On Oscillation of Second-Order "Sublinear" Differential Equations

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1. Introduction

The behaviour of solutions, as oscillatory or non-oscillatory, of differential equations of the form

\[ x''(t) + a(t) f(x) = 0, \quad t \in [0, \infty), \]

where \( a(t) \) and \( f(x) \) are functions satisfying certain prescribed general conditions, have received the attention of many authors in recent years. A fairly extensive bibliography of the earlier works can be found, for instance, in the survey article by Wong [1]. An example of particular interest is the differential equation

\[ x'' + t \lambda \sin t \ |x|^\gamma \text{sgn} x = 0, \quad (1.1) \]

where \( \lambda \) and \( \gamma \) are constants, with \( \gamma > 0 \). In a series of papers [2–4] Butler showed that, with \( \gamma > 1 \), the solutions of (1.1) are oscillatory for \( \lambda \geq -1 \). However, with \( 0 < \gamma < 1 \), i.e., in the sublinear case, he succeeded only in proving the oscillation of the equation for \( \lambda \geq 1 \). He made the conjecture that, with \( 0 < \gamma < 1 \), the equation is oscillatory for \( \lambda \geq -\gamma \). We are indebted to Dr. James S. W. Wong for drawing our attention to this problem. We learned from him that the conjecture which remained unsettled for quite some time has just been proved by him and M. K. Kwong [7].

In this paper, we consider the question of oscillation for differential equations of the more general form:

\[ x''(t) + t^\lambda p(t) f(x) = 0, \quad t \in [0, \infty), \quad (1.2) \]

where \( p(t) \in C[0, \infty) \) is periodic, with period \( \omega \), such that \( \int_0^\omega p(t) \, dt \geq 0 \), and is not identically zero, and \( f(x) \) is an odd function subject to the conditions

\[ x^{-\alpha} f(x) \text{ is nondecreasing in } (0, \infty) \quad (1.2a) \]

\[ x^{-\beta} f(x) \text{ is nonincreasing in } (0, \infty) \quad (1.2b) \]
for some constants $\alpha, \beta$, where $0 < \alpha \leq \beta < 1$. We shall refer to the above properties of $f(x)$ by writing $f(x) \sim [\alpha, \beta]$. The properties of such functions have been studied in some detail by Chen in an earlier paper [5]. It is known that $f(x)$ is absolutely continuous in any finite interval $(0, N)$ and that

$$0 < \alpha \frac{f(x)}{x} \leq f'(x) \leq \beta \frac{f(x)}{x}, \quad x > 0,$$

where $f'(x)$ exists almost everywhere. Furthermore, if $x(t) > 0$, $t \geq t_0 > 0$ for some $t_0$, is a solution of (1.2), then

$$\frac{d}{dt} \{f(x(t))\} = f'(x(t)) x'(t)$$

almost everywhere, where the left-hand side is taken to be zero whenever $x'(t) = 0$ ([6, Lemma 1], with slight modification of the proof therein).

Equation (1.2) clearly includes (1.1) as a special case. Our main result is stated in the following theorem.

**Theorem.** If $p(t)$ and $f(x)$ satisfy the conditions given immediately after (1.2), then the continuable solutions of (1.2) are oscillatory for $0 < \lambda < 1$.

2. **Some Lemmas**

**Lemma 1.** Let $g(t)$ and $p(t)$ be functions satisfying the following conditions:

(i) $g(t)$ is continuous and strictly decreasing to zero on $[0, \infty)$ as $t \to \infty$.

(ii) $p(t) \neq 0$ is continuous and periodic on $[0, \omega)$, with period $\omega$ and zero integral mean, i.e., $\int_0^\omega p(t) \, dt = 0$.

Then there exist $t_1$ and $t_2 \in [0, \omega]$ such that

(A) $\int_{t_1}^{t_1+\omega} g(\tau) \, p(\tau) \, d\tau \geq \varepsilon_1 > 0$, for $t \geq t_1 + \omega$ and a constant $\varepsilon_1$,

(B) $\int_{t_2}^{t_2+\omega} g(\tau) \, p(\tau) \, d\tau \leq -\varepsilon_2 < 0$, for $t \geq t_2 + \omega$ and a constant $\varepsilon_2$, and

(C) $|\int_{t_0}^{\infty} g(\tau) \, p(\tau) \, d\tau| \leq K g(t_0)$ for some constant $K > 0$ and all $t_0 \in [0, \infty)$. 

Proof: Let \( p_1(t) = \int_0^t p(\tau) \, d\tau \). Then \( p_1(t) \) is continuous and periodic, with period \( \omega \). Hence there exist \( t_1 \) and \( t_2 \) such that

\[
m = p_1(t_1) \leq p_1(t) \leq p_1(t_2) = M, \quad \text{for all } t \geq 0,
\]

where \( m \) and \( M \) are, respectively, the proper minimum and proper maximum of \( p_1(t) \), with \( m < M \) since \( p(t) \neq 0 \).

Also let

\[
\bar{p}_1(t) = \int_{t_1}^{t_1 + t} p(\tau) \, d\tau.
\]

Then \( \bar{p}_1(t) = \bar{p}_1(t + \omega) \) and \( \bar{p}_1(t) \geq 0 \) for \( t \geq 0 \).

To prove (A), we note that

\[
\int_{t_1}^t g(\tau) \, p(\tau) \, d\tau = \int_0^{t - t_1} g(t_1 + \tau) \, d\bar{p}_1(\tau)
\]

\[
= g(t) \bar{p}_1(t - t_1) - \int_0^\omega \bar{p}_1(\tau) \, dg(t_1 + \tau) - \int_0^{t - t_1} \bar{p}_1(\tau) \, dg(t_1 + \tau).
\]

Since \( \bar{p}_1(\tau) \geq 0 \) for \( \tau \geq 0 \) and \( g(\tau) \downarrow 0 \) (strictly decreasing) and \( \bar{p}_1(\tau) \neq 0 \) in \([0, \omega]\), we have

\[
\int_{t_1}^t g(\tau) \, p(\tau) \, d\tau \geq -\int_0^\omega \bar{p}_1(\tau) \, dg(t_1 + \tau) = \epsilon_1 > 0,
\]

for \( t \geq t_1 + \omega \).

For the proof of (B), it is easily seen that (B) is just equivalent to (A) when we replace \( p(t) \) by \(-p(t)\) and \( t_1 \) by \( t_2 \).

To prove (C), we make use of the second mean value theorem so that

\[
\int_{t_0}^t g(\tau) \, p(\tau) \, d\tau = g(t_0) \int_{t_0}^{t'} p(\tau) \, d\tau + g(t) \int_{t'}^t p(\tau) \, d\tau
\]

for some \( t' \in [t_0, t] \).

Hence

\[
\int_{t_0}^t g(\tau) \, p(\tau) \, d\tau = p_1(t')(g(t_0) - g(t)) + p_1(t) g(t) - g(t_0) p_1(t_0)
\]

and

\[
\left| \int_{t_0}^t g(\tau) \, p(\tau) \, d\tau \right| \leq 3Mg(t_0), \quad \text{for } t \geq t_0.
\]
From the above inequality, we conclude that the integral

$$\mathcal{Y}(t_0, t) = \int_{t_0}^{t} g(\tau) p(\tau) d\tau$$

converges and

$$\mathcal{Y}(t_0, t) = O(g(t_0)), \quad \text{as} \quad t \to \infty.$$

**Remark.** It is easy to see that results similar to (A) and (B) can also be obtained for $t$ in any finite interval $[0, N]$ if $g(t) \uparrow A \leq \infty$ (strictly increasing) as $t \to \infty$.

**Lemma 2.** Consider the functions

$$\phi(t) = \int_{t_0}^{t} \tau^\lambda p(\tau) d\tau, \quad 0 < \lambda < 1,$$

and

$$\phi_1(t) = \int_{t_0}^{t} \phi(\tau) d\tau,$$

where $p(t)$ satisfies the conditions of Lemma 1. Then

(i) $\phi(t) \geq K t^\lambda$ in "half-measure" for some constant $K > 0$. By this, we mean that on any interval $[\bar{t}, \bar{t} + \omega]$, for $\bar{t}$ sufficiently large, there exists a subset $E$ of the interval, such that $m(E) \geq \sigma \omega$ for some $\sigma$, $0 < \sigma < 1$, independent of $\bar{t}$.

(ii) $|\phi_1(t)| \leq K' t$, for some constant $K' > 0$ and $t$ sufficiently large.

**Proof.** We introduce the following functions:

(a) $p_1(t) = \int_{t_0}^{t} p(\tau) d\tau$.

(b) $p_1^*(t) = p_1(t) - c_0$, with constant $c_0$ chosen so that $p_1^*(t)$ is of zero integral mean.

(c) $p_2^*(t) = \int_{t_0}^{t} p_1^*(\tau) d\tau$.

(d) $p_2^*(t) = p_2^*(t) - c_1$, with constant $c_1$ chosen so that $p_2^*(t)$ is of zero integral mean.

Then, on integrating by parts, we have

$$\phi(t) = c_2 + t^\lambda p_1^*(t) - \lambda \int_{t_0}^{t} t^{\lambda - 1} p_1^*(\tau) d\tau,$$

where $c_2$ is a constant depending only on $t_0$ for given $\lambda$ and $p(t)$. Since $t^{\lambda - 1}$
decreases steadily to zero in \([1, \infty)\), by Lemma 1, there exist \(t_1\) and \(t_2\) such that

\[
\int_{t_1}^{t_2} \tau^{\lambda - 1} p^{*}_1(\tau) \, d\tau > 0 \quad \text{and} \quad \int_{t_1}^{t_2} \tau^{\lambda - 1} p^{*}_2(\tau) \, d\tau < 0
\]

for all \(t \geq \max(t_1 + \omega, t_2 + \omega)\). Hence for such \(t\) we have

\[
c'_3 + t^\lambda p^*_1(t) \leq \phi(t) \leq c'_3 + t^\lambda p^*_2(t),
\]

where

\[
c'_3 = c_2 - \lambda \int_{t_0}^{t_1} \tau^{\lambda - 1} p^*_1(\tau) \, d\tau
\]

and

\[
c_3 = c_2 - \lambda \int_{t_0}^{t_1} \tau^{\lambda - 1} p^*_2(\tau) \, d\tau.
\]

Since \(p^*_1(t)\) is periodic, with period \(\omega\) and zero integral mean, and is not identically zero, there exists on any arbitrary interval \([\bar{t}, \bar{t} + \omega], \bar{t} \in [0, \infty)\), a subset \(E\) such that

(i) \(p^*_1(t) > \delta > 0, \, t \in E,\)

(ii) \(m(E) = \sigma \omega, \, 0 < \sigma < 1,\)

where \(\delta, \sigma\) are constants independent of \(\bar{t}\). Thus for sufficiently large \(\bar{t}\), say, \(\bar{t} \geq T\), we have from (2.1)

\[
\phi(t) \geq c'_3 + t^\lambda p^*_1(t) \geq \frac{1}{2} t^\lambda p^*_1(t) > K t^\lambda,
\]

for some constant \(K > 0\), whenever \(t \in E \subset [\bar{t}, \bar{t} + \omega]\).

Next, consider \(\phi_1(t)\). On integrating by parts and expressing the functions in terms of \(p^*_1(t)\) and \(p^*_2(t)\), we have

\[
\phi_1(t) = c_4 + c_5 t + (1 + \lambda) \int_{t_0}^{t} \tau^{\lambda - 1} p^*_2(t) \, d\tau - \lambda (1 + \lambda) \int_{t_0}^{t} \tau^{\lambda - 1} p^*_2(\tau) \, d\tau
\]

\[
- \lambda t \int_{t_0}^{t} \tau^{\lambda - 1} p^*_1(\tau) \, d\tau,
\]

where \(c_4\) and \(c_5\) are constants. By (C) of Lemma 1, we have

\[
\int_{t_0}^{t} \tau^{\lambda - 1} p^*_1(\tau) \, d\tau \leq K t_0^{\lambda - 1} \quad \text{and} \quad \int_{t_0}^{t} \tau^{\lambda - 1} p^*_2(\tau) \, d\tau \leq K t_0^{\lambda - 1}
\]
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for some constant $K > 0$. Also $p^2(t)$ is bounded for all $t$. Hence

$$|\phi_1(t)| \leq K't$$

for some constant $K' > 0$ and $t$ sufficiently large, say, $t \geq T$.

3. PROOF OF MAIN THEOREM

Proof: Suppose that, on the contrary, there is a continuable solution $x(t)$ of (1.2) which is non-oscillatory. Then, without loss of generality, we may assume that $x(t) > 0$ for $t$ sufficiently large. In terms of the transformation $z = x'/f'(x)$, (1.2) can be written as

$$z'(t) = -t^4p(t) - f'(x(t)) z^2(t). \quad (3.1)$$

Consider first the proof of the theorem for $p(t)$ with zero integral mean. Integrating (3.1), we have

$$z(t) = z(t_0) - \phi(t) - \mathcal{H}(t), \quad (3.2)$$

where

$$\phi(t) = \int_{t_0}^{t} t^4p(\tau) \, d\tau$$

$$\mathcal{H}(t) = \int_{t_0}^{t} f'(x(\tau)) z^2(\tau) \, d\tau.$$

Again integrating (3.2), we have

$$\int_{t_0}^{t} z(\tau) \, d\tau = z(t_0)(t - t_0) - \phi_1(t) - \mathcal{H}_1(t), \quad (3.3)$$

where

$$\phi_1(t) = \int_{t_0}^{t} \phi(\tau) \, d\tau \quad \text{and} \quad \mathcal{H}_1(t) = \int_{t_0}^{t} \mathcal{H}(\tau) \, d\tau.$$

In view of (1.2c), we have two possible cases:

Case I. $\mathcal{H}(t) \uparrow \infty$ as $t \to \infty$. Then, for $t$ sufficiently large, we have

$$\mathcal{H}_1(t) > Nt$$

for any arbitrarily large positive constant $N$. By Lemma 2, $|\phi_1(t)| \leq K't$, for $t$ sufficiently large. Hence (3.3) implies that
where $\overline{N}$ can be any arbitrarily large positive constant. However,

$$\int_{x_0}^{x} \frac{du}{f(u)} = \frac{x}{f(x)} - \frac{x_0}{f(x_0)} + \int_{x_0}^{x} \frac{uf''(u)}{f^2(u)} du$$

so that using (1.2c) we have

$$\frac{1}{1-\alpha} \left( \frac{x}{f(x)} - \frac{x_0}{f(x_0)} \right) \leq \int_{x_0}^{x} \frac{du}{f(u)} \leq \frac{1}{1-\beta} \left( \frac{x}{f(x)} - \frac{x_0}{f(x_0)} \right).$$

Hence it follows from (3.4) that

$$\frac{1}{1-\alpha} \left( \frac{x}{f(x)} - \frac{x_0}{f(x_0)} \right) \leq -\overline{N}t, \quad \text{as} \quad t \to \infty.$$ 

By assumption, $x(t) > 0$ and $f(x) > 0$, and we have a contradiction.

**Case II.** $\mathcal{R}(t) \uparrow A < \infty$ for $A > 0$ as $t \to \infty$. From (3.3) and Lemma 2, again we have

$$\int_{t_0}^{t} z(\tau) d\tau = 0(t), \quad \text{as} \quad t \to \infty.$$ 

Therefore it follows from (3.5) that

$$\frac{x}{f(x)} \leq K_1 t \quad \text{and} \quad \frac{f(x)}{x} \geq K_1' t^{-1}$$

for some constants $K_1, K_1' > 0$ and $t$ sufficiently large. On the other hand from (3.2) and Lemma 2, we have

$$|z(t)| \geq K_2 t^\lambda \quad \text{in "half-measure"}$$

for $K_2 > 0$ and $t$ sufficiently large, say, $t \geq T$. Hence

$$\mathcal{R}(t) = \int_{t_0}^{t} f'(x(\tau)) z^2(\tau) d\tau \geq \alpha \int_{t_0}^{t} \frac{f(x(\tau))}{x(\tau)} z^2(\tau) d\tau \geq K_3 \int_{t_0}^{t} \tau^{2\lambda-1} d\tau, \quad t \geq T.$$
for some constant $K_3 > 0$. Therefore $\mathcal{H}(t) \uparrow \infty$ as $t \to \infty$ and we have a contradiction.

If $(1/\omega) \int_0^\infty p(\tau) \, d\tau = c > 0$, define $p^*(t)$ by $p^*(t) = p(t) - c$ so that $p^*(t)$ is of zero integral mean. Equation (3.2) now becomes

$$z(t) = z(t_0) - \frac{c}{1 + \lambda} (t^{\lambda+1} - t_0^{\lambda+1}) - \phi(t) - \mathcal{H}(t),$$  \hspace{1cm} (3.6)

where $\phi(t)$, now defined with respect to $p^*(t)$, satisfies (2.1). Hence

$$\lim_{t \to \infty} z(t) = \lim_{t \to \infty} \frac{x'(t)}{f(x(t))} = -\infty.$$

By an argument similar to that for Case I, we again arrive at a contradiction. Hence the theorem is proved.

Remark. When $p(t)$ is of zero integral mean, our result is actually true also for an even function $f(x)$, because in that case Eq. (1.2) remains unchanged when $x(t)$ and $p(t)$ are replaced by $-x(t)$ and $-p(t)$ simultaneously so that in the proof we can still assume, without loss of generality, that $x(t) > 0$ for $t$ sufficiently large, and the proof remains valid.

REFERENCES