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On Mixed Boundary-Value Problems for Axially-Symmetric Potentials

A. E. HEINS*

The University of Michigan, Ann Arbor, Michigan

R. C. MACCAMY†

Carnegie Institute of Technology, Pittsburgh, Pa.

Submitted by C. L. Dolph

This paper concerns the determination of an axially symmetric harmonic function that is a function $u(r, y)$ satisfying the equation

$$u_{rr} + r^{-1} u_r + u_{yy} = 0 \qquad \text{in} \qquad y < 0 \quad 0 \leqslant r, \tag{E}$$

when u is subject to the mixed boundary conditions,

$$u(r, 0) = g(r) \qquad 0 \leqslant r < 1, \tag{A}$$

$$u_y(r, 0) = 0 \qquad r > 1. \tag{B}$$

This problem, which physically corresponds to the potential of a disk in an external field, has a long history dating back to Kelvin. There are several known methods of solution all quite complicated. Of these we mention the dual integral equation method of Titchmarsh [1] and a later method of Copson [2]. It is the purpose of this note to point out that these methods are not necessary but that the problem is easily reduced to an elementary boundary-value problem for a harmonic function of two variables.

We suppose,

$$u(r, y) \to 0 \qquad \text{as} \qquad r^2 + y^2 \to \infty.$$

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Then $u(r, y)$ can be represented in the integral form,

$$u(r, y) = (2\pi)^{-1} \int_0^1 \int_0^{2\pi} [r^2 + \rho^2 - 2r\rho \cos \theta + y^2]^{-1/2} \rho u_y(\rho,0)\, d\theta d\rho.$$

From this we have

$$u(0, y) = \int_0^1 (\rho^2 + y^2)^{-1/2} \rho u_y(\rho, 0)\, d\rho. \tag{1}$$

While Eq. (1) is derived initially for y real we observe that the integral on the right side is an analytic function for complex $y = \eta + i\tau$. Accordingly $u(0, y)$ is extended analytically to complex y. Let us denote the extension by

$$u(0, y) = U(y) = \varphi(\eta, \tau) + i\psi(\eta, \tau)$$

From Eq. (1) we find immediately then that

$$\operatorname{Re} U(y) = \varphi(\eta, \tau) = 0 \qquad \text{for} \qquad \eta = 0 \quad |\tau| > 1. \tag{2}$$

Note also that $U(y)$ equals $u(0, \eta)$ on $\tau = 0$ hence is real there and we have,

$$U(\bar{y}) = \overline{U(y)} \qquad \text{or} \qquad \varphi(\eta, -\tau) = \varphi(\eta, \tau), \psi(\eta, -\tau) = -\psi(\eta, \tau). \tag{3}$$

Finally, it follows from Eq. (1) that

$$U(y) \to 0 \qquad \text{as} \qquad |y| \to \infty. \tag{4}$$

Now it is known that solutions of equation (E) are determined by their values on $r = 0$, the axis of symmetry, provided those values are analytic. In fact the answer may be written down [3] in the integral form,

$$u(r, y) = \pi^{-1} \int_{-r}^{+r} (r^2 - \tau^2)^{-1/2}\, U(y + i\tau)\, d\tau,$$

or, using (3),

$$u(r, y) = 2(\pi)^{-1} \int_0^r (r^2 - \tau^2)^{-1/2} \varphi(y, \tau)\, d\tau. \tag{5}$$

Let us now enter the boundary condition (A). We find,

$$g(r) = 2(\pi)^{-1} \int_0^r (r^2 - \tau^2)^{-1/2} \varphi(0, \tau)\, d\tau \qquad 0 \leqslant r < 1.$$

This Volterra integral equation is easily inverted to yield (c.f. [4])

$$\varphi(0, \tau) = \frac{d}{d\tau}\int_0^\tau \rho g(\rho)\,(\tau^2 - \rho^2)^{-1/2}\,d\rho \equiv h(\tau) \qquad 0 \leqslant \tau < 1. \tag{6}$$

By Eq. (3) we have also

$$\varphi(0, \tau) = h(-\tau) \qquad -1 < \tau \leqslant 0. \tag{7}$$

The harmonic function of two variables $\varphi(\eta, \tau)$ is then known from Eqs. (2), (6), and (7) on the entire τ axis. Further, according to (4) it must vanish as $\eta^2 + \tau^2 \to \infty$. The required function is then given by the Poisson integral,

$$\varphi(\eta, \tau) = (\pi^{-1})\eta\int_{-1}^{+1} h(t)\,[(\eta^2 + (\tau - t)^2]^{-1}\,dt.$$

Having determined $\varphi(\eta, \tau)$ we then find $u(r, y)$ from Eq. (5).

The use of the integral representation (5) for an axially-symmetric function in terms of its values on the axis of symmetry seems to be a rather powerful tool. We remark that the case of a circular disk in a non-symmetric field can be handled similarly. One separates the angular dependence in a Fourier series. The coefficients of $\cos n\theta$ or $\sin n\theta$ are then functions $u^n(r, \theta)$ which satisfy instead of (E) equations of the form,

$$u^n_{rr} + r^{-1}u^n_r + u^n_{yy} - n^2r^{-2}u^n = 0.$$

Analogs of formula (5) exist for these equations (see [3]) and the whole technique can be duplicated. Quite similar although somewhat more complicated methods have been successfully employed by the authors in the study of acoustic wave diffraction by a circular disk [5].

References

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