} \quad \delta(x) \to 0. \tag{3.13}
\]

**Theorem 3.4.** Let \( g \in C(\Omega) \) satisfy (3.13). Suppose \( f \) satisfies both the Keller–Osserman and the Bandle–Marcus conditions. Then for any blow-up solution \( u \) of (1.3) we have
\[
\limsup_{\delta(x) \to 0} \frac{u(x)}{\phi(g^{1/p}(x)\delta(x))} \leq 1.
\]

**Proof.** Suppose the conclusion of the above theorem is not true. Then there is a sequence of points \( x_j \in \Omega \) with \( \delta(x_j) \to 0 \) as \( j \to \infty \) such that
\[
\frac{u(x_j)}{\phi(g^{1/p}(x_j)\delta(x_j))} > 1 + \alpha
\]

for all \( j = 1, 2, \ldots \) and for some \( \alpha > 0 \). By continuity, for each \( j = 1, 2, \ldots, \) there is a ball \( B_j(x_j) \subseteq \Omega \) centered at \( x_j \) such that
\[
u(x) \geq (1 + \alpha)\phi(g^{1/p}(x)\delta(x)), \quad x \in B_j(x_j). \tag{3.14}
\]

Corresponding to \( \gamma := 1 + \alpha/2 \) we choose \( \eta_\gamma \) and \( \rho_\alpha \) such that inequality (3.3) of Lemma 3.2 holds for all \( \eta \in [0, \eta_\gamma] \) and all \( \rho \in [0, \rho_\alpha] \). Set
\[
\Omega_j := \{ x \in \Omega : \delta(x) > \delta(x_j) \}.
\]

Let \( \nu > 0 \) be chosen such that for all \( x \in \Omega \) with \( \delta(x) < \nu \) and all sufficiently large \( j \) we have
\[
g^{1/p}(x)\delta(x) \in [0, \rho_\alpha], \quad \frac{d(x, \partial \Omega) - d(x, \partial \Omega_j)}{d(x, \partial \Omega)} \in [0, \eta_\gamma].
\]

Then
\[ \phi(g(x)^{1/p}d(x, \partial \Omega_j)) = \phi \left( g(x)^{1/p}d(x, \partial \Omega) \left( 1 - \frac{d(x, \partial \Omega) - d(x, \partial \Omega_j)}{d(x, \partial \Omega)} \right) \right) \leq (1 + \alpha/2) \phi(g(x)^{1/p}d(x, \partial \Omega)). \] (3.15)

Let \( w_j \) be a blow-up solution of (1.3) on the set \( \Omega_j \). By the comparison principle,
\[ w_j(x) \geq u(x), \quad x \in \Omega_j, \]
and this together with (3.14) implies that
\[ w_j(x) \geq (1 + \alpha) \phi(g(x)^{1/p}d(x, \partial \Omega_j)), \quad x \in B_j(x_j) \cap \Omega_j. \]
This last inequality along with the inequality in (3.15) shows that for sufficiently large \( j \),
\[ w_j(x) \geq (1 + \alpha/(\alpha + 2)) \phi(g(x)^{1/p}d(x, \partial \Omega_j)), \quad x \in B_j(x_j) \cap (\Omega_j \setminus \Omega_\nu). \]
But this contradicts the fact that
\[ \limsup_{d(x, \partial \Omega_j) \to 0} w_j(x) \phi(g(x)^{1/p}d(x, \partial \Omega_j)) \leq 1, \]
which holds by Theorem 3.3.

4. Some examples

Let \( f \) satisfy the Keller–Osserman condition. In this section we provide a class of functions \( g \in C(\Omega) \) for which (1.3) has a solution and also a class of functions for which (1.3) has no solution. We say the \( w \in W^{1,p}(\Omega) \) is a supersolution of \( \Delta_p w = -g \) on \( \Omega \) iff
\[ \int_{\Omega} |\nabla w|^{p-2} \nabla w \cdot \nabla \psi \geq \int_{\Omega} g \psi \]
for all \( 0 \leq \psi \in W^{1,p}_0(\Omega) \).

4.1. A class of weights for which (1.3) admits a solution

Let \( B = B(x_0, R) \subseteq \mathbb{R}^n \) be a ball centered at \( x_0 \) and of radius \( R > 0 \). Let \( g \in C(B) \) be positive near \( \partial B \) and suppose
\[ \sup_{x \in B} g(x) (R - |x - x_0|)^{\lambda} |x - x_0|^{2-p} \leq M < \infty \] (4.1)
for some \( 1 < \lambda < p \). We show below that (1.3) admits a local weak solution and therefore Theorem 2.2 shows that the corresponding problem (1.3) has a non-negative blow-up solution on \( B \).

Let \( \beta := (p - \lambda)/(p - 1) \) so that \( 0 < \beta < 1 \). We let
\[ z(x) = (R^2 - |x - x_0|^2)^\beta, \quad x \in B. \]
Direct computation shows that for all \( x \in B \) (\( x \in B \setminus \{x_0\} \) if \( 1 < p < 2 \)),

\[ \text{div}(|\nabla z|^{p-2} \nabla z) = -(2\beta)p^{-1}(R^2 - |x - x_0|^2)\left(\frac{p-1}{p-2}\right)^{\frac{p-1}{p-2}} |x - x_0|^{p-2} \]
\[ \times (2(1 - \beta)(p - 1)|x - x_0|^2 + (p - 2 + n)(R^2 - |x - x_0|^2)). \]

Since \( p - 2 + n \geq (1 - \beta)(p - 1) \) we see that
\[ 2(1 - \beta)(p - 1)|x - x_0|^2 + (p - 2 + n)(R^2 - |x - x_0|^2) \geq (1 - \beta)(p - 1)R^2. \]

Now let \( v(x) = \gamma z(x) \) for \( \gamma > 0 \). Using the hypothesis
\[ g(x) \leq M(R - |x - x_0|)^{(p-1)(p-1)}|x - x_0|^{p-2}, \quad x \in B, \]
we make the estimations
\[ -\text{div}(|\nabla v|^{p-2} \nabla v) - g(x) \]
\[ \geq (2\gamma \beta)^{p-1}(1 - \beta)(1 - p)(R^2 - |x - x_0|^2)\left(\frac{p-1}{p-2}\right)^{\frac{p-1}{p-2}} |x - x_0|^{p-2} - g(x) \]
\[ \geq (R - |x - x_0|)^{(p-1)(p-1)}|x - x_0|^{p-2} \]
\[ \times (2\gamma \beta)^{p-1}(1 - \beta)(1 - p)R^{p-1}(p-1)^{-1} - M \]
\[ \geq 0 \]
for a sufficiently big choice of \( \gamma \).

Thus if \( p \geq 2 \) we see that \( v = \gamma z \) is a supersolution of (2.3). If \( 1 < p < 2 \) then \( v \) is supersolution of (2.3) on \( B \setminus \{x_0\} \). Actually it is also a supersolution on \( B \). To see this let \( \eta \in C_0^\infty(B(0, 1)) \), \( 0 \leq \eta \leq 1 \), such that \( \eta \equiv 1 \) on \( B(0, 1/4) \). For each \( \alpha > 0 \), let
\[ \eta_\alpha(x) = \eta\left(\frac{\alpha}{R}(x - x_0)\right), \quad x \in B(x_0, R). \]

Since \( v \) is a supersolution of (2.3), then for any \( 0 \leq \varphi \in W_0^{1,p}(B) \) we have
\[ \int_B |\nabla v|^{p-2} \nabla v \cdot \nabla [(1 - \eta_\alpha)\varphi] \geq \int_B g(1 - \eta_\alpha)\varphi. \]

that is
\[ \int_B |\nabla v|^{p-2} \nabla v \cdot \nabla \varphi \geq \int_B g \varphi - \int_B g \eta_\alpha \varphi + \int_B |\nabla v|^{p-2} \nabla v \cdot \nabla (\eta_\alpha \varphi). \tag{4.2} \]

A simple calculation shows that
\[ \lim_{\alpha \to \infty} \left( -\int_B g \eta_\alpha \varphi + \int_B |\nabla v|^{p-2} \nabla v \cdot \nabla (\eta_\alpha \varphi) \right) = 0. \]

Therefore, on taking the limit in (4.2) as \( \alpha \to \infty \), we obtain
\[ \int_B |\nabla v|^{p-2} \nabla v \cdot \nabla \varphi \geq \int_B g \varphi, \quad 0 \leq \varphi \in W_0^{1,p}(B). \]

Thus indeed, \( v \) is a supersolution of (2.3) on \( B \).
We now show that for any $h \in L^\infty(B)$ with $h \leq g$, and any $w \in W^{1,p}_0(B)$ such that $\Delta_p w = -h$ we have
\[ w \leq v \quad \text{on} \quad B. \tag{4.3} \]

Note that such a solution $w$ is in fact in $W^{1,p}_0(B) \cap L^\infty(B)$. Hence it follows that $w \in C^{1,\beta}(B)$ for some $0 < \beta < 1$ (see [14]). Note that for any given $\varepsilon > 0$ there is $\rho_\varepsilon > 0$ such that $w \leq v + \varepsilon$ on $\partial B(x_0, \rho)$ for all $\rho_\varepsilon < \rho < R$. Since $v + \varepsilon \in W^{1,p}(B(x_0, \rho))$ and $\Delta_p (v + \varepsilon) \leq -h$ we now conclude, by the comparison principle, that $w \leq v + \varepsilon$ in $B(x_0, \rho)$ for any $\rho_\varepsilon < \rho < R$. Thus $w \leq v + \varepsilon$ on $B$. Since $\varepsilon$ is arbitrary we get the desired inequality (4.3).

We will exploit this fact below to show that (2.3) admits a solution when $g$ is as in (4.1) above.

For each $k$, let $g_k$ be defined as
\[
g_k := \begin{cases} g & \text{if } g \leq k, \\ k & \text{if } g \geq k. \end{cases} \]

Let $\{B_k\}$ be a sequence of balls exhausting $B$ such that $g \leq k$ on $B_k$. Let $w_k \in W^{1,p}_0(B)$ be such that $\Delta_p w_k = -g_k$. Then by (4.3) above it follows that $w_k \leq v$ on $B$ so that $\{w_k\}$ is uniformly bounded. By the standard interior $C^{1,\alpha}$ estimates, applied successively to the balls $B_k$, we conclude that a subsequence converges to a solution $w \in W^{1,p}_0(B)$ of $\Delta_p w = -g$. By Theorem 2.2 it follows that (1.3) has a blow-up solution on $B(x_0, R)$.

### 4.2. A class of weights for which (1.3) fails to admit a solution

We start with the following lemma.

**Lemma 4.1.** Let $g \in L^q_{\text{loc}}(\Omega)$, where $q$ is the Hölder conjugate exponent of $p$. If (1.3) has a blow-up solution $u \in W^{1,p}(\Omega) \cap C(\Omega)$, then the Dirichlet problem (2.3) has a weak solution $w \in W^{1,p}_0(\Omega)$.

**Proof.** Suppose that $D \subseteq \Omega$ is a subdomain such that $f(u(x)) \geq 1$ for $x \in \Omega \setminus D$. Let $q$ be the Hölder conjugate exponent of $p$. Then for any $\varphi \in W^{1,p}_0(\Omega)$, we have
\[
\left| \int_\Omega g \varphi \right| \leq \int_D g|\varphi| + \int_{\Omega \setminus D} g|\varphi| \leq \|g\|_{L^q(D)}\|\varphi\|_p + \int_{\Omega \setminus D} g f(u)|\varphi|
\]
\[
\leq \|g\|_{L^q(D)}\|\varphi\|_p + \int_\Omega g f(u)|\varphi| = \|g\|_{L^q(D)}\|\varphi\|_p - \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla |\varphi|
\]
\[
\leq \|g\|_{L^q(D)}\|\varphi\|_p + \|\nabla u\|_p^q \|\nabla \varphi\|_p = (\|g\|_{L^q(D)} + \|\nabla u\|_p^q)\|\nabla \varphi\|_1. \]

Thus
\[
\varphi \mapsto \int_\Omega g \varphi
\]

is a linear functional on $W^{1,p}_0(\Omega)$ and hence $g \in W^{-1,p}(\Omega)$ as claimed. $\Box$
We use the above lemma to establish the non-existence of weak blow-up solutions for a certain class of weights \( g \).

Let \( \Omega \) be a bounded domain that has an interior sphere condition at \( x_0 \in \partial \Omega \). If \( g \in C(\Omega) \) is a non-negative function such that
\[
\liminf_{x \to x_0} g(x)|x - x_0|^p = \alpha \in (0, \infty],
\]
then (1.3) has no solution in \( W^{1,p}(\Omega) \cap C(\Omega) \).

To see this, let \( N(x_0) \) be the unit outer normal vector to \( \partial \Omega \) at \( x_0 \). Then for any non-negative solution \( w \in W^{1,p}_{\text{loc}}(\Omega) \cap C(\Omega) \) of \( \Delta_p u = -g \), there are positive constants \( K, \delta \) such that
\[
w(x_0 - rN(x_0)) \geq K
\]
for all \( 0 < r < \delta \). This can be proved as in the proof of Theorem 4.2 of [6]. We leave the details to the reader. Thus by Proposition 4.1, problem (1.3) cannot have a blow-up solution in \( W^{1,p}(\Omega) \cap C(\Omega) \).

References