On an Integral Equation Approach
for the Exterior Robin Problem
for the Helmholtz Equation

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A boundary integral equation for the exterior Robin problem for Helmholtz's equation is analyzed in this paper. This integral operator is not compact. A proof based on a suitable regularization of this integral operator and the Fredholm alternative for the regularized compact operator was given by other authors. In this paper, we will give a direct existence and uniqueness proof for the boundary non-compact integral equation in the space settings $C^{1,\lambda}(S)$ and $C^{0,\lambda}(S)$, where $S$ is a closed bounded smooth surface.

1. Introduction

The exterior Robin problem for the Helmholtz equation has been discussed, e.g., see Leis [8], Angell and Kleinman [1]. The integral equation approach used by Angell and Kleinman [1] was based on a system of two integral equations derived from Helmholtz's representation formulas. Following the idea of Burton and Miller [4] (also see Lin [9] for a rigorous proof), Angell and Kress [2] investigated a composite integral equation. The integral operator which they used involves the normal derivative of the double layer potential, which is not compact. As they mentioned in [2], a suitable regularization must be introduced before the Fredholm alternative is applicable.

In this paper, we will give a direct existence and uniqueness proof for the composite integral equation in the space settings $C^{1,\lambda}(S)$ and $C^{0,\lambda}(S)$, where $S$ is a smooth closed bounded surface; See Section 2 for definitions. Our

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analysis is an extension of our previous work for the exterior Neumann problem [9]. Unlike Angell and Kress [2], no regularization of the composite integral operator is needed.

2. Definitions and Preliminary Results

Let $S$ be a closed bounded surface in $R^3$ which belongs to the class $C^2$. Let $D_-, D_+$ denote the interior and the exterior of $S$, respectively. The exterior Robin problem for Helmholtz's equation is to determine a function $u$ such that

$$\Delta u(A) + k^2 u(A) = 0, \quad A \in D_+, \quad (2.1)$$

$$\frac{\partial u}{\partial n_p} + \sigma(p) u(p) = f(p), \quad p \in S, \quad (2.2)$$

the Sommerfeld radiation condition

$$\left( \frac{\partial}{\partial r} - ik \right) u = o \left( \frac{1}{r} \right), \quad as \quad r = |A| \to \infty, \quad (2.3)$$

where $f$ and $\sigma$ are given functions in $C_{0, \lambda}^0(S)$, $0 < \lambda < 1$, $\text{Im} k \geq 0$, $\text{Im} \sigma \geq 0$, and $n_p$ is the outward drawn normal with respect to $D_-$. If a function $f$ is $l$ times continuous differentiable on $S$ and if the $l$th order derivatives are Holder continuous with exponent $\lambda$, we say $f \in C_{1, \lambda}^0(S)$. (See [5, p. 97].) We call

$$L_k u(p) = \int_S u(q) \frac{e^{ik|p-q|}}{|p-q|} dS_q, \quad p \in R^3,$$

a single layer function, and $u(q)$ is called the single layer density function. We call

$$M_k u(p) = -\int_S u(q) \frac{\partial}{\partial n_q} \frac{e^{ik|p-q|}}{|p-q|} dS_q, \quad p \in R^3,$$

a double layer function, and $u(q)$ is called the double layer density function. For simplicity, sometimes we write $Lu$ and $Mu$ only. We also define

$$L_+ u(p) = \lim_{A \to D_+} Lu(A), \quad p \in S,$$

$$L_- u(p) = \lim_{A \to D_-} Lu(A), \quad p \in S,$$
We can similarly define
\[ M_e u(p), \quad M_i u(p), \quad \frac{\partial}{\partial n} M_e u(p), \quad \frac{\partial}{\partial n} M_i u(p). \]

When \( u \) is fixed, we only write \( L_e, L_i, (\partial / \partial n) L_e \), etc.

If \( u \) satisfies (2.1) and (2.3), then we have the following well-known Helmholtz formula:
\[
-4\pi u(p), \quad p \in D_+,
-2\pi u(p), \quad p \in S,
0, \quad p \in D_-,
\]
where \( r = |p - q| \).

We proceed formally to obtain our boundary integral equation. From (2.4), we have
\[
\int_S \left\{ u(q) \left( -\frac{\partial}{\partial n_q} e^{ikr} \right) + \frac{e^{ikr}}{r} \frac{\partial}{\partial n_q} u(q) \right\} dS_q = \begin{cases} 
-4\pi u(p), & p \in D_+, \\
-2\pi u(p), & p \in S, \\
0, & p \in D_-,
\end{cases}
\]

From (2.4), computing normal derivatives (from the interior), from the jump discontinuity of the normal derivative of single layer we obtain
\[
-2\pi u(p) - M_u(p) + L(\sigma(p) \cdot u(p)) = Lf(p), \quad p \in S.
\]

From (2.4), computing normal derivatives (from the interior), from the jump discontinuity of the normal derivative of single layer
\[
\left( \frac{\partial}{\partial n} L_i u(p) = \frac{\partial}{\partial n} Lu(p) + 2\pi u(p), p \in S \right),
\]
we obtain
\[
\frac{\partial}{\partial n} M_i u(p) - 2\pi \sigma(p) u(p) - \frac{\partial}{\partial n} L(\sigma(p) \cdot u(p))
= -2\pi f(p) - \frac{\partial}{\partial n} Lf(p), \quad p \in S.
\]
From the idea used by Burton and Miller [4] for the Neumann problem, linearly combining (2.6) and (2.7), we obtain our integral equation

\[-2\pi u(p) - M u(p) + L(\sigma(p) \cdot u(p)) + \eta \left( \frac{\partial}{\partial n} M_i u(p) - 2\pi \sigma(p) u(p) - \frac{\partial}{\partial n} L(\sigma(p) u(p)) \right) \]

\[= Lf(p) + \eta \left( -2\pi f(p) - \frac{\partial}{\partial n} Lf(p) \right), \quad p \in S, \quad (2.8)\]

where \(\eta\) is a nonzero real number and such that

\[\eta \cdot R, k \leq 0. \quad (2.9)\]

Because \(\partial M_i/\partial n\) is not defined (in the usual sense) on all of \(C(S)\) (see Günter [5, pp. 71–76]), we choose \(C^{1,\lambda}(S)\) and \(C^{0,\lambda}(S), 0 < \lambda < 1\), instead. We had chosen these spaces for the Burton and Miller integral equation approach for the exterior Neumann problem for the Helmholtz equation; see Burton and Miller [4] and Lin [9]. We also remark that Kussmaul [7] also used these spaces for his integral equation approach for the exterior Neumann problem. From [12, Lemma 4],

\[\frac{\partial M_i}{\partial n} : C^{1,\lambda}(S) \to C^{0,\lambda}(S),\]

and hence from [11]

\[-2\pi I - M + L \sigma + \eta \left( \frac{\partial}{\partial n} M_i - 2\pi \sigma - \frac{\partial}{\partial n} L \sigma \right) : C^{1,\lambda}(S) \to C^{0,\lambda}(S).\]

Remark. Angell and Kress [2] use the condition \(\eta \cdot \text{Re } k \geq 0\) (which evidently was motivated by Brakhage and Werner [3] for their combined single and double layer potentials approach for the exterior Dirichlet problem) which corresponds to our condition (2.9), because our definition of the double layer potential (following Günter [5]) is different from theirs in sign.

Angell and Kleinman [1] used a system of integral equations (2.6) and (2.7) for their approach. Angell and Kress [2] proved the existence and uniqueness theorems for (2.8); their proof depends on the Fredholm alternative. Since the operator \((\partial/\partial n)M_i\) is not compact, a suitable regularization is necessary for their proof. In the next section, we will give a
direct existence and uniqueness proof in the space settings $C^{1,\lambda}(S)$ and $C^{0,\lambda}(S)$, without invoking the regularization and the Fredholm alternative of the composite integral operator.

3. EXISTENCE AND UNIQUENESS

We adopt the same framework as Lin [9]. For the existence, we are motivated by Angell and Kleinman [1] and Kleinman and Roach [6].

**Lemma 3.1.** For each $f \in C^{0,\lambda}(S)$, there exists a solution $u \in C^{1,\lambda}(S)$ of the integral equation (2.6).

*Proof.* From [1, Lemma 4.3], there exists a solution $z \in L^2(S)$ of (2.6). Since $f$ is continuous on $S$, the Regularity Theorem [10, p. 178] implies $u$ is also continuous on $S$. From (2.6), we obtain

$$u(p) = -(1/2\pi)(Mu(p) - L(\sigma(p) u(p)) + Lf(p)).$$

From [11, Lemmas 2–7], $Mu \in C^{0,\lambda}(S)$, $L(\sigma \cdot u) \in C^{0,\lambda}(S)$, and $Lf \in C^{1,\lambda}(S)$. From (3.1), $u \in C^{0,\lambda}(S)$. From [11, Lemmas 2–7], $Mu \in C^{1,\lambda}(S)$ and $L(\sigma \cdot u) \in C^{1,\lambda}(S)$, therefore $u \in C^{1,\lambda}(S)$.

**Lemma 3.2.** Suppose that $k$ is not an eigenvalue of the interior Dirichlet problem for the Helmholtz equation. Then for each $f \in C^{0,\lambda}(S)$, there exists a solution of (2.8) in $C^{1,\lambda}(S)$.

*Proof.* From Lemma 3.1, there exists a solution $u$ of the integral equation

$$-2\pi u(p) - Mu(p) + L(\sigma(p) u(p)) = Lf(p), \quad p \in S, \quad u \in C^{1,\lambda}(S).$$

Define a function $V$ on $D_-$ by

$$V(A) = Mu(A) - L(\sigma(A) u(A)) + Lf(A), \quad A \in D_-.$$

Letting $A \to p^-$, from the continuity of the single layer potential and the jump discontinuity of the double layer potential, we have

$$\lim_{A \to p^-} V(A) = Mu(p) + 2\pi u(p) - L(\sigma(p) u(p)) + Lf(p), \quad p \in S.$$

From (3.2), $\lim_{A \to p^-} V(A) = 0$, $p \in S$; this means that $V$ is a solution of the interior homogeneous Dirichlet problem for Helmholtz's equation with zero boundary data. Since $k$ is not an eigenvalue of the interior Dirichlet
problem, this function \( V \) must vanish identically in \( \overline{D} \) and \( \partial V / \partial n = 0 \) on \( S \). We rewrite (3.3) as
\[
Mu(A) = V(A) + L(\sigma(A) u(A)) - Lf(A), \quad A \in D_-
\]
From this, we obtain
\[
\frac{\partial M_i}{\partial n} u(p) = \frac{\partial}{\partial n} L(\sigma(p) u(p)) + 2\pi\sigma(p) u(p) - \left( \frac{\partial}{\partial n} Lf(p) + 2\pi f(p) \right),
\]
which means that \( u \) is also a solution of (2.7). Linearly combining (3.2) and (3.4), \( u \) is a solution of (2.8).

**Lemma 3.3** [1, Lemma 4.6]. The function \( V \) is an eigenfunction of the interior Dirichlet problem with zero boundary data if and only if \( V \) can be represented in the form
\[
V(A) = \frac{1}{2\pi} Mw(A) - \frac{1}{2\pi} L(\sigma(A) w(A)), \quad A \in D_-
\]
where \( w \) satisfies the homogeneous boundary integral equation
\[
2\pi w(p) + Mw(p) - L(\sigma(p) w(p)) = 0, \quad p \in S.
\]

**Theorem 3.4** (Existence Theorem). For each \( f \in C^{0,\lambda}(S) \), there exists a solution of (2.8) in \( C^{0,\lambda}(S) \).

*Proof.* If \( k \) is not an eigenvalue of the interior homogeneous Dirichlet problem, the result follows from Lemma 3.2. Now we only consider the case in which \( k \) is an eigenvalue of the interior homogeneous Dirichlet problem. From Lemma 3.1, there exists a solution \( u_1 \in C^{1,\lambda}(S) \) of the integral equation
\[
-2\pi u(p) - Mu(p) + L(\sigma(p) u(p)) = Lf(p), \quad p \in S.
\]
Define a function \( V \) on \( D_- \) by
\[
V(A) = Mu_1(A) - L(\sigma(A) \cdot u_1(A)) + Lf(A), \quad A \in D_-
\]
As in the proof of Lemma 3.2, \( V \) is either identically zero or is a nontrivial
solution of the interior homogeneous Dirichlet problem. From Lemma 3.3, in either case, we may represent the function \( V \) in the form

\[
V(A) = \frac{1}{2\pi} Mw(A) - \frac{1}{2\pi} L(\sigma(A) w(A)), \quad A \in D_-, \tag{3.7}
\]

where \( w \) satisfies the boundary integral equation

\[
2\pi w(p) + Mw(p) - L(\sigma(p) w(p)) = 0, \quad p \in S. \tag{3.8}
\]

From (3.6) and (3.7), we have

\[
\frac{1}{2\pi} Mw(A) - \frac{1}{2\pi} L(\sigma(A) w(A)) = Mu_1(A) - L(\sigma(A) \cdot u_1(A)) + Lf(A), \quad A \in D_- \tag{3.9}
\]

Define a function \( u \) on \( S \) by

\[
u = u_1 - \left( \frac{1}{2\pi} \right) w. \tag{3.10}\]

We will prove that \( u \) is a common solution of (2.6) and (2.7). From (3.5) and (3.8), we have

\[
-2\pi u(p) - Mu(p) + L(\sigma(p) \cdot u(p)) = Lf(p), \quad p \in S, \tag{3.11}
\]

i.e., \( u \) is a solution of (2.6). From (3.9) and (3.10), we obtain

\[
Mu(A) = L(\sigma(A) u(A)) - Lf(A), \quad A \in D_- \tag{3.12}
\]

From (3.8), we have

\[
w(p) = \frac{1}{2\pi} \left( L(\sigma(p) w(p)) - Mw(p) \right), \quad p \in S. \tag{3.13}
\]

From the regularity theorem [10, p. 178] and (3.8), \( w \) is continuous on \( S \). From [11] and (3.13), \( w \in C^{0,\lambda}(S) \). Again, from [11] and (3.13), \( w \in C^{1,\lambda}(S) \). Therefore \( \partial M_i/\partial n \) exists (e.g., see [5, 11, 12]) and from (3.12),

\[
\frac{\partial M_i}{\partial n} u(p) = \frac{\partial}{\partial n} \left( L(\sigma(p) u(p)) + 2\pi \sigma(p) u(p) \right)
\]

\[
- \left[ \frac{\partial}{\partial n} Lf(p) + 2\pi f(p) \right], \quad p \in S, \tag{3.14}
\]

i.e., \( u \) is also a solution of (2.7). From (3.11) and (3.14), \( u \) is a solution of (2.8) in \( C^{1,\lambda}(S) \).
Theorem 3.5 (Uniqueness). For each \( f \in C^{0,2}(S) \), there is at most one solution \( u \) of the integral equation (2.8) in \( C^{1,2}(S) \).

Proof. If suffices to show that the corresponding homogeneous equation

\[
-2\pi u(p) - Mu(p) + L(\sigma(p) \cdot u(p))
+ \eta \left[ \frac{\partial}{\partial n} M_i u(p) - 2\pi \sigma(p) u(p) - \frac{\partial}{\partial n} L(\sigma(p) \cdot u(p)) \right] = 0 \tag{3.15}
\]

has only the trivial solution \( u = 0 \) on \( S \). With our notation, (3.15) can be written as

\[
-M_i u(p) + L(\sigma(p) \cdot u(p))
+ \eta \left[ \frac{\partial}{\partial n} M_i u(p) - \frac{\partial}{\partial n} L(\sigma(p) u(p)) \right] = 0, \quad p \in S. \tag{3.16}
\]

From (3.16) and Green's theorem,

\[
\eta \int_S \left| \frac{\partial}{\partial n} M_i u(p) - \frac{\partial}{\partial n} L(\sigma \cdot u)(p) \right|^2 dS
= -\int_S \left( M_i u(p) - L(\sigma \cdot u)(p) \right) \cdot \left( \frac{\partial}{\partial n} M_i u(p) - \frac{\partial}{\partial n} L(\sigma \cdot u)(p) \right) dS
= -\left[ \int_{D_-} |\nabla(Mu(A) - L(\sigma \cdot u)(A))|^2 dV 
+ \int_{D_-} (Mu(A) - L(\sigma \cdot u)(A)) \cdot A(Mu(A) - L(\sigma \cdot u)(A)) dV \right]
= -\left[ \int_{D_-} |\nabla(Mu(A) - L(\sigma \cdot u)(A))|^2 dV 
- k^2 \int_{D_-} |Mu(A) - L(\sigma \cdot u)(A)|^2 dV \right].
\]

Equating the imaginary part of the above equation, we have

\[
\eta \int_S \left| \frac{\partial}{\partial n} M_i u(p) - \frac{\partial}{\partial n} L(\sigma \cdot u)(p) \right|^2 dS
= 2 \cdot \text{Re} \ k \cdot \text{Im} \ k \cdot \int_{D_-} |Mu(A) - L(\sigma \cdot u)(A)|^2 dV.
\]
Since $\eta$ is chosen such that $\eta \neq 0$ and $\eta \cdot \Re k \leq 0$, therefore

$$\frac{\partial}{\partial n} M_\eta u(p) - \frac{\partial}{\partial n} L_\eta (\sigma \cdot u)(p) = 0 \quad \text{on } S.$$  

From (3.16),

$$-M_\eta u(p) + L(\sigma \cdot u)(p) = 0 \quad \text{on } S.$$  

From [1, Lemma 4.4], we have $u = 0$ on $S$.

REFERENCES


