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World-line deviation and spinning particles

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Abstract

A set of world-line deviation equations is derived in the framework of Mathisson–Papapetrou–Dixon description of pseudo-classical spinning particles. They generalize the geodesic deviation equations. We examine the resulting equations for particles moving in the space–time of a plane gravitational wave.

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1. Introduction

The geodesic deviation equation,

$$\frac{D^2 n^\mu}{D\tau^2} = -R^\mu{}_{\alpha\nu\beta} v^\alpha n^\nu v^\beta, \tag{1}$$

is one of the well-known equations of general relativity. This equation shows the role of the space–time curvature on the motion of test particles and has important applications, namely, it is used for calculating relative accelerations of nearby particles in an observer-independent manner, and may be integrated to give the Lyapunov exponent in the study of chaotic behaviour of particle's orbits.

There are several ways to derive this equation. One standard derivation is as follows. Consider a two parameter family of geodesic $x^\mu(\tau;\lambda)$ in which τ is the parameter along the orbits and λ characterize different orbits. To obtain the separation of two nearby geodesics, one of them is taken as "fiducial" geodesic described by, say, x^μ, and the other by $x^\mu + \lambda n^\mu$, where λn^μ represents the separation. The equation for n^μ is then obtained by inserting $x^\mu + \lambda n^\mu$ into the geodesic equation, comparing terms linear in λ and neglecting $O(\lambda^2)$. This idea have been used in [1] to obtain generalized geodesic deviation equations by considering expansions containing higher orders of λ. These generalized equations have been applied, for example, in [1] to the problem of closed orbital motion of test particles in the Kerr space–time and in [2] to the orbital motion in Schwarzchild metric. Other interesting generalizations may be found in [3,4]. Another way to derive the equation is by varying a suitable action [5]. This method was used in [6] to generalize the equation to the so-called "string deviation equation". The quantization of the geodesic

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equation was discussed in [7] and generalizing the equation to Kaluza–Klein theories was done in [8]. Some other aspects or applications of the equation have been discussed in [9].

When forces other than gravity are present, or the particle has some internal structure, it would no longer move along geodesics in general. In these situations a "world-line deviation" equation may be obtained by modifying the geodesic deviation equation by taking the effect of the matter field or internal structure into account. Thus using methods described in the previous paragraph, a world-line deviation equation was obtained in [10] for the of motion of charged particles in the framework of Einstein–Maxwell theory. Another generalization of this type was made in [11] (see also [12]) for describing spinning particles and was used there to study epicycles.

The equations used in [11] to determine the particle's trajectories are simplified version of Mathisson–Papapetrou–Dixon (MPD) equations [13]. The later equations have been widely used to study the motion of spinning particles (e.g., see [14] and references therein). The aim of the present work is to obtain a world-line deviation in the framework of MPD description of spinning particles. In the following sections we apply the prescription mentioned earlier to obtain the equations, and provide an example in which they could be integrated analytically. Throughout the work, $1, 2, 3, 4$ would stand for u, v, x, y as indices,

$$R^{\mu}{}_{\nu\alpha\beta} = \partial_\alpha \Gamma^{\mu}{}_{\nu\beta} - \partial_\beta \Gamma^{\mu}{}_{\nu\alpha} + \Gamma^{\mu}{}_{\alpha\delta}\Gamma^{\delta}{}_{\nu\beta} - \Gamma^{\mu}{}_{\beta\delta}\Gamma^{\delta}{}_{\nu\alpha},$$

$D/D\tau$ and ∇ represent covariant derivative, the space–time signature would be $(-+++)$, and $[\mu, \nu]$ stands for $\mu\nu - \nu\mu$.

2. The equations

The motion of a spinning particle is described by MPD equations

$$\frac{Dp^\mu}{D\tau} = -\frac{1}{2} R^{\mu}{}_{\nu\kappa\lambda} v^\nu s^{\kappa\lambda}, \tag{2}$$

$$\frac{Ds^{\mu\nu}}{D\tau} = p^{[\mu} v^{\nu]}, \tag{3}$$

$$p_\mu s^{\mu\nu} = 0, \tag{4}$$

in which $v^\mu = dx^\mu/d\tau$ is the particle's four-velocity, p^μ its four-momentum and $s^{\mu\nu}$ its spin tensor. We confine ourselves to time-like orbits with

$$v_\mu v^\mu = -1. \tag{5}$$

MPD equations guarantee that the particle's mass and spin are conserved

$$p_\mu p^\mu = \text{const} = -m^2, \tag{6}$$

$$\frac{1}{2} s_{\mu\nu} s^{\mu\nu} = \text{const} = s^2. \tag{7}$$

Now consider a set $x^\mu(\tau; \lambda)$ describing the world-lines of spinning particles of the same spin-to-mass ratios and define

$$\frac{Ds^{\mu\nu}}{D\lambda} = J^{\mu\nu},$$

$$\frac{Dp^\mu}{D\lambda} = j^\mu.$$

Inserting $x^\mu(\tau) + \lambda n^\mu(\tau)$ into (2)–(4) and looking for λ terms, we obtain the following equations:

$$\frac{Dj^\mu}{D\tau} = -R^{\nu}{}_{\beta\alpha\kappa} v^\kappa n^\alpha p^\beta - \frac{1}{2} R^{\mu}{}_{\nu\alpha\beta} \frac{Dn^\nu}{D\tau} s^{\alpha\beta} - \frac{1}{2} R^{\mu}{}_{\nu\alpha\beta} v^\nu J^{\alpha\beta} - \frac{1}{2} \nabla_\kappa R^{\mu}{}_{\nu\alpha\beta} n^\kappa v^\nu s^{\alpha\beta}, \tag{8}$$

$$\frac{DJ^{\mu\nu}}{D\tau} = s^{\kappa[\mu} R^{\nu]}{}_{\kappa\alpha\beta} n^\alpha v^\beta + p^{[\mu} \frac{Dn^{\nu]}}{D\tau} + j^{[\mu} v^{\nu]}, \tag{9}$$

$$s_{\mu\nu} j^\nu + J_{\mu\nu} p^\nu = 0, \tag{10}$$

respectively. If we turn off the spin, all terms except the first in the right-hand side of (8) vanish and the geodesic deviation equation results. If we set $p^\mu = m v^\mu$, Eqs. (8)–(10) reduce to those of [11]. It can be seen from the above equations that there is no evolution equation for n^μ, it should be obtained indirectly from Eqs. (8)–(10). The situation resembles the case of MPD equations in which no direct equation exists for v^μ. The following equations

$$v_\mu \frac{Dn^\mu(\tau)}{D\tau} = 0, \tag{11}$$

$$p_\mu j^\mu = 0, \tag{12}$$

$$s_{\mu\nu} J^{\mu\nu} = 0 \tag{13}$$

are helpful in this regard. They stem from (5)–(7), respectively. The latter two equations can be used for world-lines describing nearby particles of the same spins and masses.

3. The motion in a GW space–time

In this section we consider the world-line deviations of spinning particles in the space–time described by the following metric

$$ds^2 = -du\,dv - K(u,x,y)\,du^2 + dx^2 + dy^2, \qquad (14)$$

in which $u = t - z$, $v = t + z$ are light-cone coordinates and

$$K(u,x,y) = f(u)(x^2 - y^2) + 2g(u)xy.$$

This metric represents a plane gravitational wave of arbitrary polarization and profile characterized by $f(u)$, $g(u)$ propagating in z-direction. In this space–time, the MPD equations admit the following solution:

$$v^\mu = (1,1,0,0),$$
$$p^\mu = (m,m,0,0),$$
$$s^{1\mu} = s^{2\mu} = 0, \qquad s^{34} = \sigma. \qquad (15)$$

This solution describes a particle of mass m and of spin σ,

$$s^\mu := \frac{1}{2\sqrt{-g}}\epsilon^\mu{}_{\nu\kappa\rho}p^\nu s^{\kappa\rho}, \quad \text{with } \epsilon^{1234} = -1,$$

sitting in the origin of the coordinates with its spin along the z direction. For the above world-line, the geodesic deviation equation leads to the following equations:

$$\frac{d^2n^3(\tau)}{d\tau^2} = -f(u)n^3(\tau) - g(u)n^4(\tau), \qquad (16)$$
$$\frac{d^2n^4(\tau)}{d\tau^2} = f(u)n^4(\tau) - g(u)n^3(\tau). \qquad (17)$$

We now consider two particles of the same spins and masses, one initially at the origin and the other at a nearby point with a specific separation. We aim to calculate this separation at any value of the parameter τ. We take (15) with $(\tau,\tau,0,0)$ as the fiducial world-line. Now Eq. (10) results in

$$J^{12}(\tau) = 0, \qquad (18)$$
$$J^{13}(\tau) + J^{23}(\tau) = -\frac{2\sigma}{m}j^4(\tau), \qquad (19)$$
$$J^{14}(\tau) + J^{24}(\tau) = \frac{2\sigma}{m}j^3(\tau). \qquad (20)$$

Eqs. (13) and (14) result in

$$\frac{dn^1(\tau)}{d\tau} + \frac{dn^2(\tau)}{d\tau} = 0, \qquad (21)$$
$$j^1(\tau) + j^2(\tau) = 0, \qquad (22)$$
$$J^{34}(\tau) = 0, \qquad (23)$$

respectively. From Eq. (8) we obtain

$$j^1(\tau) = \text{const} = \alpha, \qquad (24)$$
$$j^2(\tau) = \text{const} = \beta, \qquad (25)$$
$$\frac{dj^3(\tau)}{d\tau} = f(u)\big(J^{13}(\tau) - mn^3(\tau)\big) + g(u)\big(J^{14}(\tau) - mn^4(\tau)\big), \qquad (26)$$
$$\frac{dj^4(\tau)}{d\tau} = -f(u)\big(J^{14}(\tau) - mn^4(\tau)\big) + g(u)\big(J^{13}(\tau) - mn^3(\tau)\big). \qquad (27)$$

Eq. (9) leads to

$$\frac{dJ^{12}(\tau)}{d\tau} = m\left(\frac{dn^2(\tau)}{d\tau} - \frac{dn^1(\tau)}{d\tau}\right) + 2\alpha, \qquad (28)$$
$$\frac{dJ^{13}(\tau)}{d\tau} = m\frac{dn^3(\tau)}{d\tau} - j^3(\tau), \qquad (29)$$
$$\frac{dJ^{14}(\tau)}{d\tau} = m\frac{dn^4(\tau)}{d\tau} - j^4(\tau), \qquad (30)$$
$$\frac{dJ^{23}(\tau)}{d\tau} = 2\sigma g(u)n^3(\tau) - 2\sigma f(u)n^4(\tau) + m\frac{dn^3(\tau)}{d\tau} - j^3(\tau), \qquad (31)$$
$$\frac{dJ^{24}(\tau)}{d\tau} = -2\sigma g(u)n^4(\tau) - 2\sigma f(u)n^3(\tau) + m\frac{dn^4(\tau)}{d\tau} - j^4(\tau), \qquad (32)$$
$$\frac{dJ^{34}(\tau)}{d\tau} = 0. \qquad (33)$$

Now, comparing (24), (25), and (22) we get $\beta = -\alpha$. If we use the same parameter to describe both of the world-lines, then $n^t = 0$. So we set $n^1(\tau) = -n^2(\tau)$ which is consistent with (21). Taking this into account, Eqs. (18) and (28) give us

$$-n^1(\tau) = n^2(\tau) = \frac{\alpha}{m}\tau + \gamma,$$

in which γ and α are the initial values of n^2 and $dn^2/d\tau$, respectively. The transverse components of n^μ can be obtained from Eqs. (19), (20), (26), (27), and (29)–(33) if one knows the second particle's world-line and spin orientation. Two interesting cases are as follow.

Case 1. The second particle moves on a nearby geodesic. We set

$$J^{13}(\tau) = 0, J^{14}(\tau) = 0.$$

Thus

$$j^3(\tau) = m\frac{dn^3(\tau)}{d\tau}, \qquad j^4(\tau) = m\frac{dn^4(\tau)}{d\tau},$$

and

$$J^{23}(\tau) = -2\sigma\frac{dn^4(\tau)}{d\tau}, \qquad J^{24}(\tau) = 2\sigma\frac{dn^3(\tau)}{d\tau}.$$

It follows that

$$\frac{d^2n^3(\tau)}{d\tau^2} = -f(u)n^3(\tau) - g(u)n^4(\tau),$$
$$\frac{d^2n^4(\tau)}{d\tau^2} = f(u)n^4(\tau) - g(u)n^3(\tau),$$

as we expected.

Case 2. The second particle's spin is such that

$$j^3(\tau) = j^4(\tau) = 0,$$

and

$$J^{13}(\tau) = mn^3(\tau), \qquad J^{14}(\tau) = mn^4(\tau).$$

It follows that

$$\frac{dn^3(\tau)}{d\tau} = \frac{2\sigma}{m}\big(f(u)n^4(\tau) - g(u)n^3(\tau)\big), \tag{34}$$
$$\frac{dn^4(\tau)}{d\tau} = \frac{2\sigma}{m}\big(f(u)n^3(\tau) + g(u)n^4(\tau)\big), \tag{35}$$

which can be solved for $n^3(\tau), n^4(\tau)$ if $f(u), g(u)$ is given explicitly in terms of $u = \tau$.

4. Conclusions

The equations we have found describe the world-line deviations in the framework of MPD equations determining the effect of the spin–curvature coupling on relative accelerations of nearby particles. They may be helpful in different applications including the study of chaotic behaviour of spinning particles in certain space–times [15]. Another important application of these equations is to find approximate solutions to MPD equations by using a known solution, an example is the case two of the previous section. We applied the equations to the case of motion in the space–time of a gravitational wave and showed that they can be integrated analytically. A linear approximation of these equations may be useful in some applications. In more complex situations they may be solved at least numerically. In situations in which a higher accuracy is needed one can use more sophisticated equations containing higher orders of λ in a systematic way. If the particles experience extra forces like the Lorentz force, the equations can be modified by adding suitable terms. The example of the previous section deserves a more extended study.

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