Convergence of finite difference method for the parabolic problem with concentrated capacity and variable operator

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Abstract

We investigate the convergence of difference schemes for the one-dimensional heat equation with variable (time-dependent) operator and the coefficient at the time derivative (heat capacity) containing Dirac delta distribution. An abstract operator method is developed for analyzing this equation. Estimate for the rate of the convergence in special discrete energetic Sobolev norm, compatible with the smoothness of the solution is obtained.

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1. Introduction

The finite-difference method is one of the basic tools for the numerical solution of partial differential equations. In the case of the problems with discontinuous coefficients and concentrated factors (Dirac delta functions, free boundaries, etc.) the solution has weak global regularity and it is impossible to establish convergence of the finite difference schemes using the classical Taylor expansion. Often, the Bramble–Hilbert lemma takes the role of the Taylor formula for the functions from the Sobolev spaces\cite{4,5,9}.

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Following Lazarov et al. [9], a convergence rate estimate of the form
\[ \|u - v\|_{W^{s,k}_{2,0}} \leq C h^{s-k} \|u\|_{W^{s}_2}, \quad s > k \] (1)
is called compatible with the smoothness (regularity) of the boundary value problem solution \( u \). Here
\( v \) is discrete problem solution, \( h \)—space mesh step, \( W^{s}_{2,0} \) and \( W^{k}_{2,0} \), are Sobolev spaces of functions with continuous and discrete argument, respectively, \( C \) is a constant which does not depend on \( u \) and \( h \). For the parabolic case typical estimates are of the form
\[ \|u - v\|_{W^{s,k/2}_{2,h}} \leq C (h + \sqrt{\tau})^{s-k} \|u\|_{W^{s/2}_2}, \quad s > k, \] (2)
where \( \tau \) is the time step. In the case of the equations with variable coefficients the constant \( C \) in the error bounds depends on norms of the coefficients (see, for example, [1,2,5,14]).

One interesting class of parabolic problems model processes in heat-conduction media with concentrated capacity in which the heat capacity coefficient contains a Dirac delta function, or equivalently, the jump of the heat flow in the singular point is proportional to the time derivative of the temperature [11]. Therefore, the current problems are not standard and the classical analysis is difficult to apply to convergence analysis.

In the present paper a finite-difference scheme approximating the one-dimensional initial-boundary value problem for the heat equation with concentrated capacity and variable (time-dependent) operator is derived. Special energetic Sobolev norm (corresponding to the norm \( W^{1,1/2}_2 \) for classic heat-conduction problem) is constructed. In this norm, convergence rate estimate compatible with the smoothness of the boundary value problem solution is obtained.

Analogous result for the parabolic problem with concentrated capacity and constant coefficient at the space derivative is obtained in [7]. Note that the convergence on classical solutions is studied in [3,16]. Parabolic problem with variable operator (without Dirac delta function) is studied in [1,2]. Convergence of difference schemes for hyperbolic problem with concentrated mass is studied in [8].

2. Preliminary results

Let \( H \) be a real separable Hilbert space endowed with inner product \((\cdot, \cdot)\) and norm \(\|\cdot\|\) and \( S \)-unbounded self-adjoint positive definite linear operator, with domain \( D(S) \) dense in \( H \). It is easy to see that the product \((u, v)_S = (Su, v) (u, v \in D(S))\) satisfies the inner product axioms. Closure of \( D(S) \) in the norm \(\|u\|_S = (u, u)_S^{1/2}\) is a Hilbert space \( H_S \subset H \). The inner product \((u, v)\) continuously extends to \( H^*_S \times H_S \), where \( H^*_S = H_{S^{-1}} \) is the dual space for \( H_S \). Spaces \( H_S, H \) and \( H_{S^{-1}} \) represent a Gelfand triple \( H_S \subset H \subset H_{S^{-1}} \), with continuous imbeddings. Operator \( S \) extends to the map \( S : H_S \to H^*_S \). There exists unbounded self-adjoint positive definite linear operator \( S^{1/2} \), such that \( D(S^{1/2}) = H_S \) and \((u, v)_S = (Su, v) = (S^{1/2}u, S^{1/2}v)\). We also define the Sobolev spaces \( W^{s}_2(a, b; H), W^{0}_2(a, b; H) = L^2(a, b; H) \), of the functions \( u = u(t) \) mapping interval \((a, b) \subset R \) into \( H \) (see [10,17]).

Let \( A \) and \( B \) be unbounded self-adjoint positive definite linear operators, \( A = A(t), \quad B \neq B(t), \) in Hilbert space \( H \), in general noncommutative, with \( D(A) \)-dense in \( H \) and \( H_A \subset H_B \). We consider an abstract Cauchy problem (comp. [12,17])
\[ B \frac{du}{dt} + Au = f(t), \quad 0 < t < T, \quad u(0) = u_0, \] (3)
where \( f(t) \) and \( u_0 \) are given and \( u(t) \) is the unknown function with values in \( H \). Because \( A = A(t) \), this problem is known as problem with variable operator. Let us also assume that \( A_0 \leq A(t) \leq cA_0 \) where \( c = \text{const.} > 0 \) and \( A_0 \) is a constant self-adjoint positive definite linear operator in \( H \). The following proposition holds.

**Lemma 1.** The solution of the problem (3) satisfies a priori estimate

\[
\int_0^T \| u(t) \|^2 A_0 \, dt + \int_0^T \int_0^T \frac{\| u(t) - u(t') \|^2 B}{|t - t'|^2} \, dt \, dt' \leq C \left( \| u_0 \|^2_B + \int_0^T \| f(t) \|^2 A_0^{-1} \, dt \right)
\]

if \( u_0 \in H_B \) and \( f \in L^2(0, T; H A_0^{-1}) \).

**Proof.** It follows from the abstract parabolic initial-value problems theory (see [12,17]) that for \( u_0 \in H_B \) and \( f \in L^2(0, T; H A_0^{-1}) \) the problem (3) has unique solution \( u \in L^2(0, T; H A_0) \) with \( du/dt \in L^2(0, T; H A_0^{-1} B) \). We take the inner product of (3) with \( 2u \), and estimate the right-hand side by Cauchy–Schwartz inequality

\[
\frac{d}{dt}(\| u \|^2_B) + 2\| u \|^2_A \leq \| f \|^2 A_0^{-1} + \| u \|^2 A_0.
\]

Since \( A(t) \geq A_0 \), an integration with respect to \( t \) gives

\[
\int_0^T \| u(t) \|^2 A_0 \, dt + \| u(T) \|^2_B \leq \| u_0 \|^2_B + \int_0^T \| f(t) \|^2 A_0^{-1} \, dt.
\]

We shall now construct an estimate in a fractional order seminorm with respect to \( t \). We begin with deriving of Fourier series for the function \( u(t) : [0, T] \to H \):

\[
u(t) = a_0 + \sum_{j=1}^{\infty} a_j \cos \frac{j\pi t}{T}, \quad u(t) = \sum_{j=1}^{\infty} b_j \sin \frac{j\pi t}{T},
\]

where

\[
a_j = a_j[u] = 2 \frac{T}{\pi} \int_0^T u(t') \cos \frac{j\pi t'}{T} \, dt', \quad b_j = b_j[u] = 2 \frac{T}{\pi} \int_0^T u(t') \sin \frac{j\pi t'}{T} \, dt'.
\]

It is easy to check, that

\[
\int_0^T \| u(t) \|^2 \, dt = \frac{T}{2} \left( \frac{\| a_0[u] \|^2}{2} + \sum_{j=1}^{\infty} \| a_j[u] \|^2 \right) = \frac{T}{2} \sum_{j=1}^{\infty} \| b_j[u] \|^2.
\]

We multiply (3) by \( \sin k\pi t / T \) and integrate with respect to \( t \) from 0 to \( T \). Using the expansion

\[
B \frac{du(t)}{dt} = \frac{d[Bu(t)]}{dt} = -\sum_{j=1}^{\infty} a_j[Bu] \frac{j\pi}{T} \sin \frac{j\pi t}{T}
\]

and the orthogonality of sines, we obtain

\[
k\pi \frac{T}{a_k[Bu] = b_k[Au] - b_k[f]}.
\]
We take the inner product of this equality with $a_k[u]$ and summing with respect to $k$ to get

$$\frac{\pi}{T} \sum_{k=1}^{\infty} k(a_k[Bu], a_k[u]) = \sum_{k=1}^{\infty} (b_k[Au], a_k[u]) - \sum_{k=1}^{\infty} (b_k[f], a_k[u]). \quad (7)$$

Further, we have

$$(a_k[Bu], a_k[u]) = (a_k[B^{1/2}u], a_k[B^{1/2}u])$$

and analogously

$$(b_k[Au], a_k[u]) = (b_k[A_0^{-1/2}Au], a_k[A_0^{1/2}u])$$

and

$$(b_k[f], a_k[u]) = (b_k[A_0^{-1/2}f], a_k[A_0^{1/2}u]).$$

Therefore, it follows from (7), that

$$\frac{\pi}{T} \sum_{k=1}^{\infty} k\|a_k[B^{1/2}u]\|^2 \leq \sum_{k=1}^{\infty} (\|b_k[A_0^{-1/2}Au]\|\|a_k[A_0^{1/2}u]\| + \|b_k[A_0^{-1/2}f]\|\|a_k[A_0^{1/2}u]\|)$$

$$\leq \frac{1}{2} \sum_{k=1}^{\infty} (\|b_k[A_0^{-1/2}Au]\|^2 + 2\|a_k[A_0^{1/2}u]\|^2 + \|b_k[A_0^{-1/2}f]\|^2).$$

Further, using (6), we find

$$\sum_{k=1}^{\infty} k\|a_k[B^{1/2}u]\|^2 \leq \frac{1}{\pi} \int_0^T [(2 + c^2)\|A_0^{1/2}u(t)\|^2 + \|A_0^{-1/2}f(t)\|^2] \, dt$$

$$= \frac{1}{\pi} \int_0^T [(2 + c^2)\|u(t)\|^2_{A_0} + \|f(t)\|^2_{A_0^{-1}}] \, dt. \quad (8)$$

Let us consider the expression

$$\int_{-T}^{T} \int_{-T}^{T} \frac{\|u(t) - u(t - \tau)\|^2_B}{|\tau|^2} \, d\tau \, dt,$$

where we assume that $u(t)$ is periodically extended outside $[0, T]$:

$$\tilde{u}(t) = \begin{cases} u(t), & t \in [0, T], \\ u(-t), & t \in [-T, 0], \\ u(2T - t), & t \in [T, 2T] \\ \end{cases}$$

and so on.
Using the periodicity of \( u(t) \) and the expansion with respect to cosines, we obtain
\[
\int_{-T}^{T} \int_{-T}^{T} \frac{\|u(t) - u(t - \tau)\|_{B}^2}{|\tau|^2} \, d\tau \, dt = \int_{-T}^{T} \left[ \int_{-T}^{T} (u(t), -u(t + \tau) + 2u(t) - u(t - \tau)) \, dt \right] d\tau \frac{1}{\tau^2}.
\]
\[
= \int_{-T}^{T} \left[ \int_{-T}^{T} (B^{1/2}u(t), -B^{1/2}u(t + \tau) + 2B^{1/2}u(t) - B^{1/2}u(t - \tau)) \, dt \right] d\tau \frac{1}{\tau^2}
= 4T \sum_{k=1}^{\infty} \|a_k [B^{1/2}u]\|^2 \int_{-T}^{T} \sin^2 \frac{k\pi \tau}{2T} \, d\tau.
\]
Next, we have
\[
\int_{-T}^{T} \sin^2 \frac{k\pi \tau}{2T} \, d\tau = 2 \int_{0}^{T} \sin^2 \frac{k\pi \tau}{2T} \, d\tau \leq \frac{k\pi}{T} \int_{0}^{\infty} \frac{\sin^2 \theta}{\theta^2} \, d\theta = \frac{k\pi^2}{2T}
\]
and we get
\[
\int_{-T}^{T} \int_{-T}^{T} \frac{\|u(t) - u(t - \tau)\|_{B}^2}{|\tau|^2} \, d\tau \, dt \leq 2T^2 \sum_{k=1}^{\infty} k\|a_k [B^{1/2}u]\|^2.
\]
Consider the expression
\[
\int_{0}^{T} \int_{0}^{T} \frac{\|u(t) - u(t')\|_{B}^2}{|t - t'|^2} \, dt \, dt'.
\]
From (8), (9) and the inequality
\[
\int_{0}^{T} \int_{0}^{T} \frac{\|u(t) - u(t')\|_{B}^2}{|t - t'|^2} \, dt \, dt' \leq \frac{1}{2} \int_{-T}^{T} \int_{-T}^{T} \frac{\|u(t) - u(t - \tau)\|_{B}^2}{|\tau|^2} \, d\tau \, dt
\]
we obtain
\[
\int_{0}^{T} \int_{0}^{T} \frac{\|u(t) - u(t')\|_{B}^2}{|t - t'|^2} \, dt \, dt' \leq \pi \int_{0}^{T} [(2 + c^2)\|u(t)\|_{A_0}^2 + \|f(t)\|_{A_{-1}}^2) \, dt.
\]
Hence, taking into account (5), we have
\[
\int_{0}^{T} \int_{0}^{T} \frac{\|u(t) - u(t')\|_{B}^2}{|t - t'|^2} \, dt \, dt' \leq (2 + c^2)\pi\|u_0\|_{B}^2 + (3 + c^2)\pi \int_{0}^{T} \|f(t)\|_{A_{-1}}^2 \, dt.
\]
From (5) and (10) follows a priori estimate (4). The lemma is proved. \( \square \)

Letting in (3) \( f(t) = \frac{dg(t)}{dt} \) we get
\[
B \frac{du}{dt} + Au = \frac{dg}{dt}, \quad 0 < t < T, \quad u(0) = u_0.
\]
Applying similar techniques one can prove the following assertion.
Lemma 2. The solution of the problem (11) satisfies a priori estimates

\[
\int_0^T \|u(t)\|_{A_0}^2 dt + \int_0^T \int_0^T \frac{\|u(t) - u(t')\|^2_B}{|t - t'|^2} dt' dt \\
\leq C \left[ \|u_0\|_{B_0}^2 + \int_0^T \int_0^T \frac{\|g(t) - g(t')\|^2_{B^{-1}}}{|t - t'|^2} dt' dt + \int_0^T \left( \frac{1}{t} + \frac{1}{T-t} \right) \|g(t)\|_{B^{-1}}^2 dt \right]
\]

if \(u_0 \in H_B\) and \(g \in W^{1/2}_2(0, T; H_{B^{-1}})\).

Analogous results hold for operator–difference schemes. Let \(H_h\) be finite-dimensional real Hilbert space with inner product \((\cdot, \cdot)_h\) and norm \(\|\cdot\|_h\). Let \(A_h = A_h(t)\) and \(B_h \neq B_h(t)\) be self-adjoint positive linear operators defined on \(H_h\), in the general case—noncommutative. By \(H_{S_h}\), where \(S_h = S^*_h > 0\), we denote the space with inner product \((\cdot, \cdot)_{S_h} = (S_h \cdot, \cdot)_h\) and norm \(\|\cdot\|_{S_h} = (S_h \cdot, \cdot)_h^{1/2}\).

Let \(\omega_{\tau}\) be a uniform mesh on \((0, T)\) with the step \(\tau = T/m\), \(\omega_{\tau}^- = \omega_{\tau} \cup \{0\}, \omega_{\tau}^+ = \omega_{\tau} \cup \{T\}\) and \(\bar{\omega}_{\tau} = \omega_{\tau} \cup \{0, T\}\). Further, we shall use standard notation of the theory of the difference [13,14]. In particular we set

\[
v_{t_1} = v_{t_1}(t) = \frac{v(t) - v(t - \tau)}{\tau}, \quad v_t = v_t(t) = \frac{v(t + \tau) - v(t)}{\tau} = v_{t}(t + \tau).
\]

We will consider the simplest implicit operator-difference scheme

\[
B_h v_{t_1} + A_h v = \varphi(t), \quad t \in \omega_{\tau}^+, \quad v(0) = v_0,
\]

where \(v_0\) is a given element \(H_h\), \(\varphi(t)\) is known and \(v(t)\)—unknown mesh function with values in \(H_h\). Let us also consider the scheme

\[
B_h v_{t_1} + A_h v = \psi_{t_1}, \quad t \in \omega_{\tau}^+, \quad v(0) = v_0,
\]

where \(\psi(t)\) is a given mesh function with values in \(H_h\). Analogously, as in the previous case we assume that \(A_{0h} \leq A_h(t) \leq c A_{0h}\) where \(c = \text{const.} > 0\) and \(A_{0h}\) is a constant self-adjoint positive linear operator in \(H_h\). The following analogs of Lemmas 1 and 2 are true (comp. [5,6]).

Lemma 3. For the solution of the problem (12) the next estimate is valid

\[
\tau \sum_{t \in \omega_{\tau}}' \|v(t)\|_{A_{0h}}^2 + \tau^2 \sum_{t \in \omega_{\tau}, t' \in \omega_{\tau}, t' \neq t}^\prime \frac{\|v(t) - v(t')\|_{B_h}^2}{|t - t'|^2} \leq C \left( \|v_0\|_{B_h}^2 + \tau \|v_0\|_{A_{0h}}^2 + \tau \sum_{t \in \omega_{\tau}^+} \|\varphi(t)\|_{A_{0h}}^2 \right),
\]

where we denoted

\[
\sum_{t \in \omega_{\tau}}^\prime w(t) = \frac{w(0)}{2} + \sum_{t \in \omega_{\tau}} w(t) + \frac{w(T)}{2}.
\]
Lemma 4. For the solution of the problem (13) the next estimate is valid

\[ \tau \sum_{t \in \bar{\omega}_T} \| v(t) \|_{A_{0h}}^2 + \tau^2 \sum_{t \in \bar{\omega}_T} \sum_{t' \in \bar{\omega}_T, t' \neq t} \frac{\| v(t) - v(t') \|_{B_h}^2}{|t - t'|^2} \leq C \left[ \| v_0 \|_{B_h}^2 + \tau \| v_0 \|_{A_{0h}}^2 \right] + \tau^2 \sum_{t \in \bar{\omega}_T} \sum_{t' \in \bar{\omega}_T, t' \neq t} \frac{\| \psi(t) - \psi(t') \|_{B_{h^{-1}}}^2}{|t - t'|^2} + \tau \sum_{t \in \bar{\omega}_T} \left( \frac{1}{t} + \frac{1}{T - t} \right) \| \psi(t) \|_{B_{h^{-1}}}^2. \]

3. Heat equation with concentrated capacity

Let us consider the initial-boundary value problem for the heat equation with the presence of concentrated capacity at interior point \( x = \xi \)

\[ [1 + K \delta(x - \xi)] \frac{\partial u}{\partial t} - \frac{\partial}{\partial x} \left( a(x, t) \frac{\partial u}{\partial x} \right) = f(x, t), \quad (x, t) \in Q, \]

\[ u(0, t) = 0, \quad u(1, t) = 0, \quad 0 < t < T, \quad u(x, 0) = u_0(x), \quad x \in (0, 1), \]

where \( Q = (0, 1) \times (0, T), K \) is a positive constant, \( \delta(x) \) is the Dirac delta generalized function and equality is considered in the sense of the theory of distributions [15]. Our aim is to investigate the singularity of the solution of the problem (14) caused by the presence of singular coefficient \( K \delta(x - \xi) \), therefore we restrict ourselves to the simplest Dirichlet boundary conditions. Analogous problem with constant coefficient \( a(x, t) \) is considered in [7].

In a standard manner one obtains the weak form of initial-boundary value problem (14):

\[ \int_0^T \int_0^1 \frac{\partial u}{\partial t} v \, dx \, dt + \int_0^T \frac{\partial u}{\partial t}(\xi, t) v(\xi, t) \, dt + \int_0^T \int_0^1 a(x, t) \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} \, dx \, dt = \int_0^T \int_0^1 f v \, dx \, dt, \]

\[ \forall v \in \dot{W}_2^{1,0}(Q) = \{ v \in \dot{W}_2^{1,0}(Q) : v = 0 \text{ on } \{0, 1\} \times (0, T) \}. \]

The same weak form (15) corresponds to the following initial-boundary value problem

\[ \frac{\partial u}{\partial t} - \frac{\partial}{\partial x} \left( a(x, t) \frac{\partial u}{\partial x} \right) = f(x, t), \quad (x, t) \in Q_1 \cup Q_2, \]

\[ [u]_{x = \xi} = u(\xi + 0, t) - u(\xi - 0, t) = 0, \quad \left[ a(x, t) \frac{\partial u}{\partial x} \right]_{x = \xi} = K \frac{\partial u}{\partial t}(\xi, t), \]

\[ u(0, t) = 0, \quad u(1, t) = 0, \quad 0 < t < T, \quad u(x, 0) = u_0(x), \quad x \in (0, 1), \]

where \( Q_1 = (0, \xi) \times (0, T) \) and \( Q_1 = (\xi, 1) \times (0, T) \). In this sense, initial-boundary value problems (14) and (16) are equivalent.

Letting \( H = L_2(0, 1) \) it is easy to see that the initial boundary value problem (14) can be written in the form (3), where

\[ Au = -\frac{\partial}{\partial x} \left( a(x, t) \frac{\partial u}{\partial x} \right), \quad Bu = [1 + K \delta(x - \xi)]u(x, t) \]
and

\[ A_0u = -c_1 \frac{\partial^2 u}{\partial x^2}, \quad c = \frac{c_2}{c_1} \]

or

\[ (Av, w) = \int_0^1 a(x, t)v'(x)w'(x) \, dx, \quad (A_0v, w) = \int_0^1 v'(x)w'(x) \, dx \]

for \( v, w \in \hat{W}_2^1(0, 1) \) and

\[ (Bv, w) = \int_0^1 v(x) w(x) \, dx + K v(\xi) w(\xi). \]

Assuming

\[ a(x, t) \in L_\infty(Q), \quad 0 < c_1 \leq a(x, t) \leq c_2, \text{ } \text{a.e. in } Q \]

we immediately obtain

\[ \|w\|_{A_0}^2 = \int_0^1 a(x, t)|w'(x)|^2 \, dx = \|w\|^2_{W_2^1(0, 1)}, \quad w \in \hat{W}_2^1(0, 1) \]

and

\[ \|w\|_{A_0}^2 = c_1 \int_0^1 |w'(x)|^2 \, dx = \|w\|^2_{W_2^1(0, 1)}, \quad w \in \hat{W}_2^1(0, 1), \]

so, we can put \( H_A = H_{A_0} = \hat{W}_2^1(0, 1) \) and \( H_{A^{-1}} = H_{A_0^{-1}} = W_{2}^{-1}(0, 1) \).

The operator \( B \) is defined on the subset \( H_B \) of functions in \( L_2(0, 1) \) with finite norm

\[ \|w\|_B^2 = \|w\|^2_{L_2(0, 1)} + K \, w^2(\xi) \preceq \|w\|^2_{L_2(0, 1)} + w^2(\xi) = \|w\|^2_{L_2(0, 1)}, \]

so, we can put \( H_B = \hat{L}_2(0, 1) = \) closure of the set \( C[0, 1] \) in the norm \( \| \cdot \|_{L_2(0, 1)} \). Obviously, \( H_A = H_{A_0} = \hat{W}_2^1(0, 1) \subset C[0, 1] \subset \hat{L}_2(0, 1) = H_B \). The “negative” norm \( \|w\|_{B^{-1}} \) satisfies the relation

\[ \|w\|_{B^{-1}} = (B^{-1}w, w)^{1/2} = \sup_{0 \neq v \in H_B} \frac{|(w, v)|}{\|v\|_B}. \]

From Lemma 1 we immediately obtain the following assertion.

**Lemma 5.** Let \( a(x, t) \in L_\infty(Q), 0 < c_1 \leq a(x, t) \leq c_2 \) almost everywhere in \( Q \), \( f \in W_2^{-1, -1/2}(Q) \) and \( u_0 \in \hat{L}_2(0, 1) \). Then the weak solution of the initial-boundary value problem (14) belongs to the space \( W_2^{1, 1/2}(Q), u(\xi, t) \in W_2^{1/2}(0, T) \) and the following a priori estimate:

\[ \|u\|^2_{W_2^{1, 1/2}(Q)} + \|u(\xi, \cdot)\|^2_{W_2^{1/2}(0, T)} \leq C \left( \|u_0\|^2_{L_2(0, 1)} + \|f\|^2_{W_2^{-1, -1/2}(Q)} \right) \]

is satisfied.
Under stronger assumptions on the input data the solution of the problem (14) possesses extra derivatives.

**Lemma 6.** Let $a(x, t) \in W^{2,1}_\infty(Q)$, $0 < c_1 \leq a(x, t) \leq c_2$, $f \in W^{1+\varepsilon,(1+\varepsilon)/2}_2(Q)$ ($\varepsilon > 0$), $\frac{\partial f}{\partial t} \in W^{-1,-1/(2)}_2(Q)$ and $u_0 \in W^2_2(0, \xi) \cap W^{2,3/2}_2(\xi, 1) \cap C[0, 1]$. Then the solution of the initial-boundary value problem (14) belongs to the space $W^{3,3/2}_2(Q_1) \cap W^{3,3/2}_2(Q_2)$, $\frac{\partial u}{\partial t} \in W^{1,1/2}_2(Q)$ and $\frac{\partial u}{\partial t}(\xi, t) \in W^{1/2}_1(0, T)$.

**Proof.** Formally differentiating Eq. (3) in $t$ and applying a priori estimate (4) we obtain

$$
\int_0^T \left\| \frac{du}{dt}(t) \right\|_{A_0}^2 dt + \int_0^T \int_0^T \left\| \frac{du}{dt}(t) - \frac{du}{dt}(t') \right\|_B^2 dt \, dt' 
\leq C \left( \| A(0)u_0 \|_{B^{-1}}^2 + \| f(0) \|_{B^{-1}}^2 + \int_0^T \left\| \frac{dA}{dt}(t) u(t) \right\|_{A_0^{-1}}^2 dt + \int_0^T \left\| \frac{df}{dt}(t) \right\|_{A_0^{-1}}^2 dt \right).
$$

In the case of the problem (14) we have

$$
\int_0^T \left\| \frac{du}{dt}(t) \right\|_{A_0}^2 dt + \int_0^T \int_0^T \left\| \frac{du}{dt}(t) - \frac{du}{dt}(t') \right\|_B^2 dt \, dt' 
\times \left\| \frac{\partial u}{\partial t} \right\|_{W^{1,1/2}_2(Q)}^2 + \left\| \frac{\partial u}{\partial t}(\xi, \cdot) \right\|_{W^{1/2}_2(0,T)}^2,
$$

$$
\| f(0) \|_{B^{-1}} \leq C \| f(\cdot, 0) \|_{L^2(0,1)} \leq C \| f \|_{W^{1+\varepsilon,(1+\varepsilon)/2}_2(Q)},
$$

$$
\int_0^T \left\| \frac{df}{dt}(t) \right\|_{A_0^{-1}}^2 dt \times \int_0^T \left\| \frac{\partial f}{\partial t}(\cdot, t) \right\|_{W^{1,1}_2(0,1)}^2 dt \leq C \left\| \frac{\partial f}{\partial t} \right\|_{W^{2,1,-1/2}_2(Q)}^2,
$$

$$
\| A(0)u_0 \|_{B^{-1}} \leq \| (a(\cdot, 0) u_0')' \|_{L^2(0,\xi)}^2 + \frac{[a(\cdot, 0) u_0']^2_{\xi=\xi}}{K} + \| (a(\cdot, 0) u_0')' \|_{L^2(\xi,1)}^2,
$$

$$
\leq C(\| u_0 \|_{W^{2,1}_2(0,1)}^2 + \| u_0'' \|_{L^2(0,\xi)} + \| u_0'' \|_{L^2(\xi,1)}),
$$

$$
\left\| \frac{dA}{dt}(t) u(t) \right\|_{A_0^{-1}} = \sup_{v \in W^{2,1}_2(0,1)} \frac{\left\| f_0 \frac{\partial a}{\partial t} \frac{\partial u}{\partial x} \frac{dv}{dx} \right\|_{H^1}}{\left( c_1 \int_0^1 \left\| \frac{dv}{dx} \right\|_{L^2}^2 dx \right)^{1/2}} \leq C \left\| \frac{\partial u}{\partial x}(\cdot, t) \right\|_{L^2(0,1)},
$$

and

$$
\int_0^T \left\| \frac{dA}{dt}(t) u(t) \right\|_{A_0^{-1}}^2 dt \leq C(\| u_0 \|_{W^{2,1}_2(0,1)}^2 + \| f \|_{W^{2,1,-1/2}_2(Q)}^2) \leq C(\| u_0 \|_{W^{2,1}_2(0,1)}^2 + \| f \|_{L^2(0,1)}^2).$$
Using Eq. (16) we obtain the following representations of derivatives \( \partial^3 u / \partial x^3 \) and \( \partial^2 u / \partial x^2 \) for \((x, t) \in Q_i, \ i = 1, 2:\)

\[
\begin{align*}
\frac{\partial^3 u}{\partial x^3} &= \frac{1}{a} \left[ \frac{\partial^2}{\partial x^2} \left( \frac{\partial u}{\partial x} \right) - 2 \frac{\partial a}{\partial x} \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 a}{\partial x^2} \frac{\partial u}{\partial x} \right] = \frac{1}{a} \left( \frac{\partial^2 u}{\partial x \partial t} - f - \frac{\partial a}{\partial x} \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 a}{\partial x^2} \frac{\partial u}{\partial x} \right), \\
\frac{\partial^2 u}{\partial x^2} &= \frac{1}{a} \left[ \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right) - \frac{\partial a}{\partial x} \frac{\partial u}{\partial x} \right] = \frac{1}{a} \left( \frac{\partial u}{\partial t} - f - \frac{\partial a}{\partial x} \frac{\partial u}{\partial x} \right).
\end{align*}
\]

From here follows

\[
\left\| \frac{\partial^3 u}{\partial x^3} \right\|_{L^2(Q_i)} + \left\| \frac{\partial^2 u}{\partial x^2} \right\|_{L^2(Q_i)} \leq C \left( \left\| f \right\|_{W_2^{1/2}(Q)} + \left\| \frac{\partial^2 u}{\partial x \partial t} \right\|_{L^2(Q)} + \left\| \frac{\partial u}{\partial t} \right\|_{L^2(Q)} + \left\| \frac{\partial u}{\partial x} \right\|_{L^2(Q)} \right).
\]

From previous inequalities using Lemma 5 and Poincare inequality we obtain the result. \( \square \)

4. The difference problem

Let \( \omega_h \) be a uniform mesh on \((0, 1)\) with step \( h = 1/n, \omega_h^- = \omega_h \cup \{0\} \) and \( \bar{\omega}_h = \omega_h \cup \{0, 1\} \). Suppose for the simplicity that \( \bar{\xi} \) is a rational number. Then one can choose the step \( h \), so that \( \bar{\xi} \in \omega_h \). Define finite differences on the usual way:

\[
\begin{align*}
v_{\tau}(x, t) &= \frac{v(x, t) - v(x, t - \tau)}{\tau} = v_t(x, t - \tau), \\
v_{\bar{\xi}}(x, t) &= \frac{v(x, t) - v(x - h, t)}{h} = v_x(x - h, t), \\
v_{\bar{x}\bar{\xi}}(x, t) &= \frac{v(x + h, t) - 2v(x, t) + v(x - h, t)}{h^2}.
\end{align*}
\]

The problem (14) can be approximated on the mesh \( Q_{h, \tau} = \bar{\omega}_h \times \bar{\omega}_\tau \) by the implicit difference scheme with averaged right-hand side

\[
[1 + K \delta_h(x - \bar{\xi})] v_{\tau} - \frac{1}{2} ((av_x)_{\bar{\xi}} + (av_{\bar{\xi}})_x) = T_x^2 T_t^- f, \quad (x, t) \in \omega_h \times \omega_t^+, \\
v(0, t) = 0, \quad v(1, t) = 0, \quad t \in \omega_t^+, \quad v(x, 0) = u_0(x), \quad x \in \bar{\omega}_h,
\]

where

\[
\delta_h(x - \bar{\xi}) = \begin{cases} 
0, & x \in \omega_h \setminus \{\bar{\xi}\}, \\
1/h, & x = \bar{\xi},
\end{cases}
\]

is the mesh Dirac function, and \( T_x, T_t^- \) are Steklov averaging operators [15]:

\[
T_x f(x, t) = T_x^- f(x + h/2, t) = T_x^+ f(x - h/2, t) = \frac{1}{h} \int_{x-h/2}^{x+h/2} f(x', t) \, dx',
\]

\[
T_t^- f(x, t) = T_t^+ f(x, t - \tau) = \frac{1}{\tau} \int_{t-\tau}^{t} f(x, t') \, dt'.
\]
Notice that these operators are commutative and map derivatives into finite differences, for example
\[ T^2_x \frac{\partial^2 u}{\partial x^2} = u_{xx}, \quad T^-_t \frac{\partial u}{\partial t} = u_t. \]

Let \( H_h \) denote the set of functions defined on the mesh \( \mathcal{O}_h \) and equal to zero at \( x = 0 \) and 1. We define the inner product and norms
\[
(v, u)_h = \sum_{x \in \mathcal{O}_h} v(x) u(x),
\]
\[
\|v\|_h = \|v\|_{L^2} = \left( \sum_{x \in \mathcal{O}_h} v^2(x) \right)^{1/2},
\]
\[
\|v\|_{\tilde{W}^{1,1/2}_h(Q_h)} = \left( \tau \sum_{t \in \mathcal{O}_t} |v_x(\cdot, t)|^2 + \tau^2 \sum_{t \in \mathcal{O}_t} \sum_{t' \in \mathcal{O}_t, t' \neq t} \frac{\|v(\cdot, t) - v(\cdot, t')\|_{B_h}^2}{|t - t'|^2} \right)^{1/2}.
\]

The difference scheme (17) can be reduced to the form (12) setting \( A_h v = -0.5((a v_x)_x + (a v_\bar{x})_x), B_h v = [1 + K \delta_h (x - \bar{\xi})] v \) and \( A_{0h} v = -c_1 v_x \bar{x} \). For each \( v \in H_h \) we have
\[
\|v\|_{A_h}^2 = (A_h v, v)_h \leq \|v\|_{A_h}^2 = c_1 \|v_x\|_h^2, \quad \|v\|_{A_{0h}}^2 = \|v\|_{A_h}^2 + K v^2(\bar{\xi}),
\]
\[
\|v\|_{B_h^{-1}}^2 = (B_{h}^{-1} v, v)_h = h \sum_{x \in \mathcal{O}\backslash\{\bar{\xi}\}} v^2(x) + \frac{h^2}{K+h} v^2(\bar{\xi}).
\]

5. Convergence of the difference scheme

In this section, we shall prove convergence of the difference (17) in \( \tilde{W}^{1,1/2}_2 \) norm. Let \( u \) be the solution of the boundary value problem (14) and \( v \)—the solution of the difference problem (17). The error \( z = u - v \) satisfies finite difference scheme
\[
[1 + K \delta_h (x - \bar{\xi})]z_t + A_h z = \eta_{\bar{x}} + \psi_t, \quad (x, t) \in \mathcal{O}_h \times \mathcal{O}_t, \]
\[
z(0, t) = z(1, t) = 0, \quad t \in \mathcal{O}_t, \quad z(x, 0) = 0, \quad x \in \mathcal{O}_h,
\]
where
\[
\eta = T^+_x T^-_t \left( \frac{\partial u}{\partial x} \right) - \frac{1}{2} (a + a^+) u_x, \quad \psi = u - T^+_x u
\]
and \( a^+(x, t) = a(x + h, t). \)
Using Lemmas 3, 4, and the equality \(\|z\|_{A_{0h}}^2 = c_1 \|z_x\|_{B_h}^2\), we directly obtain the following a priori estimate for the solution of difference (18):

\[
\|z\|_{\bar{W}_2^{1,1/2}(Q_{h+1})} \leq C \left[ \tau \sum_{t \in t_0} \|\eta(\cdot, t)\|_{B_h}^2 + \tau^2 \sum_{t \in t_0} \sum_{t' \in t_0, t' \neq t} \frac{\|\psi(\cdot, t) - \psi(\cdot, t')\|_{B_{h-1}}^2}{|t - t'|^2} \right]^{1/2} + \tau \sum_{t \in t_0} \left( \frac{1}{t} + \frac{1}{T - t} \right) \|\psi(\cdot, t)\|_{B_{h-1}}^2.
\]

(19)

Therefore, in order to estimate the rate of convergence of the difference scheme (17), it is sufficient to estimate the right-hand sides of the inequality (19). In the sequel we shall assume that

\[a(x, t) \in W_2^{2,1}(Q), \quad 0 < c_1 \leq a(x, t) \leq c_2\]

and

\[u \in W_2^{3,3/2}(Q_1) \cap W_2^{3,3/2}(Q_2)\]

Note that the condition (20) is weaker than the corresponding assumption in Lemma 6.

First of all, we decompose the term \(\eta = \eta_1 + \eta_2 + \eta_3\), where

\[\eta_1 = T_x^+ T_t^- \left( a \frac{\partial u}{\partial x} - (T_x^+ T_t^- a) \left( T_x^+ T_t^- \frac{\partial u}{\partial x} \right) \right),\]

\[\eta_2 = \left[ T_x^+ T_t^- a - \frac{1}{2} (a + a^+) \right] \left( T_x^+ T_t^- \frac{\partial u}{\partial x} \right),\]

\[\eta_3 = \frac{1}{2} (a + a^+) \left( T_x^+ T_t^- \frac{\partial u}{\partial x} - u_x \right) \]

Also, we set \(\eta_1 = \eta_{1,1} + \eta_{1,2} + \eta_{1,3}\), where

\[\eta_{1,1}(x, t) = \frac{1}{2h^2 \tau^2} \int_a^x \int_x^{x+1} \int_t^{t+1} \int_{x'}^{x''} \int_{t'}^{t''} \frac{\partial a(x', t''')}{\partial t} \frac{\partial u(x', t'')}{\partial x} \frac{\partial u(x', t'')}{\partial x'} \, dt'''' \, dt''' \, dt'' \, dx'' \, dx',\]

\[\eta_{1,2}(x, t) = \frac{1}{2h^2 \tau^2} \int_a^x \int_x^{x+1} \int_t^{t+1} \int_{x'}^{x''} \int_{t'}^{t''} \int_{t'}^{t''} \left[ a(x', t'') - a(x'', t'') \right] \frac{\partial^2 u(x', t'')}{\partial t \partial x} \, dt''' \, dt'' \, dt' \, dx'' \, dx',\]

\[\eta_{1,3}(x, t) = \frac{1}{2h^2 \tau^2} \int_a^x \int_x^{x+1} \int_t^{t+1} \int_{x'}^{x''} \int_{x'}^{x''} \int_{t'}^{t''} \int_{t'}^{t''} \frac{\partial a(x'', t'')}{\partial x} \frac{\partial^2 u(x'', t')}{\partial x^2} \, dx''' \, dx'''' \, dx''' \, dx',\]
From the first integral representation we immediately obtain

$$|\eta_{1,1}(x, t)| \leq \frac{\tau}{2\sqrt{h\tau}} \left\| \frac{\partial a}{\partial t} \right\|_{L_2(g)} \left\| \frac{\partial u}{\partial x} \right\|_{C(\Bar{g})},$$

where $g = (x, x + h) \times (t - \tau, t)$. Summing over the nodes of the mesh $Q_{h\tau}$ and using imbedding $W^{2,1}_2(Q_i) \subset C(\Bar{Q}_i)$ ($i = 1, 2$) we obtain

$$\tau \sum_{t \in \mathcal{O}_i^2} ||\eta_{1,1}(|, t)||^2_h \leq C \tau^2 \|a\|^2_{W^{2,1}_2(Q)} \left( \|u\|^2_{W^{3,2}_2(Q_1)} + \|u\|^2_{W^{3,2}_2(Q_2)} \right).$$

Analogously one obtains

$$|\eta_{1,2}(x, t)| \leq \frac{\tau}{2\sqrt{h\tau}} \|a\|_{C(\Bar{g})} \left\| \frac{\partial^2 u}{\partial t \partial x} \right\|_{L_2(g)}$$

and

$$\tau \sum_{t \in \mathcal{O}_i^2} ||\eta_{1,2}(|, t)||^2_h \leq C \tau^2 \|a\|^2_{W^{2,1}_2(Q)} \left( \left\| \frac{\partial^2 u}{\partial t \partial x} \right\|^2_{L_2(Q)} \right).$$

From the third integral representation, using Hölder inequality one obtains

$$|\eta_{1,3}(x, t)| \leq \frac{h^2}{2\sqrt{h\tau}} \left\| \frac{\partial a}{\partial x} \right\|_{L_4(g)} \left\| \frac{\partial^2 u}{\partial x^2} \right\|_{L_4(g)}.$$

Summing over the nodes of the mesh $Q_{h\tau}$ and using imbeddings $W^{1,1/2}_2(Q) \subset L_4(Q)$, $W^{1,1/2}_2(Q_i) \subset L_4(Q_i)$ ($i = 1, 2$) one obtains

$$\tau \sum_{t \in \mathcal{O}_i^2} ||\eta_{1,3}(|, t)||^2_h \leq C h^4 \|a\|^2_{W^{2,1}_2(Q)} \left( \|u\|^2_{W^{3,2}_2(Q_1)} + \|u\|^2_{W^{3,2}_2(Q_2)} \right).$$

In such a way we finally obtain

$$\tau \sum_{t \in \mathcal{O}_i^2} ||\eta_1(|, t)||^2_h \leq C (h^4 + \tau^2) \|a\|^2_{W^{2,1}_2(Q)} \left( \|u\|^2_{W^{3,2}_2(Q_1)} + \|u\|^2_{W^{3,2}_2(Q_2)} \right). \quad (22)$$

Further,

$$\eta_2 = \left[ T^+ T^- a - \frac{1}{2} (a + a^+ \right) \left( T^+ T^- \frac{\partial u}{\partial x} \right) = \eta_{2,1} \left( T^+ T^- \frac{\partial u}{\partial x} \right).$$

The term $\eta_{2,1}$ is bounded linear functional of the argument $a \in W^{2,1}_2(g)$; further, $\eta_{2,1} = 0$ whenever $a$ is a polynomial of degree one in $x$ and constant function in $t$. Applying the Bramble–Hilbert lemma [4], we get

$$|\eta_{2,1}(x, t)| \leq \frac{C(h^2 + \tau)}{\sqrt{h\tau}} |a|_{W^{2,1}_2(g)}.$$
Applying imbedding \( W^{2,1}_2(Q) \subset C(\tilde{Q}) \) and summing over the mesh, we obtain

\[
\tau \sum_{t \in \Omega_t^+} |\eta_2(\cdot, t)|^2_h \leq C(h^4 + \tau^2) \|a\|^2_{W^{2,1}_2(Q)} \left( \|u\|^2_{W^{3,3/2}_2(Q_1)} + \|u\|^2_{W^{3,3/2}_2(Q_2)} \right).
\]  

(23)

The term \( \eta_3 \) can be represented on the following way:

\[
\eta_3(x, t) = \frac{1}{2h^2} [a(x, t) + a(x + h, t)] \int_x^{x+h} \int_{t-h}^t \int_t^{t-h} \frac{\partial^2 u(x', t'')}{\partial x' \partial t''} \, dt'' \, dt' \, dx'.
\]

From this representation, using Cauchy–Schwartz inequality, we have

\[
|\eta_3(x, t)| \leq \frac{\tau}{\sqrt{h^2}} \|a\|_{C(\tilde{Q})} \left\| \frac{\partial^2 u}{\partial x' \partial t} \right\|_{L^2(g)}.
\]

Summing over the mesh we have

\[
\tau \sum_{t \in \Omega_t^+} |\eta_3(\cdot, t)|^2_h \leq C \tau^2 \|a\|^2_{W^{2,1}_2(Q)} \left( \|u\|^2_{W^{3,3/2}_2(Q_1)} + \|u\|^2_{W^{3,3/2}_2(Q_2)} \right).
\]

(24)

Finally, from (22)–(24) we get the next estimate of the term \( \eta \)

\[
\tau \sum_{t \in \Omega_t^+} |[\eta(\cdot, t)]|^2 \leq C(h^4 + \tau^2) \|a\|^2_{W^{2,1}_2(Q)} \left( \|u\|^2_{W^{3,3/2}_2(Q_1)} + \|u\|^2_{W^{3,3/2}_2(Q_2)} \right).
\]  

(25)

Further, we directly obtain the following integral representations of the term \( \psi \):

\[
\psi(x, t) = \frac{1}{h} \int_{x-h}^{x} \int_{x'}^{x'} \int_x^{x''} \left( 1 - \frac{|x' - x|}{h} \right) \frac{\partial^2 u(x''', t)}{\partial x^2} \, dx''' \, dx' \, dx' \quad \text{for } x \neq \xi,
\]

(26)

\[
\psi(\xi, t) = \frac{1}{h} \int_{\xi-h}^{\xi} \int_{\xi}^{\xi} \int_{\xi}^{x''} \left( 1 + \frac{x' - \xi}{h} \right) \frac{\partial^2 u(x'''', t)}{\partial x^2} \, dx''' \, dx' + \frac{1}{h} \int_{\xi}^{\xi+h} \int_{x'}^{\xi} \int_{\xi}^{x''} \left( 1 - \frac{x' - \xi}{h} \right) \frac{\partial^2 u(x'''', t)}{\partial x^2} \, dx''' \, dx' - \frac{h}{6} \left[ \frac{\partial u}{\partial x} \right]_{(\xi, t)}.
\]

(27)
We estimate the fractional order seminorm of $\psi$ in (19) as follows:

$$
\tau^2 \sum_{t=0}^{T} \sum_{t'=0}^{T} \frac{\|\psi(\cdot, t) - \psi(\cdot, t')\|^2_{B_{h-1}^{-1}}}{|t-t'|^2}
\leq 2 \tau^2 \sum_{t=0}^{T} \sum_{t'=0}^{T} \frac{\|\psi(\cdot, t) - \psi(\cdot, t')\|^2_{B_{h-1}^{-1}}}{|t-t'|^2}
\leq 4 \tau^2 \sum_{t=0}^{T} \sum_{t'=0}^{T} \frac{\|T_t^- \psi(\cdot, t) - T_{t'}^+ \psi(\cdot, t')\|^2_{B_{h-1}^{-1}}}{|t-t'|^2} + 8 \tau^2 \sum_{t=0}^{T} \sum_{t'=0}^{T} \frac{\|\psi(\cdot, t) - T_t^- \psi(\cdot, t)\|^2_{B_{h-1}^{-1}}}{|t-t'|^2}.
$$

Further, we have

$$
\|T_t^- \psi(\cdot, t) - T_{t'}^+ \psi(\cdot, t')\|^2_{B_{h-1}^{-1}} = \left\| \frac{1}{\tau^2} \int_{t-\tau}^{t} \int_{t'}^{t'+\tau} [\psi(\cdot, s) - \psi(\cdot, s')] \, ds' \, ds \right\|^2_{B_{h-1}^{-1}}
\leq \frac{1}{\tau^4} \int_{t-\tau}^{t} \int_{t'}^{t'+\tau} \|\psi(\cdot, s) - \psi(\cdot, s')\|^2_{B_{h-1}^{-1}} \, ds' \, ds 
\times \int_{t-\tau}^{t} \int_{t'}^{t'+\tau} |s-s'|^2 \, ds' \, ds.
$$

Hence, using the integral representations (26) and (27), and summing over the mesh $\bar{\Omega}_\tau$, we get

$$
\tau^2 \sum_{t=0}^{T} \sum_{t'=0}^{T} \frac{\|T_t^- \psi(\cdot, t) - T_{t'}^+ \psi(\cdot, t')\|^2_{B_{h-1}^{-1}}}{|t-t'|^2}
\leq C h^4 \left[ \int_0^T \int_0^T \frac{\|\partial^2 u(\cdot, t) - \partial^2 u(\cdot, t')\|^2_{L_2(0, \xi)}}{|t-t'|^2} \, dt' \, dt 
+ \int_0^T \int_0^T \frac{\|\partial^2 u(\cdot, t) - \partial^2 u(\cdot, t')\|^2_{L_2(\xi, 1)}}{|t-t'|^2} \, dt' \, dt 
+ \int_0^T \int_0^T \left[ \left( \frac{\partial u}{\partial x} \right)_{(\xi, t)} - \left( \frac{\partial u}{\partial x} \right)_{(\xi, t')} \right]^2 \, dt' \, dt \right].
$$
Using the trace theorem (see [10]) and imbedding theorem we have

\[
\int_0^T \int_0^T \left| \frac{\partial u}{\partial x} (\xi, \tau) - \frac{\partial u}{\partial x} (\xi', \tau') \right|^2 |\tau - \tau'| \, d\tau' \, d\tau = \left\| \frac{\partial u (\xi, \tau)}{\partial x} \right\|_{W_2^{1/2}(0,T)}^2.
\]

Therefore, we get

\[
\tau^2 \sum_{t = \tau}^T \sum_{\tau' = 0}^{t-\tau} \frac{\| T^-_t \psi (\cdot, \tau) - T^+_t \psi (\cdot, \tau') \|^2_{B_h^{-1}}}{|t - \tau'|^2} \leq C h^4 \left( \| u \|^2_{W_2^{3,3/2}(Q_1)} + \| u \|^2_{W_2^{3,3/2}(Q_2)} \right). \tag{28}
\]

The expression \( \psi - T^-_t \psi \) takes the form

\[
\psi(x, t) - T^-_t \psi(x, t) = \frac{1}{h} \int_{t-\tau}^t \int_{x-h}^{x+h} \int_{x'}^{x} \int_{t'}^t \left( 1 - \frac{|x' - x|}{h} \right) \frac{\partial^2 u(x'', t'')}{\partial x \partial t} \, dx'' \, dx' \, dt' \, dt,
\]

from where we get

\[
\tau^2 \sum_{t = \tau}^T \sum_{\tau' = 0}^{t-\tau} \frac{\| \psi (\cdot, \tau) - T^-_t \psi (\cdot, \tau') \|^2_{B_h^{-1}}}{|t - \tau'|^2} \leq C h^2 \tau \left\| \frac{\partial^2 u}{\partial x \partial t} \right\|_{L_2(Q)}. \tag{29}
\]

A similar estimate is valid for

\[
\tau^2 \sum_{t = \tau}^T \sum_{\tau' = 0}^{t-\tau} \frac{\| \psi (\cdot, \tau') - T^+_t \psi (\cdot, \tau) \|^2_{B_h^{-1}}}{|t - \tau'|^2}.
\]

Finally, from (26) and (27) it follows

\[
\tau \sum_{t \in (0, T]} \left( \frac{1}{t} + \frac{1}{T - t} \right) \| \psi (\cdot, t) \|^2_{B_h^{-1}} \leq C h^4 \log \frac{1}{\tau} \max_{t \in (0, T]} \left( \left\| \frac{\partial^2 u (\cdot, t)}{\partial x^2} \right\|_{L_2(0, \xi)}^2 + \left\| \frac{\partial^2 u (\cdot, t)}{\partial x^2} \right\|_{L_2(\xi, 1)}^2 + \left| \frac{\partial u (\cdot, t)}{\partial x} \right|_{L_2(\xi, 1)}^2 \right)
\]

\[
\leq C h^4 \log \frac{1}{\tau} \left( \| u \|^2_{W_2^{3,3/2}(Q_1)} + \| u \|^2_{W_2^{3,3/2}(Q_2)} \right). \tag{30}
\]

Now from (19) and (25)–(30) we get the following result.
Theorem. Assuming that the data and the corresponding solution of the initial-boundary value problem (14) satisfies conditions (20) and (21), the difference scheme (17) satisfies the error bound
\[
\|u - v\|_{\tilde{W}^{1/2}_{1/2}(Q_{h\tau})} \leq C(h^2 + \tau) \left( \|a\|_{W^{2,1}_2(Q)} + \sqrt{\log \frac{1}{\tau}} \right) \left( \|u\|_{W^{3,3/2}_{2}(Q_1)} + \|u\|_{W^{3,3/2}_{2}(Q_2)} \right).
\] (31)

This estimate is “almost” compatible with the smoothness of the coefficient and solution of the differential problem (14). (The compatibility is spoiled only by the term \(\sqrt{\log \frac{1}{\tau}}\), which slowly increases when \(\tau \to 0\)). In the case \(c_1 h^2 \leq \tau \leq c_2 h^2\), the error bound (31) takes the form
\[
\|u - v\|_{\tilde{W}^{1/2}_{1/2}(Q_{h\tau})} \leq Ch^2 \left( \|a\|_{W^{2,1}_2(Q)} + \sqrt{\log \frac{1}{h}} \right) \left( \|u\|_{W^{3,3/2}_{2}(Q_1)} + \|u\|_{W^{3,3/2}_{2}(Q_2)} \right).
\]

Remark 1. An analogous result also holds for the operator \(B\) defined with
\[
Bu = [b(x) + K \delta(x - \xi)]u, \quad 0 < k_1 \leq b(x) \leq k_2.
\]

Remark 2. It is possible to derive analogous convergence rate estimate in the energetic norm corresponding to the norm \(W^{2,1}_2\).

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