Differential equations of infinite order for Sobolev-type orthogonal polynomials

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Abstract

Assume that \( \{P_n(x)\}_{n=0}^{\infty} \) are orthogonal polynomials relative to a quasi-definite moment functional \( \sigma \), which satisfy a differential equation of spectral type of order \( D \) (\( 2 \leq D \leq \infty \)):

\[
L_D[y](x) = \sum_{i=1}^{D} \lambda_i(x)y^{(i)}(x) = \lambda_n y(x),
\]

where \( \lambda_i(x) \) are polynomials of degree \( \leq i \). Let \( \phi \) be the symmetric bilinear form of discrete Sobolev type defined by

\[
\phi(p, q) = (\sigma, pq) + Np^{(k)}(c)q^{(k)}(c),
\]

where \( N \neq 0 \) and \( c \) are real constants, \( k \) is a non-negative integer, and \( p \) and \( q \) are polynomials.

We first give a necessary and sufficient condition for \( \phi \) to be quasi-definite and then show: If \( \phi \) is quasi-definite, then the corresponding Sobolev-type orthogonal polynomials \( \{R_n^{k,c}(x)\}_{n=0}^{\infty} \) satisfy a differential equation of infinite order of the form

\[
N \left\{ a_0(x,n)y(x) + \sum_{i=1}^{\infty} a_i(x)y^{(i)}(x) \right\} + L_D[y](x) = \lambda_n y(x),
\]

where \( \{a_i(x)\}_{i=0}^{\infty} \) are polynomials of degree \( \leq i \), independent of \( n \) except \( a_0(x) := a_0(x,n) \). We also discuss conditions under which such a differential equation is of finite order when \( \sigma \) is positive-definite, \( D < \infty \), \( N \geq 0 \), and \( k = 0 \).

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1. Introduction

In [11], Koornwinder introduced the generalized Jacobi polynomials \( \{P_n^{\alpha, \beta, M, N}(x)\}_{n=0}^{\infty} \), which are orthogonal on \([-1, 1]\) relative to the Jacobi weight plus two point masses at \( x = \pm 1 \):

\[
\frac{\Gamma(\alpha + \beta + 2)}{2^{\alpha + \beta + 1}\Gamma(\alpha + 1)\Gamma(\beta + 1)}(1 - x)^{\alpha}(1 + x)^{\beta} + M\delta(x + 1) + N\delta(x - 1),
\]

where \( \alpha > -1, \beta > -1, M \geq 0, \) and \( N \geq 0. \) As a limiting case, he also found the generalized Laguerre polynomials \( \{L_n^{M}(x)\}_{n=0}^{\infty} \), which are orthogonal on \([0, \infty)\) relative to the Laguerre weight plus a point mass at \( x = 0 \):

\[
\frac{1}{\Gamma(\alpha + 1)}x^\alpha e^{-x} + M\delta(x),
\]

where \( \alpha > -1 \) and \( M \geq 0. \) Koekoek and Meijer [10] introduced the Sobolev-type Laguerre polynomials \( \{L_n^{M, N}(x)\}_{n=0}^{\infty} \), which are orthogonal relative to the Sobolev inner product

\[
\phi(p, q) = \frac{1}{\Gamma(\alpha + 1)} \int_{0}^{\infty} p(x)q(x)x^\alpha e^{-x} \, dx + Mp(0)q(0) + Np'(0)q'(0),
\]

where \( \alpha > -1, M \geq 0, \) and \( N \geq 0. \)

On the other hand, Koekoek and Koekoek [8] (see also [1]) showed that \( \{L_n^{M}(x)\}_{n=0}^{\infty} \) satisfy a unique differential equation of the form

\[
M\sum_{i=0}^{\infty} a_i(x)y^{(i)}(x) + xy''(x) + (\alpha + 1 - x)y'(x) + ny(x) = 0. \tag{1.1}
\]

For symmetric generalized ultraspherical polynomials \( \{P_n^{\alpha, M, M}(x)\}_{n=0}^{\infty} \), Koekoek [7] found a differential equation of the form

\[
M\sum_{i=0}^{\infty} a_i(x)y^{(i)}(x) + (1 - x^2)y''(x) - 2(\alpha + 1)xy'(x) + n(n + 2\alpha + 1)y(x) = 0. \tag{1.2}
\]

Some special cases of (1.1) and (1.2) can also be found in [12, 16]. Recently, Koekoek et al. [9] (see also [6]) construct all differential equations of the form

\[
M\sum_{i=0}^{\infty} a_i(x)y^{(i)}(x) + N\sum_{i=0}^{\infty} b_i(x)y^{(i)}(x) + MN\sum_{i=0}^{\infty} c_i(x)y^{(i)}(x)
\]

\[
+ xy''(x) + (\alpha + 1 - x)y'(x) + ny(x) = 0 \tag{1.3}
\]

satisfied by the polynomials \( \{L_n^{M, N}(x)\}_{n=0}^{\infty}. \)

In all the above three differential equations (1.1)–(1.3), \( a_i(x), b_i(x), \) and \( c_i(x) \) for \( i \geq 0 \) are polynomials of degree \( \leq i \) and they are independent of \( n \) for \( i \geq 1. \) Moreover for \( M > 0 \) in case of \( \{L_n^{M}(x)\}_{n=0}^{\infty} \) and \( \{P_n^{\alpha, M, M}(x)\}_{n=0}^{\infty} \) and for \( M + N > 0 \) in case of \( \{L_n^{M, N}(x)\}_{n=0}^{\infty}, \) these differential equations are of finite order only for nonnegative integer values of \( \alpha. \)

We note that the differential equation (1.1) for \( \{L_n^{M}(x)\}_{n=0}^{\infty} \) is unique but the differential equation (1.3) for \( \{L_n^{M, N}(x)\}_{n=0}^{\infty} \) is not unique in general.
We also note that in all above three cases, the inner products involved are always positive-definite and point masses are given at the end points of the interval of orthogonality for classical Laguerre or Jacobi polynomials.

Motivated by these examples, we consider any quasi-definite (not necessarily positive-definite) moment functional $\sigma$ of which the corresponding orthogonal polynomials $\{P_n(x)\}_{n=0}^{\infty}$ satisfy a differential equation of spectral type of order $D$ ($2 \leq D \leq \infty$):

$$L_D[y](x) = \sum_{i=1}^{D} \ell_i(x) y^{(i)}(x) = \lambda_n y(x), \quad (1.4)$$

where each $\ell_i(x) = \sum_{j=0}^{i} \ell_{ij} x^j$ is a polynomial of degree $\leq i$, independent of $n$ and $\lambda_n$ are eigenvalue parameters given by

$$\lambda_n = \ell_{11} n + \ell_{22} n(n - 1) + \cdots + \ell_{DD} n(n - 1) \cdots (n - D + 1).$$

Now, consider the following point mass perturbation of $\sigma$:

$$\phi(p(x), q(x)) := \langle \sigma, p(x) q(x) \rangle + Np^{(k)}(c)q^{(k)}(c), \quad (1.5)$$

where $N(\neq 0)$ and $c$ are arbitrary real numbers and $k$ is a nonnegative integer. Then, $\phi(\cdot, \cdot)$ defines a moment functional when $k = 0$ or a Sobolev-type quasi-inner product when $k > 0$ on the space of polynomials. We first find a necessary and sufficient condition for $\phi(\cdot, \cdot)$ to be quasi-definite. When $\phi(\cdot, \cdot)$ is quasi-definite, we show that the corresponding orthogonal polynomials satisfy a differential equation (not unique in general) of the form

$$N \left\{ a_0(x,n) y(x) + \sum_{i=1}^{\infty} a_i(x) y^{(i)}(x) \right\} + L_D[y](x) = \lambda_n y(x), \quad (1.6)$$

where each $a_i(x)$ is a polynomial of degree $\leq i$, independent of $n$ except $a_0(x,n)$. In particular, if $P_n^{(k)}(c) \neq 0$, $n \geq k + 1$, then we show that there is a unique such differential equation with $a_i(x) \equiv 0$, $1 \leq i \leq k$.

Finally, we find necessary conditions for the differential equation (1.6) to be of finite order when $\sigma$ is positive-definite, $D < \infty$, $N > 0$ and $k = 0$.

2. Preliminaries

All polynomials throughout this work are assumed to be real polynomials of a real variable $x$. The linear space of all such polynomials is denoted by $\mathcal{P}$. We shall denote the degree of a polynomial $\pi \in \mathcal{P}$ by $\deg(\pi)$ with the convention that $\deg(0) = -1$. By a polynomial system (PS), we mean a sequence of polynomials $\{\phi_n(x)\}_{n=0}^{\infty}$ with $\deg(\phi_n) = n$ ($n \geq 0$). We call any linear functional $\sigma : \mathcal{P} \to \mathbb{R}$ a moment functional and denote its action on a polynomial $\pi$ by $\langle \sigma, \pi \rangle$.

We say that a moment functional $\sigma$ is quasi-definite (respectively, positive-definite) if the moments

$$\sigma_n := \langle \sigma, x^n \rangle \quad (n \geq 0)$$
of $\sigma$ satisfy the Hamburger condition

$$\Delta_n(\sigma) := \det[\sigma_{i+j}]_{i,j=0}^n \neq 0 \quad \text{(respectively, } \Delta_n(\sigma) > 0)$$

for each $n \geq 0$.

More generally for any symmetric bilinear form $\phi(\cdot, \cdot)$ on $\mathcal{P} \times \mathcal{P}$, we call the double sequence

$$\phi_{m,n} := \phi(x^m, x^n) \quad (m \text{ and } n \geq 0)$$

the moments of $\phi(\cdot, \cdot)$ and say that $\phi(\cdot, \cdot)$ is quasi-definite (respectively, positive-definite) if

$$\Delta_n(\phi) := \det[\phi_{i,j}]_{i,j=0}^n \neq 0 \quad \text{(respectively, } \Delta_n(\phi) > 0)$$

for each $n \geq 0$.

A symmetric bilinear form $\phi(\cdot, \cdot)$ is quasi-definite (respectively, positive-definite) if and only if there are PS $\{R_n(x)\}_{n=0}^\infty$ and real constants $k_n \neq 0$ (respectively, $k_n > 0$) for $n \geq 0$ such that

$$\phi(R_m(x), R_n(x)) = k_n \delta_{mn} \quad (m \text{ and } n \geq 0).$$

Moreover, in this case, each $R_n(x)$ is uniquely determined up to an arbitrary nonzero factor.

When the symmetric bilinear form $\phi(\cdot, \cdot)$ is quasi-definite, we call a corresponding PS $\{R_n(x)\}_{n=0}^\infty$ as in (2.1) an orthogonal polynomial system (OPS) relative to $\phi(\cdot, \cdot)$.

3. Infinite order differential equations

Throughout this section, we consider a quasi-definite moment functional $\sigma$ on $\mathcal{P}$ and let $\{P_n(x)\}_{n=0}^\infty$ be a corresponding OPS and

$$K_n(x, y) = \sum_{i=0}^n \sigma_i(x) \sigma_i(y), \quad n \geq 0$$

the kernel polynomial of order $n$ associated to $\{P_n(x)\}_{n=0}^\infty$. We also set

$$K^{(r,s)}_n(x, y) = \frac{\partial^{r+s}}{\partial x^r \partial y^s} K_n(x, y).$$

Proposition 3.1. The symmetric bilinear form $\phi(\cdot, \cdot)$ in (1.5) is quasi-definite if and only if $1 + NK^{(k,k)}_n(c, c) \neq 0$, $n \geq 0$. If $\phi(\cdot, \cdot)$ is quasi-definite, then a PS $\{R_n(x)\}_{n=0}^\infty$, where

$$R_n(x) = (1 + NK^{(k,k)}_{n-1}(c, c)) P_n(x) - NP^{(k)}_n(c) K^{(0,k)}_{n-1}(x, c), \quad n \geq 0$$

is an OPS relative to $\phi(\cdot, \cdot)$ and then

$$\phi(R_n(x), R_n(x)) = (1 + NK^{(k,k)}_n(c, c))(1 + NK^{(k,k)}_{n-1}(c, c))(\sigma, P^{2}_n), \quad n \geq 0,$$

where $K^{(1,1)}_{n-1}(x, y) \equiv 0$.

Proof. Assume that $\phi(\cdot, \cdot)$ is quasi-definite and let $\{\tilde{R}_n(x)\}_{n=0}^\infty$ be an OPS relative to $\phi(\cdot, \cdot)$. Then we can express $\tilde{R}_n(x)$ as

$$\tilde{R}_n(x) = \sum_{j=0}^n C_{n,j} P_j(x).$$
From the orthogonality of \( \{P_n(x)\}_{n=0}^{\infty} \) and \( \{\tilde{R}_n(x)\}_{n=0}^{\infty} \), we obtain

\[
C_{n,j} = \frac{\langle \sigma, \tilde{R}_nP_j \rangle}{\langle \sigma, P_j^2 \rangle} = \begin{cases} 
-\frac{N\tilde{R}_n^{(k)}(c)P_j^{(k)}(c)}{\langle \sigma, P_j^2 \rangle}, & 0 \leq j \leq n-1, \\
\frac{\langle \sigma, \tilde{R}_nP_n \rangle}{\langle \sigma, P_n^2 \rangle}, & j = n 
\end{cases}
\]

and so

\[
\tilde{R}_n(x) = \frac{\langle \sigma, \tilde{R}_nP_n \rangle}{\langle \sigma, P_n^2 \rangle} P_n(x) - N\tilde{R}_n^{(k)}(c)K_{n-1}^{(0,k)}(x,c).
\]

Differentiating (3.3) \( k \) times and then evaluating at \( x = c \), we obtain

\[
\tilde{R}_n^{(k)}(c)(1 + NK_{n-1}^{(k,k)}(c,c)) = \frac{\langle \sigma, \tilde{R}_nP_n \rangle}{\langle \sigma, P_n^2 \rangle} P_n^{(k)}(c).
\] (3.4)

Now we claim that \( 1 + NK_{n-1}^{(k,k)}(c,c) \neq 0, n \geq 1 \). If \( 1 + NK_{n-1}^{(k,k)}(c,c) = 0 \) for some \( n \geq 1 \), then, by (3.4), \( P_n^{(k)}(c) = 0 \) and so \( 1 + NK_{n}^{(k,k)}(c,c) = 0 \). Thus we obtain \( P_m^{(k)}(c) = 0, m \geq n \) by induction. By differentiating \( k \) times the three term recurrence relation satisfied by \( \{P_n(x)\}_{n=0}^{\infty} \), we obtain \( P_n^{(k-1)}(c) = 0, m \geq n + k \), which is a contradiction since any two consecutive polynomials from \( \{P_n(x)\}_{n=0}^{\infty} \) cannot have a common zero.

By substituting \( \tilde{R}_n^{(k)}(c) \) from (3.4) into (3.3) and letting

\[
R_n(x) = \frac{\langle \sigma, \tilde{R}_nP_n \rangle}{\langle \sigma, P_n^2 \rangle} x(n) - N\tilde{R}_n^{(k)}(c)K_{n-1}^{(0,k)}(x,c),
\]

we obtain expression (3.1). Conversely, let \( 1 + NK_{n}^{(k,k)}(c,c) \neq 0 \) for all \( n \geq 0 \). Define \( R_n(x) \) by (3.1). Then \( \{R_n(x)\}_{n=0}^{\infty} \) is a PS and it is easy to show the orthogonality (3.2).

Proposition 3.1 for \( k = 0 \) and \( k = 1 \) is proved in [17] and [18] respectively. Let

\[
G_n(x, y) = \sum_{i=0}^{n} \frac{R_i(x)R_i(y)}{\phi(R_i(x), R_i(y))}
\]

be the kernel polynomial of order \( n \) associated to \( \{R_n(x)\}_{n=0}^{\infty} \).

Proposition 3.2. For any nonnegative integers \( r \) and \( s \), we have

\[
\phi(G_n^{(r,s)}(x, y), \phi(x)) = \phi^{(r)}(y)
\]

for any polynomial \( \phi(x) \) of degree \( \leq n \) (reproducing property) and

\[
G_n^{(r,s)}(x, y) = K_n^{(r,s)}(x, y) - N\frac{K_n^{(r,k)}(x, c)K_n^{(s,k)}(c, y)}{1 + NK_{n}^{(k,k)}(c,c)}.
\] (3.5)

Proof. The reproducing property of \( G_n(x, y) \) is easy to obtain (cf. [2, 19]).
We can write \( G_n(x, y) \) as
\[
G_n(x, y) = \sum_{i=0}^{n} C_i(y) P_i(x),
\]
where \( C_i(y) \) are polynomials in \( y \). By using the orthogonality of \( \{P_n(x)\}_{n=0}^{\infty} \) and the reproducing property of \( G_n(x, y) \), we obtain
\[
C_i(y) = \frac{P_i(y)}{\langle \sigma, P_i^2 \rangle} - N \frac{G_n^{(k,0)}(c, y) P_i^{(k)}(c)}{\langle \sigma, P_i^2 \rangle}, \quad 0 \leq i \leq n
\]
and so
\[
G_n(x, y) = K_n(x, y) - NG_n^{(k,0)}(c, y) K_n^{(0,k)}(x, c).
\]
Differentiating (3.6) \( k \) times with respect to \( x \) and evaluating at \( x = c \), we obtain
\[
G_n^{(k,0)}(c, y) (1 + NK_n^{(k,k)}(c, c)) = K_n^{(k,0)}(c, y)
\]
and so
\[
G_n(x, y) = K_n(x, y) - N \frac{K_n^{(0,k)}(x, c) K_n^{(k,0)}(c, y)}{1 + NK_n^{(k,k)}(c, c)},
\]
from which (3.5) follows. \( \square \)

From now on, we always assume that the OPS \( \{P_n(x)\}_{n=0}^{\infty} \) relative to \( \sigma \) satisfy the differential equation (1.4) and \( \phi(\cdot, \cdot) \) in (1.5) is quasi-definite and let \( \{R_n^{N,k}(x)\}_{n=0}^{\infty} = \{R_n(x)\}_{n=0}^{\infty} \) be an OPS relative to \( \phi(\cdot, \cdot) \).

In the following, all the summations are understood to be equal to 0 if the upper limit of the sum is less than the lower limit of the sum.

**Theorem 3.3.** There exists a sequence \( \{a_i(x)\}_{i=0}^{\infty} \) of polynomials such that
(i) for each \( i \geq 0 \), \( \deg(a_i) \leq i \);
(ii) all \( a_i(x) \) for \( i \geq 1 \) are independent of \( n \) and \( a_0(x) = a_0(x, n) \);
(iii) for each \( n \geq 0 \), \( R_n(x) \) satisfies the differential equation:
\[
N \left\{ a_0(x,n)y(x) + \sum_{i=1}^{\infty} a_i(x)y^{(i)}(x) \right\} + L_D[y](x) = \lambda_n y(x).
\]
In fact, we may choose \( \{a_i(x)\}_{i=0}^{\infty} \) by
\[
a_0(x, 0) = 0;
a_0(x, n) \text{ is an arbitrary constant for } n = 1, 2, \ldots, k \text{ (if } k \geq 1);\n\]
\[
a_0(x, n) = a_0(x, n - 1) - K_{n-1}^{(k,k)}(c, c)(\lambda_n - \lambda_{n-1}), \quad n \geq k + 1
= a_0(x, k) - \sum_{i=k}^{n-1} K_i^{(k,k)}(c, c)(\lambda_{i+1} - \lambda_i), \quad n \geq k + 1,
\]
and
\[ a_i(x) = \frac{-1}{P_{i}^{(i)}(x)} \left\{ a_0(x,i)P_{i}(x) + \sum_{j=1}^{i-1} a_j(x)P_{i}^{(j)}(x) + P_{i}^{(k)}(c) \sum_{j=k}^{i-1} (\lambda_j - \lambda_i)P_{i}(x)P_{j}^{(k)}(c) \right\}, \quad i \geq 1. \]

(3.9)

**Proof.** When \( y(x) = R_n(x), n \geq 0 \), (3.7) becomes by (3.1)
\[
N \sum_{i=1}^{\infty} a_i(x)R_n^{(i)}(x) + L_0[R_n(x)] + (Na_0(x,n) - \lambda_n)R_n(x)
= N \left\{ a_0(x,n)P_n(x) + \sum_{i=1}^{\infty} a_i(x)P_n^{(i)}(x) + P_n^{(k)}(c) \sum_{i=k}^{n-1} (\lambda_n - \lambda_i)P_{i}(x)P_{i}^{(k)}(c) \right\}
+ N^2 \left\{ K_{n-1}^{(k,k)}(c,c)(a_0(x,n)P_n(x) + \sum_{i=1}^{\infty} a_i(x)P_n^{(i)}(x)) \right\} = 0.
\]

(3.10)

Since \( N \neq 0 \) can be any real number satisfying \( 1 + NK_n^{(k,k)}(c,c) \neq 0, n \geq 0 \), Eq. (3.10) is equivalent to
\[
a_0(x,n)P_n(x) + \sum_{i=1}^{\infty} a_i(x)P_n^{(i)}(x) + P_n^{(k)}(c) \sum_{i=k}^{n-1} (\lambda_n - \lambda_i)P_{i}(x)P_{i}^{(k)}(c) = 0
\]
and
\[
K_{n-1}^{(k,k)}(c,c) \left\{ a_0(x,n)P_n(x) + \sum_{i=1}^{\infty} a_i(x)P_n^{(i)}(x) \right\} - P_n^{(k)}(c) \left\{ a_0(x,n)K_{n-1}^{(0,k)}(x,c) + \sum_{i=1}^{\infty} a_i(x)K_{n-1}^{(i,k)}(x,c) \right\} = 0
\]
for all \( x \in \mathbb{R} \) and \( n \geq 0 \). Thus to prove the theorem, it is sufficient to show that \( \{a_i(x)\}_{i=0}^{\infty} \) defined by (3.8) and (3.9) satisfy (3.11) and (3.12). When \( K_{n-1}^{(k,k)}(c,c) \neq 0 \), after multiplying (3.11) by \( K_{n-1}^{(k,k)}(c,c) \) and subtracting (3.12), we obtain
\[ P_n^{(k)}(c) \left\{ a_0(x,n)K_{n-1}^{(0,k)}(x,c) + \sum_{i=1}^{\infty} a_i(x)K_{n-1}^{(i,k)}(x,c) + K_{n-1}^{(k,k)}(c,c) \sum_{i=k}^{n-1} (\lambda_n - \lambda_i)P_{i}(x)P_{i}^{(k)}(c) \right\} = 0.
\]
Hence, it is sufficient to show that \( \{a_i(x)\}_{i=0}^{\infty} \) satisfy Eqs. (3.11) and
\[
a_0(x,n)K_{n-1}^{(0,k)}(x,c) + \sum_{i=1}^{\infty} a_i(x)K_{n-1}^{(i,k)}(x,c) + K_{n-1}^{(k,k)}(c,c) \sum_{i=k}^{n-1} (\lambda_n - \lambda_i)P_{i}(x)P_{i}^{(k)}(c) = 0.
\]
(3.13)

Note that Eq. (3.13) implies (3.12) even when \( K_{n-1}^{(k,k)}(c,c) = 0 \).
When \( 0 \leq n \leq k \), Eqs. (3.9) and (3.11) become
\[ a_i(x) = \frac{-1}{P_{i}^{(i)}(x)} \left\{ a_0(x,i)P_{i}(x) + \sum_{j=1}^{i-1} a_j(x)P_{i}^{(j)}(x) \right\}, \quad 1 \leq i \leq n
\]
and
\[ a_0(x, n)P_n(x) + \sum_{i=1}^{n} a_i(x)P_n^{(i)}(x) = 0, \quad 0 \leq n \leq k, \]
respectively. Hence Eq. (3.11) holds for \( 0 \leq n \leq k \). On the other hand, for \( 0 \leq n \leq k \), Eq. (3.13) holds trivially.

Assume now that Eqs. (3.11) and (3.13) hold up to \( n = m \) for some \( m \geq k \). Then Eq. (3.11) for \( n = m + 1 \) is
\[ a_0(x, m + 1)P_{m+1}(x) + \sum_{i=1}^{m+1} a_i(x)P_{m+1}^{(i)}(x) + P_{m+1}(c) \sum_{i=k}^{m} \left( \lambda_{m+1} - \lambda_i \right) P_i(x)P_i^{(k)}(c) \frac{\langle \sigma, P_i^2 \rangle}{\langle \sigma, P_i^2 \rangle} = 0, \]
which is the same equation as (3.9) for \( i = m + 1 \).

For \( n = m + 1 \), the left-hand side of Eq. (3.13) becomes in view of (3.8)
\[ a_0(x, m + 1)K_m^{(0,k)}(x, c) + \sum_{i=1}^{\infty} a_i(x)K_m^{(i,k)}(x, c) + K_m^{(k,k)}(c, c) \sum_{i=k}^{m-1} \left( \lambda_m - \lambda_i \right) P_i(x)P_i^{(k)}(c) \frac{\langle \sigma, P_i^2 \rangle}{\langle \sigma, P_i^2 \rangle} = a_0(x, m)K_{m-1}^{(0,k)}(x, c) + \sum_{i=1}^{\infty} a_i(x)K_{m-1}^{(i,k)}(x, c) + K_{m-1}^{(k,k)}(c, c) \]
\[ \times \sum_{i=k}^{m-1} \left( \lambda_m - \lambda_i \right) P_i(x)P_i^{(k)}(c) \frac{\langle \sigma, P_i^2 \rangle}{\langle \sigma, P_i^2 \rangle} + \frac{P_m^{(k)}(c)}{\langle \sigma, P_m^2(x) \rangle} \left\{ a_0(x, m)P_m(x) \right\} \]
\[ + \sum_{i=1}^{\infty} a_i(x)P_m^{(i)}(x) + P_m(x) \sum_{i=k}^{m-1} \left( \lambda_m - \lambda_i \right) P_i(x)P_i^{(k)}(c) \frac{\langle \sigma, P_i^2 \rangle}{\langle \sigma, P_i^2 \rangle} \right\}, \]
which is equal to 0 by the induction hypothesis for \( n = m \). \( \square \)

Note that there are infinitely many differential equations of the form (3.7) which have \( \{R_n(x)\}_{n=0}^{\infty} \) as solutions. However, we may have the uniqueness in certain cases if \( k = 0 \) or if we require \( a_i(x) \equiv 0 \) for \( 1 \leq i \leq k \) when \( k \geq 1 \).

**Theorem 3.4.** If \( P_n^{(k)}(c) \neq 0, n \geq k + 1 \), then there is a unique set of continuous functions \( a_0(x) = a_0(x, n) \) and \( \{a_i(x)\}_{i=k+1}^{\infty} \) such that
\( \text{(i) all } a_i(x), \ i = k + 1, k + 2, \ldots, \text{ are independent of } n; \)
\( \text{(ii) for each } n \geq 0, \ R_n(x) \text{ satisfies the differential equation:} \)
\[ N \left\{ a_0(x) y(x) + \sum_{i=k+1}^{\infty} a_i(x) y^{(i)}(x) \right\} + L_D[y](x) = \lambda_n y(x). \]  
(3.14)
In fact, each \( a_i(x) \) turns out to be a polynomial of degree \( \leq i \) given by

\[
a_0(x,n) = \begin{cases} 
0, & n = 0, 1, \ldots, k \\
\sum_{i=k}^{n-1} K_i^{(k)}(c,c)(\lambda_i - \lambda_{i+1}), & n \geq k + 1,
\end{cases}
\]

\( n \geq 0 \) \hspace{1cm} (3.15)

\[
a_i(x) = -\frac{1}{P_i^{(i)}(x)} \left\{ a_0(x,i)P_i(x) + \sum_{j=k+1}^{i-1} a_j(x)P_i^{(j)}(x) + P_i^{(k)}(c) \sum_{j=k}^{i-1} (\lambda_i - \lambda_j)P_j(x) P_j^{(k)}(c) \langle \sigma, P_j^2 \rangle \right\}, \quad i \geq k + 1,
\]

\( (3.16) \)

where \( a_0(x,-1) \equiv 0 \) and \( \lambda_{-1} = 0 \).

**Proof.** Let \( \{a_i(x)\}_{i=0}^{\infty} \) be the ones given by (3.15), (3.16), and \( a_i(x) \equiv 0, 1 \leq i \leq k \) (if \( k \geq 1 \)). Then, by Theorem 3.3, \( \{R_\sigma(x)\}_{n=0}^{\infty} \) satisfy Eq. (3.14). In order to prove the uniqueness, we first observe that Eqs. (3.11) and (3.12) are equivalent to Eqs. (3.11) and (3.13) since \( P_n^{(k)}(c) = 0, n \geq k + 1 \) and Eqs. (3.12) and (3.13) hold trivially for \( 0 \leq n \leq k \). Hence, we only need to show that the two equations

\[
a_0(x,n)P_n(x) + \sum_{i=k+1}^{\infty} a_i(x)P_n^{(i)}(x) = 0, \quad n \geq 0
\]

and

\[
a_0(x,n)K_n^{(0,k)}(x,c) + \sum_{i=k+1}^{\infty} a_i(x)K_n^{(i,k)}(x,c) = 0, \quad n \geq 0
\]

have only the trivial solutions for continuous functions \( a_0(x,n) \) and \( \{a_i(x)\}_{i=k+1}^{\infty} \).

First, by substituting \( n = 0, 1, \ldots, k \) into (3.17), we obtain \( a_0(x,i) \equiv 0 \) for \( 0 \leq i \leq k \). If we set \( n = k + 1 \) in (3.18), then \( a_0(x,k+1)P_k(x) \equiv 0 \) so that \( a_0(x,k+1) \equiv 0 \) by the continuity of \( a_0(x,k+1) \) and so \( a_{k+1}(x) \equiv 0 \) by (3.17) for \( n = k + 1 \). Repeating the same process, we obtain \( a_0(x,n) \equiv 0 \) for all \( n \geq 0 \) and \( a_i(x) \equiv 0 \) for all \( i \geq k + 1 \). \( \Box \)

**Remark 3.5.** By using (3.8) and (3.9), we can easily find the leading coefficient \( c_k \) of the polynomial \( a_k(x) \) for \( k \geq 1 \) as

\[
\begin{cases}
\ c_1 = -a_0(x,1) \\
\ c_k = -a_0(x,k)\frac{1}{k!} + \sum_{j=0}^{k-1} \frac{c_j}{(k-j)!}, \quad k \geq 2.
\end{cases}
\]

\( (3.19) \)

Below, we give examples which illustrate Theorems 3.3 and 3.4.

**Example 3.6.** Let \( \sigma \) be the moment functional defined by the weight function (or distribution)

\[
w(x) = \frac{1}{\Gamma(\alpha + 1)} x^\alpha e^{-x} \quad \text{on } [0,\infty), \quad \alpha \neq -1, -2, \ldots.
\]
In this case, the corresponding orthogonal polynomials are the Laguerre polynomials

\[
\left\{ L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!} \right\}_{n=0}^{\infty}
\]

satisfying

\[ xy''(x) + (\alpha + 1 - x)y'(x) + ny(x) = 0. \]

Then

\[
\frac{1}{\Gamma(\alpha + 1)} \langle x^\alpha e^{-x}, L_n^{(\alpha)} \rangle = \binom{n+\alpha}{n}, \quad n \geq 0
\]

and the \(n\)th kernel polynomial is

\[
J_n(x, y) = \sum_{i=0}^{n} \binom{i+\alpha}{i}^{-1} L_i^{(\alpha)}(x)L_i^{(\alpha)}(y).
\]

Hence,

\[
\phi(p, q) = \frac{1}{\Gamma(\alpha + 1)} \langle x^\alpha e^{-x}H(x), pq \rangle + Mp(c)q(c)
\]

is quasi-definite if and only if \(1 + MJ_n(c, c) \neq 0, \quad n \geq 0\). When \(\phi(\cdot, \cdot)\) is quasi-definite, its corresponding OPS \(\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}\) is given by

\[
L_n^{(\alpha)}(x) = (1 + MJ_{n-1}(c, c))L_n^{(\alpha)}(x) - ML_n^{(\alpha)}(c)L_{n-1}(x, c). \quad (3.20)
\]

By Theorem 3.3, \(\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}\) satisfy a differential equation:

\[
M \sum_{i=0}^{n} a_i(x)y^{(i)}(x) + xy''(x) + (\alpha + 1 - x)y'(x) + ny(x) = 0, \quad n \geq 0, \quad (3.21)
\]

where

\[
a_0(x, 0) = 0;
\]

\[
a_0(x, n) = \sum_{i=0}^{n-1} J_i(c, c), \quad n = 1, 2, \ldots, \quad (3.22)
\]

and

\[
a_1(x) = -x + c;
\]

\[
a_k(x) = (-1)^{k+1} \left\{ \sum_{i=0}^{k-1} J_i(c, c)L_k^{(\alpha)}(x) + \sum_{i=1}^{k-1} a_i(x)(L_k^{(\alpha)})^{(i)}(x) \right. \\
+ \left. L_k^{(\alpha)}(c) \sum_{i=0}^{k-1} \frac{(i-k)L_i^{(\alpha)}(c)L_i^{(\alpha)}(x)}{(i+\alpha)} \right\}, \quad k = 2, 3, \ldots. \quad (3.23)
\]

Note that by Theorem 3.4, if \(L_n^{(\alpha)}(c) \neq 0, \quad n \geq 1\), (e.g. it is so when \(c \leq 0\)) then \(\{a_i(x)\}_{i=0}^{\infty}\) are uniquely determined.
When $M > 0$, $\alpha > -1$ (so that $\phi(\cdot, \cdot)$ is positive-definite), and $c = 0$, $\{L_n^{N}(x) := L_n^{N;0}(x)\}_{n=0}^\infty$ was first introduced in [11]. Koekoek and Koekoek [8] showed that they satisfy the differential equation (3.21) and evaluated the coefficients $a_i(x)$ for $i \geq 0$ explicitly as

$$a_0(x) = \binom{n + \alpha + 1}{n - 1}$$

$$a_i(x) = \frac{1}{i!} \sum_{j=1}^{i} (-1)^{i+j+1} \binom{\alpha + 1}{j - 1} \binom{\alpha + 2}{i - j} (x + 3)^{i-j} x^j, \quad i = 1, 2, 3, \ldots.$$

In particular, for nonnegative integer values of $\alpha$ we have $a_i(x) = 0$ for $i \geq 2\alpha + 5$ so that the differential equation (3.21) becomes of finite-order $2\alpha + 4$.

**Example 3.7.** Let $\sigma$ be the moment functional defined by the weight function

$$w(x) = \frac{1}{\sqrt{\pi}} e^{-x^2} \quad \text{on} \quad (-\infty, \infty).$$

In this case, the corresponding orthogonal polynomials are the Hermite polynomials

$$H_n(x) = n! \sum_{k=0}^{[n/2]} \frac{(-1)^k (2x)^{n-2k}}{(n-2k)!k!}$$

satisfying

$$y''(x) - 2xy'(x) + 2ny(x) = 0.$$

Since $\langle \sigma, H_n^2 \rangle = 2^n n!, \ n \geq 0,$

$$K_n(0,0) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(2i)!}{2^i (i!)^2}, \quad n \geq 0 \ ([2, 19]).$$

For any $N$ such that $1 + NK_n(0,0) \neq 0$, $n \geq 0$,

$$\phi(p, q) := \langle \sigma, pq \rangle + Np(0)q(0)$$

is quasi-definite and its corresponding OPS $\{H_n^N(x)\}_{n=0}^\infty$ is given by

$$H_n^N(x) = \begin{cases} 
(1 + \sum_{i=0}^{m-1} \frac{(2i)!}{2^i (i!)^2}) H_{2m}(x) + (-1)^{m+1} \frac{(2m)!}{m!} NK_{2m-1}(x,0), \quad n = 2m \ (m \geq 0) \\
(1 + \sum_{i=0}^{m} \frac{(2i)!}{2^i (i!)^2}) H_{2m+1}(x), \quad n = 2m + 1 \ (m \geq 0).
\end{cases}$$

By Theorem 3.3, $\{H_n^N(x)\}_{n=0}^\infty$ satisfy a differential equation:

$$N \sum_{i=0}^{\infty} a_i(x) (H_n^N)'(x) + (H_n^N)''(x) - 2x(H_n^N)'(x) + 2nH_n^N(x) = 0, \quad n \geq 0,$$

$$n \geq 0,$$  \hspace{1cm} (3.24)
where

\[ a_0(x, 0) = 0; \]

\[ a_0(x, n) = \sum_{j=0}^{n-1} \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(2j)!}{2^{2i-1}(i!)^2}, \quad n \geq 1 \]

and

\[ a_k(x) = \begin{cases} -2x, & k = 1; \\ \frac{1}{k!2^k} \left( \sum_{j=0}^{k-1} \sum_{i=0}^{\left\lfloor \frac{j}{2} \right\rfloor} \frac{(2j)!}{2^{2i-1}(i!)^2} \right) H_k(x) + \sum_{i=1}^{k-1} a_i(x) H_k^{(i)}(x) \right), & k = 2j \ (j \geq 1); \\ \frac{1}{k!2^k} \left( \sum_{j=0}^{k-1} \sum_{i=0}^{\left\lfloor \frac{j}{2} \right\rfloor} \frac{(2j)!}{2^{2i-1}(i!)^2} \right) H_k(x) + \sum_{i=1}^{k-1} a_i(x) H_k^{(i)}(x) \right), & k = 2j - 1 \ (j \geq 2). \end{cases} \]

Note that the differential equation (3.24) is of infinite-order since we can easily show that the leading coefficients \( c_k \) of \( a_k(x) \) for \( k \geq 3 \) (see (3.19)) are

\[ c_{2k+1} = \sum_{j=1}^{k} \left( \sum_{i=0}^{2k-2j+i} \frac{(-1)^{i+1} (2k - 2j - i + 1)}{(2k - i + 1)!} \right) \frac{(2j)!}{2^{2j-1}(j!)^2} < 0, \quad k \geq 1, \]

\[ c_{2k+2} = \sum_{j=1}^{k} \left( \sum_{i=0}^{2k-2j+i+1} \frac{(-1)^{i+1} (2k - 2j - i + 2)}{(2k - i + 2)!} \right) \frac{(2j)!}{2^{2j-1}(j!)^2} > 0, \quad k \geq 1. \]

**Example 3.8.** Consider a symmetric bilinear form of Sobolev type on \( \mathcal{P} \times \mathcal{P} \) defined by

\[ \phi(p, q) := \frac{1}{I(\alpha + 1)} \left( x^\alpha e^{-x} , pq \right) + Mp(c)q(c) + Np'(c)q'(c) \quad (p, q \in \mathcal{P}), \quad (3.25) \]

where \( \alpha \neq -1, -2, \ldots \), \( M, N \), and \( c \) are real constants. We always assume that

\[ 1 + M J_n(c, c) \neq 0 \quad \text{and} \quad 1 + N K^{(1)}_{n-1}(c, c) \neq 0, \quad n \geq 0 \]

so that \( \phi(\cdot, \cdot) \) is quasi-definite (see Proposition 3.1), where \( J_n(x, y) \) and \( K_n(x, y) \) are the kernel polynomials of order \( n \) corresponding to OPS's \( \{L_n^{(\alpha)}(x)\}_{n=0}^\infty \) and \( \{L_n^{M, c}(x)\}_{n=0}^\infty \) (cf. (3.20)) respectively. We call the corresponding OPS \( \{L_n^{M, N, c}(x)\}_{n=0}^\infty \) the generalized Sobolev–Laguerre polynomials. Then by Proposition 3.1, we have

\[ L_n^{M, N, c}(x) = (1 + N K^{(1)}_{n-1}(c, c)) L_n^{M, c}(x) - N(L_n^{M, c})'(c) K^{(0,1)}_{n-1}(x, c), \quad n \geq 0. \]

The generalized Sobolev–Laguerre polynomials \( \{L_n^{M, N, c}(x)\}_{n=0}^\infty \) satisfy a differential equation:

\[ N \sum_{i=0}^{\infty} d_i(x) y^{(i)}(x) + M \sum_{i=0}^{\infty} a_i(x) y^{(i)}(x) + xy''(x) + (\alpha + 1 - x) y'(x) + ny(x) = 0, \quad n \geq 0, \quad (3.26) \]
where \{a_i(x)\}_{i=0}^{\infty} are given by (3.22) and (3.23) and

\[
\begin{cases}
  d_0(x, 0) = 0; \\
  d_0(x, 1) \text{ is an arbitrary constant;}
\end{cases}
\]

\[
d_0(x, n) = d_0(x, 1) + \sum_{i=1}^{n-1} K_i^{1, 1}(c, c) \left\{ 1 + M \sum_{j=0}^{i} \binom{j + \alpha}{j} \left\{ L_j^{(x)}(c) \right\}^2 \right\}, \quad n \geq 2
\]

and

\[
d_i(x) = (-1)^{i+1} \tilde{k}_{i-1} \left[ d_0(x, i)L_i^{x, M; c}(x) + \sum_{j=1}^{i-1} d_i(x)(L_j^{x, M; c})^{(j)}(x) - (L_i^{x, M; c})'y(c) \sum_{j=1}^{i-1} \tilde{k}_j \left( \frac{j + \alpha}{\alpha} \right)^{-1} \left\{ i - j + M \sum_{k=j}^{i-1} \sum_{m=0}^{k} \binom{m + \alpha}{m} \right\} \right] \\
\times (L_{m}^{(x)}(c))^2 \{ (L_j^{x, M; c})'y(c)L_j^{x, M; c}(x) \}, \quad i \geq 1,
\]

where

\[
\tilde{k}_n = 1 + MJ_n(c, c) = 1 + M \sum_{i=0}^{n} \binom{i + \alpha}{i} (L_i^{(x)}(c))^2, \quad n \geq 0.
\]

In particular, if

\[
(L_n^{x, M; c})'(c) \neq 0, \quad n \geq 2
\]

(3.27)

and if we choose \(d_1(x) = 0\), then the differential equation (3.26) is the unique such differential equation with polynomial coefficients, which has \{L_n^{x, M; c}(x)\}_{n=0}^{\infty} as solutions (see Theorem 3.4). For example, when \(c = 0\), we have

\[
(L_n^{x, M})(0) = - \binom{n + \alpha}{n - 1} \left[ 1 + M \sum_{i=0}^{n-1} (n - i) \binom{i + \alpha}{i} \right], \quad n > 0
\]

so that the condition (3.27) is satisfied when \(c = 0, \alpha > -1,\) and \(M \geq 0\).

Recently, Koekoek et al. [9] succeeded in finding all differential equations of the form

\[
M \sum_{i=0}^{\infty} a_i(x)y^{(i)}(x) + N \sum_{i=0}^{\infty} b_i(x)y^{(i)}(x) + MN \sum_{i=0}^{\infty} c_i(x)y^{(i)}(x) + xy''(x) + (\alpha + 1 - x)y'(x) + ny(x) = 0
\]

(3.28)

for \{L_n^{x, M, N}(x)\}_{x=0}^{\infty}, when \(\alpha > -1, M \geq 0,\) and \(N \geq 0,\) and computed all the coefficients \(a_i(x), b_i(x), c_i(x)\) explicitly. In case \(M + N \geq 0,\) they also show that the order of the differential equation (3.28) is finite only for nonnegative integer values of \(\alpha.\)
\( \{L_n^{M_0, M_1, \ldots, M_N}(x)\}_{n=0}^{\infty} \) which are orthogonal with respect to the Sobolev inner product

\[
\frac{1}{\Gamma(\alpha + 1)} \int_0^\infty f(x)g(x)x^\alpha e^{-x} \, dx + \sum_{j=0}^N M_j f^{(j)}(0)g^{(j)}(0),
\]

where \( \alpha > -1, N = 0,1,2, \ldots, \) and \( M_j \geq 0, 1 \leq j \leq N \) and found a second-order differential equation of the form

\[
a(x,n)y'' + b(x,n)y' + c(x,n)y = 0
\]
satisfied by these polynomials.

By applying Theorem 3.3 successively, one can obtain an infinite-order differential equation of spectral type

\[
\sum_{j=0}^N M_j \sum_{i=0}^\infty a_{ij}(x)y^{(i)}(x) + xy''(x) + (\alpha + 1 - x)y'(x) + ny(x) = 0,
\]

where each \( a_{ij}(x) \) is a polynomial of degree \( \leq i \), independent of \( n \) except \( a_{0i}(x) = a_{0i}(x,n) \).

4. Finite-order differential equations

From the viewpoint of spectral analysis of differential operators (see [3, 4] and references therein), it is interesting and important to know whether an OPS relative to \( \phi(\cdot, \cdot) \) in (1.5) satisfies a finite-order differential equation of spectral type.

We should be able to check it by simply looking at the coefficients \( a_i(x) \) in (3.9), but it is, in general, very difficult to compute explicitly all \( a_i(x), i \geq 0 \). See the computations in [1,7–9] for a few known cases.

Assume that the OPS \( \{P_n(x)\}_{n=0}^{\infty} \) relative to \( \sigma \) satisfies a differential equation (1.4) of a finite-order \( D \) and consider a quasi-definite point mass perturbation \( \tau \) of \( \sigma \) such that

\[
\tau = \sigma + N\delta(x - c) \quad (N(\neq 0), c \in \mathbb{R}).
\]

Then the OPS \( \{R_n(x)\}_{n=0}^{\infty} \) relative to \( \tau \) satisfies a differential equation of the form (3.7).

It is known (see [13,15]) that \( D = 2r \) \( (r \geq 1) \) must be an even integer and \( \sigma \) satisfies \( r \) equations

\[
R_k(\sigma) := \sum_{i=2k+1}^{2r} (-1)^i \binom{i-k-1}{k} (\ell_i \sigma)^{i-2k-1} = 0, \quad 0 \leq k \leq r - 1. \tag{4.1}
\]

Moreover, what is important to us is that \( \sigma \) is the only one linearly independent solution of the overdetermined system of equations (4.1) (see Theorem 3.4 in [14]). Hence, any nontrivial moment functional solution of the system (4.1) must be quasi-definite since it is a nonzero constant multiple of \( \sigma \).

If we let \( v(x) \) be a distributional representation of \( \sigma \), then \( v(x) \) satisfies \( r \) nonhomogeneous system of differential equations (called weight equations):

\[
R_k(v) = g_k(x), \quad 0 \leq k \leq r - 1, \tag{4.2}
\]

where \( g_k(x) \) are distributions having zero moments but need not be 0 in general (see [13]).
Proposition 4.1. Assume that $\sigma$ has a distributional representation $v(x)$, which satisfies homogeneous weight equations:

$$R_k(v) = 0, \quad 0 \leq k \leq r - 1. \quad (4.3)$$

If the OPS $\{R_n(x)\}_{n=0}^\infty$ relative to $\tau$ also satisfies a finite-order differential equation of the type (1.4):

$$L_{2s}[y] = \sum_{i=1}^{2s} m_i(x)y^{(i)}(x) = \mu_n y(x), \quad (4.4)$$

then $w(x) := v(x) + N\delta(x - c)$ also satisfies homogeneous weight equations

$$S_k(w) := \sum_{i=2k+1}^{2s} (-1)^i \binom{i-k-1}{k}(m_iw)^{(i-2k-1)} = 0, \quad 0 \leq k \leq s - 1. \quad (4.5)$$

Before proving Proposition 4.1, we note that the condition (4.3) does not hold in general as we can see in the case of Bessel polynomials (see [13]). However, the condition (4.3) holds for all other classical orthogonal polynomials or if $\text{supp}(v)$ is compact (see Lemma 4.2 in [14]).

**Proof of Proposition 4.1.** We know that $w(x)$ must satisfy nonhomogeneous weight equations

$$S_k(w) = h_k(x), \quad 0 \leq k \leq s - 1,$$

where each $h_k(x)$ is 0 as a moment functional. We need to show that $h_k(x) \equiv 0$ as a distribution for $0 \leq k \leq s - 1$. For $k = s - 1$,

$$h_{s-1}(x) = S_{s-1}(w) = sm_{2s}(x)v'(x) + (sm'_{2s}(x) - m_{2s-1}(x))v(x) + u(x), \quad (4.5)$$

where $u(x) = N[sm_{2s}(c)v'(x) + (sm'_{2s}(x) - m_{2s-1}(x))v(x)], \quad (4.5)$

Multiplying Eq. (4.5) by $r\ell_2(x)(x - c)^2$ and using $R_{r-1}(v) = 0$, we obtain

$$\pi(x)v(x) = r\ell_2(x)(x - c)^2h_{r-1}(x),$$

where $\pi(x) = (x - c)^2[sm_{2s}(x)(\ell_{2r-1}(x) - r\ell'_2(x)) - r\ell_2(x)(m_{2s+1}(x) - sm'_2(x))]$. Since $h_{s-1}(x) \equiv 0$ as a moment functional, $\pi(x)v(x) \equiv 0$ as a moment functional so that $\pi(x)\sigma = 0$. Hence $\pi(x) \equiv 0$ and so $r\ell_2(x)(x - c)^2h_{r-1}(x) \equiv 0$ since $\sigma$ is quasi-definite (see Lemma 2.3 in [15]). Therefore, $h_{s-1}(x)$ must be a finite linear combination of derivatives of Dirac delta distributions of the form $\delta(x - a)$, where $a$ is either $c$ or a real root of $\ell_2(x)$. Hence, $h_{s-1}(x) \equiv 0$ as a distribution since $h_{s-1}(x) \equiv 0$ as a moment functional (see Lemma 4.2 in [14]). Similarly, we can show $h_k(x) \equiv 0$ as distributions for $0 \leq k \leq r - 1$. □

For any subset $A$ of $\mathbb{R}$, we let $\partial(A)$ and $\text{Int}(A)$ denote the set of boundary points and interior points of $A$ respectively.

**Theorem 4.2.** Let $\sigma$ be the same as in Proposition 4.1. If either $c \notin \text{supp}(v)$ or $\ell_2(c) \neq 0$ and $c \notin \partial(\text{supp}(v))$, then $\{R_n(x)\}_{n=0}^\infty$ cannot satisfy any finite-order differential equation of type (4.4). In particular, any differential equation (3.7) satisfied by $\{R_n(x)\}_{n=0}^\infty$ must be of infinite order.
Proof. Assume that \( \{R_n(x)\}_{n=0}^{\infty} \) satisfies a differential equation (4.4). If \( c \notin \text{supp}(v) \), then there is a positive number \( \varepsilon \) such that \( (c - \varepsilon, c + \varepsilon) \cap \text{supp}(v) = \emptyset \) so that

\[
w(x) = N\delta(x - c) \quad \text{on} \quad (c - \varepsilon, c + \varepsilon).
\]

Then \( S_k(w) = NS_k(\delta(x - c)) = 0 \), \( 0 \leq k \leq s - 1 \), on \( (c - \varepsilon, c + \varepsilon) \) and so \( S_k(\delta(x - c)) = 0 \), \( 0 \leq k \leq s - 1 \), on \( \mathbb{R} \). Hence, \( \delta(x - c) \) must be a quasi-definite moment functional (see Theorem 3.4 in [14]), which is a contradiction.

We now assume that \( \ell_{2r}(c) \neq 0 \) and \( c \in \text{Int}(\text{supp}(v)) \). Since \( R_{r-1}(v) = \ell_{2r}(x)v'(x) + (r\ell'_{2r}(x) - \ell_{2r-1}(x))v(x) = 0 \) and \( \ell_{2r}(c) \neq 0 \), there is a positive number \( \varepsilon \) such that \( (c - \varepsilon, c + \varepsilon) \subseteq \text{Int}(\text{supp}(v)) \) and \( v(x) \in C^\infty \) in \( (c - \varepsilon, c + \varepsilon) \). Since \( S_k(w) = S_k(v) = 0 \), \( 0 \leq k \leq s - 1 \), on \( (c - \varepsilon, c + \varepsilon) \) and \( v \in C^\infty((c - \varepsilon, c + \varepsilon)) \), \( S_k(v) = 0 \), \( 0 \leq k \leq s - 1 \), on \( (c - \varepsilon, c + \varepsilon) \).

Hence, \( S_k(w) = S_k(v) + NS_k(\delta(x - c)) = NS_k(\delta(x - c)) = 0 \), \( 0 \leq k \leq s - 1 \), on \( (c - \varepsilon, c + \varepsilon) \). It leads to a contradiction as before.\[\square\]

In case \( \{P_n(x)\}_{n=0}^{\infty} \) is a classical OPS except Bessel polynomials, \( \{P_n(x)\}_{n=0}^{\infty} \) satisfies a second-order differential equation (1.5) with \( D = 2 \) and the condition (4.3) always holds. Moreover, the leading coefficient \( \ell_2(x) \) of the differential equation (1.5) has no root in \( \text{Int}(\text{supp}(v)) \). Therefore, by Theorem 4.2, \( \{R_n(x)\}_{n=0}^{\infty} \) can never satisfy a finite-order differential equation (4.4) unless \( c \) is a boundary point of \( \text{supp}(v) \).

In particular, the differential equation (3.7) satisfied by \( \{R_n(x)\}_{n=0}^{\infty} \) must be of infinite-order if \( c \notin \text{supp}(v) \). When \( c \in \partial(\text{supp}(v)) \), the differential equation (3.7) may or may not be of finite-order as we can see from Example 3.6.

For example, the differential equation (3.21) must be of infinite order for any \( c \neq 0 \).

Finally, we note that we can easily extend results in this section to the case when \( \tau \) is obtained from \( \sigma \) by adding two point masses as in [6, 7, 9, 11].

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