European Journal of Combinatorics 33 (2012) 148–167

Contents lists available at SciVerse ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Child's addition in the Stern–Brocot tree

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A R T I C L E   I N F O

Article history:
Received 12 May 2010
Accepted 14 September 2011
Available online 19 October 2011

A B S T R A C T

We use child's addition and cross-differencing to discover significant relationships for diagonals, paths and branches within the Stern–Brocot tree and the Stern–Brocot sequence. This allows us to develop results for continued fraction summation under child's addition.

1. Introduction and preliminaries

In earlier studies [2,3], we identified diagonalisation within the Stern–Brocot and Calkin–Wilf trees to define normalised additive factorisation. Such factorisation allowed us to locate and identify entries in both trees and gave us a simple method for converting terms in the Stern–Brocot tree into vertices in the Calkin–Wilf tree and vice versa.

In this study of the Stern–Brocot tree, we extensively develop two related concepts, *child's addition* and *cross-differencing*, to achieve the following goals:

(1) Discover a recursive result (Theorem 10) for determining *diagonals* within the tree.
(2) Generalise branching in the tree (Section 4) to show that traditional branches are particular cases of *offset branches* (Theorem 15).
(3) Introduce the concept of a *path* to demonstrate (Theorem 16) that path cross-differences,
- are constant for two given paths that commence from a common term, irrespective of the choice of that common term; and
- possess the same magnitude, but are different in sign, for *symmetry couplets*.

(4) Demonstrate that progressive child's addition of consecutive terms in a level of the tree (Theorem 19) yields a value that is dependent only on the last two terms of the addition; that is, the addition is not dependent on the number or size of earlier summands.
(5) Discover results (Theorem 20) for the child's addition of continued fractions.

We state some important definitions and results. Proofs of results can be found in [2].

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0195-6698/$ – see front matter
doi:10.1016/j.ejc.2011.09.038

Definition 1 (*Child's Addition*)**.** Let $\frac{a}{b}$ and $\frac{c}{d}$ be either a fraction or a pseudofraction ($\frac{0}{1}$ or $\frac{1}{0}$). Then we define the child's addition operators, $\oplus$ and $\tilde{\oplus}$, as follows

(i) $\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d}$, and

(ii) $\frac{a}{b} \tilde{\oplus} \frac{c}{d} = \frac{a+c}{b+d}$ in its reduced form.

Example 1. $\frac{1}{2} \oplus \frac{1}{4} = \frac{2}{6}$; $\frac{1}{2} \tilde{\oplus} \frac{1}{4} = \frac{1}{3}$; $\frac{1}{2} \oplus \frac{1}{3} = \frac{1}{2} \tilde{\oplus} \frac{1}{3} = \frac{2}{5}$.

Note that:

(i) $\oplus$ is associative, whereas $\tilde{\oplus}$ is not, in general, associative.

(ii) For $m \neq 1$, $\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d} \neq \frac{ma}{mb} \oplus \frac{c}{d} = \frac{ma+c}{mb+d}$.

Definition 2 (*Stern–Brocot Sequence*)**.** With $a_{0,1} = 0$ and $a_{0,2} = 1$, we define for $n \geq 0$,

$$A_n = \left\langle a_{n,1}, a_{n,2}, \ldots, a_{n,2^n+1} \right\rangle$$

as the sequence for which, for $k \geq 1$, $n > 0$,

$$a_{n,2k-1} = a_{n-1,k} \quad \text{and}$$
$$a_{n,2k} = a_{n-1,k} + a_{n-1,k+1}.$$

Similarly, with $b_{0,1} = 1$, $b_{0,2} = 0$, we define B_n. Then the sequence defined by $H_n = \left\langle h_{n,1}, h_{n,2}, \ldots, h_{n,2^n+1} \right\rangle$ where $h_{n,i} = \frac{a_{n,i}}{b_{n,i}}$ is called the *Stern–Brocot Sequence* of order n. It represents the sequence containing both the first n generations of mediants based on H_0, and the terms of H_0 itself.

Definition 3 (*Parents and Children*)**.** We call:

(a) $h_{n-1,k}$ and $h_{n-1,k+1}$, the left and right parents respectively of $h_{n,2k}$,

(b) $h_{n,2k}$, the left child of $h_{n-1,k+1} (= h_{n,2k+1})$,

(c) $h_{n,2k}$, the right child of $h_{n-1,k} \left(= h_{n,2k-1}\right)$.

Definition 4 (medH_{n-1})**.** Let H_0 be level 0 of the *Stern–Brocot tree*. For $n > 0$, level n of the Stern–Brocot tree is defined as medH_{n-1} where

$$\begin{aligned} \mathrm{med}H_{n-1} &= \left\langle \left(h_{n-1,1} \oplus h_{n-1,2}\right), \left(h_{n-1,2} \oplus h_{n-1,3}\right), \ldots, \left(h_{n-1,2^{n-1}} \oplus h_{n-1,2^{n-1}+1}\right) \right\rangle, \\ &= \left\langle h_{n,2}, h_{n,4}, h_{n,6}, \ldots, h_{n,2^n} \right\rangle. \end{aligned}$$

We note from Definition 4 that the levels of the Stern–Brocot tree contain only newly produced terms so that for $n > 0$, level n contains 2^{n-1} terms and the kth term of level n is $h_{n,2k}$.

Definition 5 (*Left and Right Branches*)**.** Let β be the left(right) child of μ. The left(right) branch of μ consists of β followed by all successive right(left) children beginning with the right(left) child of β.

Theorem 1. *We have*

(i) *the left branch of* $h_{n,k}$ *is* $\left\langle h_{n+m,2^m(k-1)} \mid m \in \mathbb{N} \right\rangle$,

(ii) *the right branch of* $h_{n,k}$ *is* $\left\langle h_{n+m,2^m(k-1)+2} \mid m \in \mathbb{N} \right\rangle$.

Proof. We have

(i) By Definition 3(b), the left child of $h_{n,k}$ is $h_{n+1,2(k-1)}$. By applying Definition 3(c), $m-1$ times, the $(m-1)$th right child of $h_{n+1,2(k-1)}$ is $h_{n+m,2^m(k-1)}$.

(ii) By Definition 3(c), the right child of $h_{n,k}$ is $h_{n+1,2k}$. By applying Definition 3(b), $m-1$ times, the $(m-1)$th left child of $h_{n+1,2k}$ is $h_{n+m,2^m(k-1)+2}$. $\square$

Theorem 2. *For* $0 < i \leq 2^n$, $a_{n,i+1}b_{n,i} - a_{n,i}b_{n,i+1} = 1$.

Definition 6 (*Left Diagonals*)**.** (i) For $k \in \mathbb{N}$, the kth *left diagonal* of the tree, $\mathcal{L}_k$, is the sequence made up of *each* kth *term from each level* beginning at the level in which a kth term is first found (that is, level $\lceil \log_2 k \rceil + 1$).

(ii) The first$^+$ *left diagonal* of the tree, $\mathcal{L}_{1+}$, is the first left diagonal preceded by the term $\frac{1}{0}$.

(Right diagonals).

(i) For $k \in \mathbb{N}$, the kth *right diagonal* of the tree, $\mathcal{R}_k$, is the sequence made up of *each* kth *term taken from the end of each level* beginning at the level in which a kth term is first found (that is, level $\lceil \log_2 k \rceil + 1$).

(ii) The first$^+$ *right diagonal* of the tree, $\mathcal{R}_{1+}$, is the first right diagonal preceded by the term $\frac{0}{1}$.

Note that left diagonals are read from upper right to lower left of the tree, whereas right diagonals are read from upper left to lower right of the tree.

Example 2. The first, first$^+$, second, third and fourth left diagonals are the respective sequences $\left\langle \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \right\rangle, \left\langle \frac{1}{0}, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots \right\rangle, \left\langle \frac{2}{1}, \frac{2}{3}, \frac{2}{5}, \frac{2}{7}, \ldots \right\rangle, \left\langle \frac{3}{2}, \frac{3}{5}, \frac{3}{8}, \frac{3}{11}, \ldots \right\rangle$ and $\left\langle \frac{3}{1}, \frac{3}{4}, \frac{3}{7}, \frac{3}{10}, \ldots \right\rangle$.

Theorem 3 (*The "$\frac{a}{a+b}$ rule" for left diagonals*)**.** *For $n > 1$, let $\frac{a}{b}$ be $h_{n-1,2k}$, the kth entry in level $n-1$ of the Stern–Brocot tree. Then*

(i) *$h_{n,2k}$, the kth entry in level n, is $\frac{a}{a+b}$.*

(ii) *$h_{n,2^n-2k+2}$, the $\left(2^{n-1}-k+1\right)$th entry in level n, is $\frac{a+b}{a}$.*

Corollary 1. *Terms on the same level of the Stern–Brocot tree that are equidistant from either end are reciprocals of each other. Such terms are styled symmetric complements. Algebraically, using the notation of Definition 2, $a_{n,k} = b_{n,2^{n-1}-k+1}$.*

The following result on continued fractions, based on Graham et al. [4], is found in Bates et al. [1].

Theorem 4. *For $a_0 \geq 0$, $a_i \geq 1$, $i = 1, 2, \ldots, k$,*

$$[a_0, a_1, a_2, \ldots, a_{k-1}, a_k + 1] = \begin{cases} R^{a_0+1} L^{a_1} R^{a_2} \cdots L^{a_k}, & \text{for } k \text{ odd} \\ R^{a_0+1} L^{a_1} R^{a_2} \cdots R^{a_k}, & \text{for } k \text{ even} \end{cases}$$

where beginning at $\frac{0}{1}$ in the tree , $R^{a_0+1}L^{a_1}\cdots$ represents a successive right-and-down movement performed a_0+1 times; followed by a left-and-down movement performed a_1 times; and so on.

2. Cross-differences

We introduce the notion of cross-differences. In later sections, cross-differences are used to identify interesting properties of the tree.

Definition 7 (*Cross-Differences*)**.** The cross-difference of $\frac{a}{b}$ and $\frac{c}{d}$ is $bc - ad$.

Theorem 2 reveals that all cross-differences of consecutive terms in H_n have cross-difference 1.

Definition 8 (*Stern–Brocot Cross-Differences*)**.** For $i = 1, 2, \ldots, 2^{n-1} - 1$, and $n > 1$, $\mathbb{C}_{n,i}$, the ith Stern–Brocot Cross-Difference in level n, is given by

$$\mathbb{C}_{n,i} = a_{n,2i+2} b_{n,2i} - a_{n,2i} b_{n,2i+2}$$

where $a_{n,2i}$, $a_{n,2i+2}$, $b_{n,2i}$ and $b_{n,2i+2}$ are terms defined in Definition 2.

We conclude this section with our main result for cross-differences.

Theorem 5 (*Main Result for Cross-Differences*)**.** *Let $i = 2^j\,(2m-1)$. Then*

$$\mathbb{C}_{n,i} = 2j + 3.$$

Proof. By Definition 8,

$$\begin{aligned}
\mathbb{C}_{n,i} &= a_{n,2i+2}b_{n,2i} - a_{n,2i}b_{n,2i+2},\\
&= \left(a_{n-1,i+1} + a_{n-1,i+2}\right)\left(b_{n-1,i} + b_{n-1,i+1}\right)\\
&\quad - \left(a_{n-1,i} + a_{n-1,i+1}\right)\left(b_{n-1,i+1} + b_{n-1,i+2}\right) \quad \text{by Definition 2},\\
&= a_{n-1,i+1}b_{n-1,i} - a_{n-1,i}b_{n-1,i+1}\\
&\quad + a_{n-1,i+2}b_{n-1,i} - a_{n-1,i}b_{n-1,i+2}\\
&\quad + a_{n-1,i+2}b_{n-1,i+1} - a_{n-1,i+1}b_{n-1,i+2},\\
&= 2 + a_{n-1,i+2}b_{n-1,i} - a_{n-1,i}b_{n-1,i+2} \quad \text{by Theorem 2}. \qquad (2.1)
\end{aligned}$$

For i odd, that is, for $j = 0$, (2.1) becomes by Definition 2 and Theorem 2,

$$\mathbb{C}_{n,i} = 2 + a_{n-2,\frac{i+1}{2}+1}b_{n-2,\frac{i+1}{2}} - a_{n-2,\frac{i+1}{2}}b_{n-2,\frac{i+1}{2}+1} = 3. \qquad (2.2)$$

For i even, that is, for $j > 0$, (2.1) becomes by Definition 8, $\mathbb{C}_{n,i} = 2 + \mathbb{C}_{n-1,\frac{i}{2}}$. Therefore

$$\begin{aligned}
\mathbb{C}_{n,i} &= 2 + \mathbb{C}_{n-1,\frac{i}{2}},\\
&= 4 + \mathbb{C}_{n-2,\frac{i}{4}},\\
&\;\;\vdots\\
&= 2j + \mathbb{C}_{n-j,\frac{i}{2^j}},\\
&= 2j + \mathbb{C}_{n-j,2m-1},\\
&= 2j + 3 \quad \text{by (2.2)}. \qquad \square
\end{aligned}$$

3. Parents, branches and diagonals

From Definition 4, any two consecutive terms in any level of the tree have a common parent in a previous level, not necessarily the previous one. For example, every middle term in every level has common parent $\frac{1}{1}$. We find that parenthood and branching are linked in interesting ways.

Lemma 1. *We have*(i) $a_{n,2^i(2m-1)} = a_{n-1,2^{i-1}(2m-1)} + a_{n-i,2m}$,
(ii) $a_{n,2^i(2m-1)} = a_{n-i-1,m} + i \cdot a_{n-i,2m}$,
(iii) $a_{n,2^i(2m-1)+2} = a_{n-i-1,m+1} + i \cdot a_{n-i,2m}$,
(iv) $a_{n,2^i(2m-1)+2} = a_{n-1,2^{i-1}(2m-1)+2} + a_{n-i,2m}$,
(v) $a_{n,2^i(2m-1)+1} = a_{n-i,2m} = a_{n-i-1,m} + a_{n-i-1,m+1}$.

Proof. By Definition 2. $\square$

Lemma 2.

$$a_{n,2^i(2m-1)} + a_{n,2^i(2m-1)+2} = (2i+1)\,a_{n-i,2m}.$$

Proof. By Lemma 1(ii) and (iii), and Definition 2,

$$\begin{aligned}
a_{n,2^i(2m-1)} + a_{n,2^i(2m-1)+2} &= a_{n-i-1,m} + a_{n-i-1,m+1} + 2i.a_{n-i,2m},\\
&= (2i+1)\,a_{n-i,2m}. \qquad \square
\end{aligned}$$

Identical results hold, when each $a_{s,t}$ is substituted with $b_{s,t}$ in Lemmas 1 and 2. We are now able to state a number of interesting properties of the Stern–Brocot sequence based on child's addition.

Theorem 6. *We have* (i) $h_{n,2^i(2m-1)} = h_{n-1,2^{i-1}(2m-1)} \oplus h_{n-i,2m}$,
(ii) $h_{n,2(2m-1)} = h_{n-2,m} \oplus h_{n-1,2m}$,
(iii) $h_{n,4m} = h_{n-2,m+1} \oplus h_{n-1,2m}$,
(iv) $h_{n,2^i(2m-1)+2} = h_{n-1,2^{i-1}(2m-1)+2} \oplus h_{n-i,2m}$,
(v) $h_{n,2^i(2m-1)+1} = h_{n-i-1,m} \oplus h_{n-i-1,m+1}$,
(vi) $h_{n-i,2m} = h_{n,2^i(2m-1)} \tilde{\oplus} h_{n,2^i(2m-1)+2}$,
(vii) *If* $k = 2^i(2m-1)$, *then* $h_{n,k+1} = h_{n-i,2m+2}$,
(viii) $h_{n,2k+3} = h_{n,2k+1} \tilde{\oplus} h_{n,2k+4}$,
(ix) $h_{n,2k+1} = h_{n,2k} \tilde{\oplus} h_{n,2k+3}$.

Proof. (i)–(v): By Lemma 1(i)–(v) respectively.
(vi): By Lemma 2.
(vii): By Lemma 1(v).
(viii): By Definition 2,

$$\begin{aligned}
a_{n,2k+1} + a_{n,2k+4} &= a_{n-1,k+1} + a_{n-1,k+2} + a_{n-1,k+3}, \\
&= \left(a_{n-1,k+1} + a_{n-1,k+3}\right) + a_{n-1,k+2}, \\
&= \left(a_{n-1,2^i(2m-1)} + a_{n-1,2^i(2m-1)+2}\right) + a_{n-1,2^i(2m-1)+1}, \\
&= (2i+1)\, a_{n-i-1,2m} + a_{n-i-1,2m} \text{by Lemma 1(v) and Lemma 2}, \\
&= (2i+2)\, a_{n-i-1,2m}, \\
&= (2i+2)\, a_{n,2^{i+1}(2m-1)+1}, \\
&= (2i+2)\, a_{n,2k+3}.
\end{aligned}$$

Similarly,

$$b_{n,2k+1} + b_{n,2k+4} = (2i+2)\, b_{n,2k+3}.$$

Therefore

$$\frac{(2i+2)\, a_{n,2k+3}}{(2i+2)\, b_{n,2k+3}} = \frac{a_{n,2k+1} + a_{n,2k+4}}{b_{n,2k+1} + b_{n,2k+4}}.$$

That is, $h_{n,2k+3} = h_{n,2k+1} \tilde{\oplus} h_{n,2k+4}$.
(ix) Let $k = 2^i\,(2m-1)$. Then by Definition 2 and Lemma 1(ii) and (v),

$$\begin{aligned}
a_{n,2k} &= a_{n-1,k} + a_{n-1,k+1}, \\
&= \left(a_{n-i-2,m} + i a_{n-i-1,2m}\right) + a_{n-i-1,2m}, \\
&= a_{n-i-2,m} + (i+1)\, a_{n-i-1,2m},
\end{aligned} \tag{3.1}$$

and by Definition 2 and Lemma 1(iii)

$$\begin{aligned}
a_{n,2k+3} &= a_{n-1,k+2}, \\
&= a_{n-i-2,m+1} + i a_{n-i-1,2m}.
\end{aligned} \tag{3.2}$$

Combining (3.1) and (3.2) by Definition 2 and Lemma 1(v),

$$\begin{aligned}
a_{n,2k} + a_{n,2k+3} &= (2i+1)\, a_{n-i-1,2m} + a_{n-i-2,m} + a_{n-i-2,m+1}, \\
&= (2i+1)\, a_{n-i-1,2m} + a_{n-i-1,2m}, \\
&= (2i+2)\, a_{n-i-1,2m}, \\
&= (2i+2)\, a_{n,2k+1}.
\end{aligned}$$

Similarly, $b_{n,2k} + b_{n,2k+3} = (2i+2)\, b_{n,2k+1}$. Thus

$$h_{n,2k} \oplus h_{n,2k+3} = \frac{a_{n,2k} + a_{n,2k+3}}{b_{n,2k} + b_{n,2k+3}} = \frac{(2i+2)\, a_{n,2k+1}}{(2i+2)\, b_{n,2k+1}}.$$

That is, $h_{n,2k} \tilde{\oplus} h_{n,2k+3} = h_{n,2k+1}$. □

The following theorem tells us that the reduced value of the child's addition of any two consecutive terms in the tree is the common parent of those terms; and that the number of levels separating these consecutive terms and their parent can be readily determined from the cross-difference of the two terms.

Lemma 3. *Let $\frac{a}{b}$ and $\frac{c}{d}$, where $\frac{a}{b} < \frac{c}{d}$, be the parents of $\frac{e}{g}$. Then for $i = 1, 2, 3, \ldots,$*

(i) *The left branch of $\frac{e}{g}$ is*

$$\left\{ \frac{(i+1)\,a + ic}{(i+1)\,b + id} \right\}.$$

(ii) *The right branch of $\frac{e}{g}$ is*

$$\left\{ \frac{ia + (i+1)\,c}{ib + (i+1)\,d} \right\}.$$

Proof. We have $\frac{e}{g} = \frac{a+c}{b+d}$. The left child of $\frac{e}{g}$ is $\frac{2a+c}{2b+d}$. It is the first term in a left branch of $\frac{e}{g}$ for which the ith term is $\frac{(i+1)a+ic}{(i+1)b+id}$, through a further $(i-1)$ additions of $\frac{a+c}{b+d}$. Similarly, the right child of $\frac{e}{g}$ is $\frac{a+2c}{b+2d}$. It is the first term in a right branch for which the ith term is $\frac{ia+(i+1)c}{ib+(i+1)d}$. □

Theorem 7. *Let $\frac{m}{k}$ and $\frac{s}{t}$ be two consecutive terms in a level of the Stern–Brocot tree, and $\frac{e}{g}$ their common parent. Then*

(i)

$$\frac{m}{k} \oplus \frac{s}{t} = \frac{(2i+1)\,e}{(2i+1)\,g}$$

That is, the set of all reduced values of the child's addition of all consecutive terms in level n, represents the set of all terms found in level 1 to level $n-1$. Algebraically,

$$h_{n,2j+1} = h_{n,2j} \tilde{\oplus} h_{n,2j+2}.$$

(ii) *$\frac{e}{g}$ is found i levels above $\frac{m}{k}$ and $\frac{s}{t}$.*

(iii) *the cross-difference of $\frac{m}{k}$ and $\frac{s}{t}$ is $2i+1$.*

Proof. Let $\frac{m}{k}$ and $\frac{s}{t}$ be found i levels below $\frac{e}{g}$. Let also $\frac{a}{b}$ and $\frac{c}{d}$, where $\frac{a}{b} < \frac{c}{d}$, be the parents of $\frac{e}{g}$. Then $\frac{e}{g} = \frac{a+c}{b+d}$.

(i) and (ii). By Lemma 3,
$\frac{m}{k} = \frac{(i+1)a+ic}{(i+1)b+id}$ and $\frac{s}{t} = \frac{ia+(i+1)c}{ib+(i+1)d}$.
Thus $\frac{m}{k} \oplus \frac{s}{t} = \frac{(2i+1)(a+c)}{(2i+1)(b+d)} = \frac{(2i+1)e}{(2i+1)g}$.

(iii) The cross-difference of $\frac{m}{k}$ and $\frac{s}{t}$ is $sk - mt = (2i+1)\,(bc - ad) = 2i+1$ by Theorem 2. □

Example 3. In level 7, $\frac{7}{18}$ and $\frac{7}{17}$ are consecutive fractions with common parent $\frac{2}{5}$ which appears three levels above in level 4.

(i) $\frac{7}{18} \oplus \frac{7}{17} = \frac{14}{35} = \frac{7(2)}{7(5)}$; and so, $\frac{7}{18} \tilde{\oplus} \frac{7}{17} = \frac{2}{5}$.

(ii) The cross-difference of $\frac{7}{18}$ and $\frac{7}{17}$ is $7 = 2.3 + 1$.

Theorem 8. *Let $j = 2^i\,(2m-1)$, for $i \geq 1$, and $m = 2^r\,(2l-1) + 1$, for $r \geq 0$. Then for $h_{n,j}$ as defined in Definition 2, if $\frac{s}{q}$ is the right parent, or an element of the right branch, of $h_{n,j}$, and $\frac{u}{v}$ is its left parent, or an element of its left branch, then*

$$\frac{s}{q} \tilde{\oplus} \frac{u}{v} = h_{n,j}.$$

That is, for $t \geq 0, p \geq 0$,

$$h_{n,j} = h_{n+t,2^t(j-1)} \tilde{\oplus} h_{n+p,2^p(j-1)+2}.$$

Proof. For $k \geq 0$, $w \geq 0$,

$$\begin{aligned}
\frac{s}{q} \oplus \frac{u}{v} &= \frac{a_{n,j+1} + ka_{n,j} + a_{n,j-1} + wa_{n,j}}{b_{n,j+1} + kb_{n,j} + b_{n,j-1} + wb_{n,j}}, \\
&= \frac{a_{n-1,\frac{j}{2}} + ka_{n,j} + a_{n-1,\frac{j}{2}+1} + wa_{n,j}}{b_{n-1,\frac{j}{2}} + kb_{n,j} + b_{n-1,\frac{j}{2}+1} + wb_{n,j}}, \\
&= \frac{(k+w+1)\, a_{n,j}}{(k+w+1)\, b_{n,j}}.
\end{aligned}$$

That is, $\frac{s}{q} \tilde{\oplus} \frac{u}{v} = h_{n,j}$.

By Theorem 1, for $t \geq 0$, $p \geq 0$, $h_{n,j} = h_{n+t,2^t(j-1)} \tilde{\oplus} h_{n+p,2^p(j-1)+2}$. □

Let $[a_0, a_1, a_2, \ldots, a_i]$ be a continued fraction expansion of μ.

- If $a_i > 1$, then $[a_0, a_1, a_2, \ldots, a_i]$ is the short form continued fraction expansion of μ.
- If $a_i = 1$, then $[a_0, a_1, a_2, \ldots, a_i]$ is the long form continued fraction expansion of μ.

Note that $[a_0, a_1, a_2, \ldots, a_{i-1}, 1] = [a_0, a_1, a_2, \ldots, a_{i-1} + 1]$.

Let $[a_0, a_1, a_2, \ldots, a_i, a_{i+1}, \ldots]$ be a finite or infinite continued fraction expansion of μ. We designate $\frac{p_i}{q_i} = [a_0, a_1, a_2, \ldots, a_i]$.

Theorem 9. *Let $[a_0, a_1, a_2, \ldots, a_i]$ be the short form of μ. Then*

(i) *for i odd,*

- *the right branch of μ is the set $\{[a_0, a_1, a_2, \ldots, a_i - 1, 1, m] \mid m \geq 1\}$ and*
- *the left branch of μ is the set $\{[a_0, a_1, a_2, \ldots, a_i, m] \mid m \geq 1\}$;*

(ii) *for i even,*

- *the right branch of μ is the set $\{[a_0, a_1, a_2, \ldots, a_i, m] \mid m \geq 1\}$ and*
- *the left branch of μ is the set $\{[a_0, a_1, a_2, \ldots, a_i - 1, 1, m] \mid m \geq 1\}$.*

Proof. We give the proof for i odd. The proof for i even is similar.

By Theorem 4, $\mu = [a_0, a_1, a_2, \ldots, a_i] = R^{a_0+1} L^{a_1} R^{a_2} \cdots L^{a_i-1}$.

The left branch of μ is the set of terms that are found from a single left-and-down movement from μ followed by an infinite set of right-and-down movements. That is, by Theorem 4, its terms can be represented as

$$\begin{aligned}
\left\{R^{a_0+1} L^{a_1} R^{a_2} \cdots L^{a_i} R^m \mid m \geq 0\right\} &= \{[a_0, a_1, a_2, \ldots, a_i, m+1] \mid m \geq 0\}, \\
&= \{[a_0, a_1, a_2, \ldots, a_i, m] \mid m \geq 1\}.
\end{aligned}$$

The right branch of μ is the set of terms that are found from a single right-and-down movement from μ followed by an infinite set of left-and-down movements. That is, by Theorem 4, its terms can be represented as

$$\left\{R^{a_0+1} L^{a_1} R^{a_2} \cdots L^{a_i-1} R^1 L^m \mid m \geq 0\right\} = \{[a_0, a_1, a_2, \ldots, a_i - 1, 1, m] \mid m \geq 1\}. \quad \square$$

We now examine the sum of left diagonals in the tree.

Definition 9 (*Sum of Diagonals*)**.** Let $\mathcal{L}_s = \langle m_{s,1}, m_{s,2}, m_{s,3}, \ldots \rangle$ and $\mathcal{L}_t = \langle m_{t,1}, m_{t,2}, m_{t,3}, \ldots \rangle$ represent the sth and tth left diagonals respectively. The sum of these two left diagonals is

$$\mathcal{L}_s \dagger \mathcal{L}_t = \langle m_{s,1} \tilde{\oplus} m_{t,1}, m_{s,2} \tilde{\oplus} m_{t,2}, m_{s,3} \tilde{\oplus} m_{t,3}, \ldots \rangle.$$

A similar definition holds for $\mathcal{R}_s \dagger \mathcal{R}_t$ where $\mathcal{R}_s$ and $\mathcal{R}_t$ represent the sth and tth right diagonals respectively.

Lemma 4. *For $k \in \mathbb{N}$, $\mathcal{L}_k = \langle h_{n,2k}, h_{n+1,2k}, h_{n+2,2k}, \ldots \rangle$ where $n = \lceil \log_2 k \rceil + 1$.*

Proof. By Definition 4, $\mathcal{L}_k = \langle h_{n,2k}, h_{n+1,2k}, h_{n+2,2k}, \ldots \rangle$ where n is the lowest level of the tree that has a kth element.

For $n > 0$, the nth level of the tree has 2^{n-1} elements. Accordingly, $2^{n-2} < k \le 2^{n-1}$, where $k \in \mathbb{N}$. That is, $n = \lceil \log_2 k \rceil + 1$. □

We conclude this section with a statement of our main result for the recursive determination of diagonals.

Theorem 10 (*Main Result for Diagonal Determination*)**.** *If $k = 2^i(2j+1)$, then*

(i) $\mathcal{L}_{2k} = \begin{cases} \mathcal{L}_k \dagger \mathcal{L}_{1^+}, & j = 0 \\ \mathcal{L}_k \dagger \mathcal{L}_{j+1}, & j > 0. \end{cases}$

(ii) $\mathcal{L}_{2k+1} = \mathcal{L}_{k+1} \dagger \mathcal{L}_{j+1}$.

(iii) *Identical results hold for right diagonals where $\mathcal{R}$ is substituted for $\mathcal{L}$ in* (i) *and* (ii).

Proof. We have

(i) By Definition 2, for $t \ge 0$,

$$\begin{aligned} h_{n+t,4k} &= h_{n-1+t,2k} \oplus h_{n-1+t,2k+1}, \\ &= h_{n-1+t,2k} \oplus h_{n-2-i+t,2j+2}. \end{aligned} \tag{3.3}$$

Now by Lemma 4, $\mathcal{L}_{2k} = \langle h_{n,4k}, h_{n+1,4k}, h_{n+2,4k}, \ldots \rangle$ where $n = \lceil \log_2 4k \rceil$. As $n - 1 = \lceil \log_2 2k \rceil$, it follows that

$$\mathcal{L}_k = \langle h_{n-1,2k}, h_{n,2k}, h_{n+1,2k}, \ldots \rangle$$

which is the set of first terms on the right hand side of (3.3). Also

$$n - 2 - i = \lceil \log_2 (2j+1) \rceil . \tag{3.4}$$

There are two cases:

(a) $j = 0$. From (3.4), $n - 2 - i = 0$ and as $\mathcal{L}_{1^+} = \langle h_{0,2}, h_{1,2}, h_{2,2}, \ldots \rangle$, (i) follows by (3.3).

(b) $j > 0$. Since $2^{n-3} < k \le 2^{n-2}$,

$$2^{n-3-i} < 2j + 1 \le 2^{n-2-i},$$
$$2^{n-4-i} < j + \frac{1}{2} \le 2^{n-3-i},$$

that is, $2^{n-4-i} < j + 1 \le 2^{n-3-i}$.

It follows from Lemma 4 that

$$\mathcal{L}_{j+1} = \langle h_{n-2-i,2j+2}, h_{n-1-i,2j+2}, h_{n-i,2j+2}, \ldots \rangle .$$

Accordingly, (i) follows from (3.3).

(ii) By Definition 2 and Theorem 6(vii),

$$\begin{aligned} h_{n+t,4k+2} &= h_{n-1+t,2k+1} \oplus h_{n-1+t,2k+2}, \\ &= h_{n-2-i+t,2j+2} \oplus h_{n-1+t,2k+2}. \end{aligned} \tag{3.5}$$

Now

$$\mathcal{L}_{2k+1} = \langle h_{n,4k+2}, h_{n+1,4k+2}, h_{n+2,4k+2}, \ldots \rangle \tag{3.6}$$

where $n = \lceil \log_2 (4k+2) \rceil$. Hence $n - 2 = \lceil \log_2 (k+1) \rceil$ and

$$\mathcal{L}_{k+1} = \langle h_{n-1,2k+2}, h_{n,2k+2}, h_{n+1,2k+2}, \ldots \rangle . \tag{3.7}$$

Also $n - i - 3 = \lceil \log_2 (j+1) \rceil$ so

$$\mathcal{L}_{j+1} = \langle h_{n-2-i,2j+2}, h_{n-1-i,2j+2}, h_{n-i,2j+2}, \ldots \rangle . \tag{3.8}$$

Using (3.6)–(3.8) in (3.5), we have (ii).

(iii) By Corollary 1 and Definition 6, consecutive terms in $\mathcal{R}_m$ are the reciprocals of consecutive terms in $\mathcal{L}_m$. Accordingly, by Definition 9, consecutive terms in $\mathcal{R}_s\dagger\mathcal{R}_t$ are the reciprocals of consecutive terms in $\mathcal{L}_s\dagger\mathcal{L}_t$. The result follows. □

4. Offset branches

We now explore a more general type of branch in the Stern–Brocot tree—one for which left and right branches are particular cases.

Definition 10 (*Offset Branches*)**.** Let $\mathcal{B}_{L(\mu)}$ and $\mathcal{B}_{R(\mu)}$ denote left and right branches respectively of some term μ in the tree.

For each term in $\mathcal{B}_{L(\mu)}$, locate a term found t left-and-down movements away. The set of all these new terms is designated as $\mathcal{B}_{L(\mu),t}$.

Similarly, for each term in $\mathcal{B}_{R(\mu)}$, locate a term found t right-and-down movements away. The set of all these new terms is designated as $\mathcal{B}_{R(\mu),t}$.

We call $\mathcal{B}_{L(\mu),t}$ the left branch with offset t of μ, and $\mathcal{B}_{R(\mu),t}$ the right branch with offset t of μ.

It follows from Definition 10 that $\mathcal{B}_{L(\mu)} = \mathcal{B}_{L(\mu),0}$ and $\mathcal{B}_{R(\mu)} = \mathcal{B}_{R(\mu),0}$.

Example 4. $\mathcal{B}_{L\left(\frac{1}{2}\right),2} = \frac{1}{5}, \frac{4}{11}, \frac{7}{17}, \ldots$ and $\mathcal{B}_{R\left(\frac{1}{2}\right),2} = \frac{4}{5}, \frac{7}{11}, \frac{10}{17}, \ldots$.

Definition 11 (*Offset Cross-Differences*)**.** Let $\frac{u}{v}$ and $\frac{r}{s}$ be the ith elements respectively of $\mathcal{B}_{L(\mu),t}$ and $\mathcal{B}_{R(\mu),t}$. The ith Cross-Difference of μ with offset t, $\mathbb{C}_{\mu,t,i}$, is given by

$$\mathbb{C}_{\mu,t,i} = rv - us.$$

Definition 12 (*Intra-Branch Cross-Differences*)**.** Let $\frac{u}{v}$ and $\frac{r}{s}$ be the ith and $(i+1)$th elements respectively in $\mathcal{B}_{L(\mu),t}$ $\left(\mathcal{B}_{R(\mu),t}\right)$. The ith intra-branch cross-difference of $\mathcal{B}_{L(\mu),t}$ $\left(\mathcal{B}_{R(\mu),t}\right)$, $\mathbb{D}_{L(\mu),t,i}$ $\left(\mathbb{D}_{R(\mu),t,i}\right)$, is given by $rv - us$.

Theorem 11. *Let $\frac{u}{v}$ and $\frac{r}{s}$ represent any two terms taken respectively from $\mathcal{B}_{L(\mu),t}$ and $\mathcal{B}_{R(\mu),t}$. Then for $\frac{a}{b}$ and $\frac{c}{d}$ the left and right parents respectively of μ,*

(i)

$$\mathcal{B}_{L(\mu),t} = \left\{ \left. \frac{(ti+i+1)\,a + (ti+i-t)\,c}{(ti+i+1)\,b + (ti+i-t)\,d} \right| i = 1, 2, \ldots \right\}$$

(ii)

$$\mathcal{B}_{R(\mu),t} = \left\{ \left. \frac{(ti+i-t)\,a + (ti+i+1)\,c}{(ti+i-t)\,b + (ti+i+1)\,d} \right| i = 1, 2, \ldots \right\}$$

(iii)

$$\frac{u}{v} \tilde{\oplus} \frac{r}{s} = \mu.$$

Proof. From Lemma 3(i), let $\frac{u'}{v'} = \frac{(i+1)a+ic}{(i+1)b+id}$ represent the left parent $\frac{a}{b}$ (for $i = 0$), and elements of the left branch (for $i > 0$), of μ. From Lemma 3(ii), let $\frac{r'}{s'} = \frac{ja+(j+1)c}{jb+(j+1)d}$ represent the right parent $\frac{c}{d}$ (for $j = 0$), and elements of the right branch (for $j > 0$), of μ. The typical element $\frac{u}{v}$ of $\mathcal{B}_{L(\mu),t}$ is therefore the $(t+1)$th entry in the right branch of $\frac{u'}{v'}$. That is, for $i = 1, 2, \ldots; t = 0, 1, \ldots,$

$$\frac{u}{v} = \frac{(i+1)\,a + ic}{(i+1)b + id} \oplus \frac{t\,(ia + (i-1)\,c)}{t\,(ib + (i-1)\,d)} = \frac{(ti+i+1)\,a + (ti+i-t)\,c}{(ti+i+1)\,b + (ti+i-t)\,d} \tag{4.1}$$

establishing (i); and the typical element $\frac{r}{s}$ of $\mathcal{B}_{R(\mu),t}$ is the $(t+1)$th entry in the left branch of $\frac{r'}{s'}$. That is, for $j = 1, 2, \ldots; t = 0, 1, \ldots,$

$$\frac{r}{s} = \frac{ja + (j+1)\,c}{jb + (j+1)\,d} \oplus \frac{t\,((j-1)\,a + jc)}{t\,((j-1)\,b + jd)} = \frac{(tj + j - t)\,a + (tj + j + 1)\,c}{(tj + j - t)\,b + (tj + j + 1)\,d} \tag{4.2}$$

establishing (ii). Accordingly,

$$\frac{u}{v} \oplus \frac{r}{s} = \frac{(ti + i + 1 + tj + j - t)\,(a + c)}{(ti + i + 1 + tj + j - t)\,(b + d)}.$$

That is, $\frac{u}{v} \tilde{\oplus} \frac{r}{s} = \frac{a+c}{b+d} = \mu$. □

Example 5. $\mathcal{B}_{L\left(\frac{1}{2}\right),2} = \frac{1}{5}, \frac{4}{11}, \frac{7}{17}, \ldots$ and $\mathcal{B}_{R\left(\frac{1}{2}\right),2} = \frac{4}{5}, \frac{7}{11}, \frac{10}{17}, \ldots$. The child's addition of any term in $\mathcal{B}_{L\left(\frac{1}{2}\right),2}$ with any term in $\mathcal{B}_{R\left(\frac{1}{2}\right),2}$ yields $\frac{1}{2}$.

The following result is a generalisation of Theorem 7(iii).

Theorem 12. *Let $\frac{a}{b}$ and $\frac{c}{d}$ be the parents of μ where $\frac{a}{b} < \frac{c}{d}$. Let also $\frac{u}{v}$ be the ith element of $\mathcal{B}_{L(\mu),t}$ and $\frac{r}{s}$ the jth element of $\mathcal{B}_{R(\mu),w}$. Their cross-difference is*

$$rv - us = 1 - tw + (i + j)\,(1 + t + w + tw)\,.$$

Proof. By Theorem 2, $bc - ad = 1$. By Theorem 11(i) and (ii),

$$\frac{u}{v} = \frac{(ti + i + 1)\,a + (ti + i - t)\,c}{(ti + i + 1)\,b + (ti + i - t)\,d} \quad \text{and}$$
$$\frac{r}{s} = \frac{(wj + j - w)\,a + (wj + j + 1)\,c}{(wj + j - w)\,b + (wj + j + 1)\,d}.$$

The result follows. □

Corollary 2. *Let $\frac{a}{b}$ and $\frac{c}{d}$ be the parents of μ where $\frac{a}{b} < \frac{c}{d}$. Let also $\frac{u}{v}$ and $\frac{r}{s}$ be the ith elements respectively of $\mathcal{B}_{L(\mu),t}$ and $\mathcal{B}_{R(\mu),t}$. Then*

$$C_{\mu,t,i} = 4ti + 2t^2 i - t^2 + 2i + 1.$$

Proof. This is the $j = i$, $w = t$ case in Theorem 12. □

Remark 1. Corollary 2 indicates that each offset cross-difference, $\mathbb{C}_{\mu,t,i}$, is not dependent upon μ.

Theorem 13. *Let $\frac{a}{b}$ and $\frac{c}{d}$ be the parents of μ where $\frac{a}{b} < \frac{c}{d}$. For all i and μ,*

$$\mathbb{D}_{L(\mu),t,i} = \mathbb{D}_{R(\mu),t,i} = (t + 1)^2\,.$$

Proof. Let $\frac{u}{v}$ and $\frac{r}{s}$ be the ith and $(i+1)$th elements respectively in $\mathcal{B}_{L(\mu),t}$. From Theorem 11(i),

$$\begin{aligned}
\mathbb{D}_{L(\mu),t,i} &= rv - us, \\
&= ((ti + t + i + 2)\,a + (ti + i + 1)\,c)\,((ti + i + 1)\,b + (ti + i - t)\,d) \\
&\quad - ((ti + i + 1)\,a + (ti + i - t)\,c)\,((ti + t + i + 2)\,b + (ti + i + 1)\,d)\,, \\
&= (t + 1)^2\,(bc - ad)\,, \\
&= (t + 1)^2 \quad \text{by Theorem 2.}
\end{aligned}$$

Similarly, from Theorem 11(ii),

$$\mathbb{D}_{R(\mu),t,i} = (t + 1)^2\,. \quad \square$$

Remark 2. Theorem 13 indicates that each intra-branch cross-difference is not dependent upon μ or i.

Theorem 14. *Let $\frac{u}{v}$, $\frac{w}{x}$, and $\frac{y}{z}$ be consecutive terms in a branch with offset t, then $\frac{w}{x} = \frac{u}{v} \tilde{\oplus} \frac{y}{z}$.*

Proof. We have two cases.

(i) Left branches: From (4.1) with $\frac{a}{b}$ and $\frac{c}{d}$ the parents of the term upon which the branch is based,

$$\begin{aligned} \frac{u}{v} \tilde{\oplus} \frac{y}{z} &= \frac{(ti+i+1)\,a + (ti+i-t)\,c}{(ti+i+1)\,b + (ti+i-t)\,d} \tilde{\oplus} \frac{(ti+2t+i+3)\,a + (ti+t+i+2)\,c}{(ti+2t+i+3)\,b + (ti+t+i+2)\,d}, \\ &= \frac{(ti+i+t+2)\,a + (ti+i+1)\,c}{(ti+i+t+2)\,b + (ti+i+1)\,d}, \\ &= \frac{w}{x}. \end{aligned}$$

(ii) Right branches: A similar argument to (i) follows by utilising (4.2). $\square$

Example 6. Using Example 5, $\frac{1}{5} \tilde{\oplus} \frac{7}{17} = \frac{4}{11}$ and $\frac{4}{5} \tilde{\oplus} \frac{10}{17} = \frac{7}{11}$.

Corollary 3. *Let $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$ be consecutive terms in a branch of the Stern–Brocot tree, then*

$$\frac{c}{d} = \frac{a}{b} \tilde{\oplus} \frac{e}{f}.$$

That is, for $m > 0$ and $k > 1$,

(i) $h_{n+m+1,2^{m+1}(k-1)} = h_{n+m,2^m(k-1)} \tilde{\oplus} h_{n+m+2,2^{m+2}(k-1)}$

(ii) $h_{n+m+1,2^{m+1}(k-1)+2} = h_{n+m,2^m(k-1)+2} \tilde{\oplus} h_{n+m+2,2^{m+2}(k-1)+2}$.

Proof. This is the $t = 0$ case in Theorem 14. The result follows from Theorem 1. $\square$

We now state our main result for generalising branches in the tree. Our usual left and right branches of Theorem 9 can then be viewed as particular cases of this general result.

Theorem 15 (*Main Result for Offset Branches*)**.** *Let $[a_0, a_1, a_2, \ldots, a_i]$ be the short form of μ and $t \geq 0$. Then*

(i) *for i odd,*

- *the right branch with offset t of μ is the set*

$$\{[a_0, a_1, a_2, \ldots, a_i - 1, 1, m - 1, t + 1] \mid m \geq 1\}$$

- *the left branch with offset t of μ is the set*

$$\{[a_0, a_1, a_2, \ldots, a_i, m - 1, t + 1] \mid m \geq 1\}$$

(ii) *for i even,*

- *the right branch with offset t of μ is the set*

$$\{[a_0, a_1, a_2, \ldots, a_i, m - 1, t + 1] \mid m \geq 1\}$$

- *the left branch with offset t of μ is the set*

$$\{[a_0, a_1, a_2, \ldots, a_i - 1, 1, m - 1, t + 1] \mid m \geq 1\}.$$

Proof. We prove the case for i odd. The proof for i even is similar.

By Theorems 4 and 9:

- The left branch of μ is the set

$$\{[a_0, a_1, a_2, \ldots, a_i, m] \mid m \geq 1\} = \left\{R^{a_0+1}L^{a_1}R^{a_2}\cdots L^{a_i}R^{m-1} \mid m \geq 1\right\}.$$

Thus the left branch with offset t of μ is the set

$$\left\{R^{a_0+1}L^{a_1}R^{a_2}\cdots L^{a_i}R^{m-1}L^{t} \mid m \geq 1\right\}$$
$$= \{[a_0, a_1, a_2, \ldots, a_i, m-1, t+1] \mid m \geq 1\} \quad \text{for some } t \geq 0.$$

- The right branch of μ is the set

$$\{[a_0, a_1, a_2, \ldots, a_i - 1, 1, m] \mid m \geq 1\} = \left\{R^{a_0+1}L^{a_1}R^{a_2}\cdots L^{a_i-1}R^{1}L^{m-1} \mid m \geq 1\right\}.$$

Thus the right branch with offset t of μ is the set

$$\left\{R^{a_0+1}L^{a_1}R^{a_2}\cdots L^{a_i-1}R^{1}L^{m-1}R^{t} \mid m \geq 1\right\}$$
$$= \{[a_0, a_1, a_2, \ldots, a_i - 1, 1, m-1, t+1] \mid m \geq 1\} \quad \text{for some } t \geq 0 \quad \square$$

5. Paths

Paths are useful in determining the value of child's addition and cross-differences derived from distant terms in the tree.

Definition 13 (*Paths*)**.** A path based on $\frac{a}{b}$ is a sequence of right-and-down and left-and-down movements in the tree that commences at $\frac{a}{b}$. A null path consists of no movements. Paths can be finite or infinite in length.

Our main discovery (Theorem 16) is an unintuitive result based on paths in the tree:

- Select two different finite paths. Base both paths on *any* term in the tree.
- Find the end terms of each of these paths and determine their cross-difference.
- This cross-difference is the same irrespective of the term chosen as the base for these paths. That is, the cross-difference is dependent on the choice of paths, not the term used as the base for these paths.

Definition 14 (*Complementary Paths*)**.** Let two paths be based on the same term $\frac{a}{b}$. If one path moves to the right (left) when the other path moves to the left (right) the paths are designated as complementary paths based on $\frac{a}{b}$.

Example 7. We have

- in level 6, $\frac{3}{11}$ and $\frac{5}{13}$ are located on different but complementary paths based on $\frac{1}{3}$;
- left and right branches of a term μ, are complementary paths based on μ.

We now generalise cross-differences associated with finite paths.

Definition 15 (*Node*)**.** Let $\frac{u}{v}$ and $\frac{r}{s}$ be two terms in the tree. The node of $\frac{u}{v}$ and $\frac{r}{s}$ is the last term that these fractions share in their respective paths emanating from $\frac{1}{1}$.

Definition 16 (*General Path Cross-Differences*)**.** Let

$$Q = \begin{cases} R^{m_0}L^{m_1}R^{m_2}\cdots(L \text{ or } R)^{m_{n-1}} \\ L^{w_0}R^{w_1}L^{w_2}\cdots(L \text{ or } R)^{w_{i-1}} \end{cases}$$

denote two different finite paths.

Also, let $\frac{r}{s}$ and $\frac{u}{v}$ be two terms with node μ where

- $\frac{r}{s}$ is the $\left(\sum_{j=0}^{n-1} m_j\right)$th term of the first path of Q emanating from μ, and
- $\frac{u}{v}$ is the $\left(\sum_{j=0}^{i-1} w_j\right)$th term of the second path of Q emanating from μ.

The general path cross-difference, $\mathbb{G}_{Q,\mu}$, is given by

$$\mathbb{G}_{Q,\mu} = rv - us.$$

Definition 17 (*Symmetry Couplets*)**.** Q and Q' are called symmetry couplets if

$$Q = \begin{cases} R^{m_0} L^{m_1} R^{m_2} \cdots (L \text{ or } R)^{m_{n-1}} \\ L^{w_0} R^{w_1} L^{w2} \cdots (L \text{ or } R)^{w_{i-1}} \end{cases} \quad \text{and}$$

$$Q' = \begin{cases} L^{m_0} R^{m_1} L^{m_2} \cdots (R \text{ or } L)^{m_{n-1}} \\ R^{w_0} L^{w_1} R^{w_2} \cdots (R \text{ or } L)^{w_{i-1}} \end{cases}.$$

The following theorem shows that $\mathbb{G}_{Q,\mu}$ is the same for any value of μ, and that symmetry couplets generate a simple cross-difference relationship.

Theorem 16 (*Main Result for Paths*)**.** *Let Q and Q' be symmetry couplets and μ be any term of the tree other than $\frac{0}{1}$ or $\frac{1}{0}$. Then*
(i) $\mathbb{G}_{Q,\mu}$ *is independent of* μ.
(ii) $\mathbb{G}_{Q,\mu} = -\mathbb{G}_{Q',\mu}$.

Proof. We have

(i) Let $\mu = \frac{e}{g}$ be the node of $\frac{r}{s}$ and $\frac{u}{v}$ as defined in Definition 16.
For k odd, let

$$\begin{aligned} \frac{e}{g} &= R^{a_0} L^{a_1} \cdots L^{a_k}, \\ &= [a_0, a_1, \ldots, a_{k-1}, a_k + 1] \quad \text{by Theorem 4}, \\ &= \frac{(a_k+1)\, p_{k-1} + p_{k-2}}{(a_k+1)\, q_{k-1} + q_{k-2}}, \quad \text{where } \frac{p_{k-1}}{q_{k-1}} = [a_0, a_1, \ldots, a_{k-1}]. \end{aligned} \tag{5.1}$$

Then

$$\begin{aligned} \frac{r}{s} &= R^{a_0} \cdots L^{a_k} R^{m_0} L^{m_1} \cdots (L \text{ or } R)^{m_{n-1}}, \\ &= [a_0, a_1, \ldots, a_k, m_0, m_1, \ldots, m_{n-2}, m_{n-1} + 1] \quad \text{by Theorem 4}, \\ &= [a_0, a_1, \ldots, a_k, m_0, \beta] \quad \text{where } \beta = [m_1, m_2, \ldots, m_{n-2}, m_{n-1} + 1], \\ &= \frac{p'_{k+2}}{q'_{k+2}}, \quad \text{say, and} \\ \frac{u}{v} &= R^{a_0} \cdots L^{a_k + w_0} R^{w_1} L^{w_2} \cdots (L \text{ or } R)^{w_{i-1}}, \\ &= [a_0, a_1, \ldots, a_{k-1}, a_k + w_0, w_1, \ldots, w_{i-2}, w_{i-1} + 1], \\ &= [a_0, a_1, \ldots, a_{k-1}, a_k + w_0, \Psi] \quad \text{where } \Psi = [w_1, w_2, \ldots, w_{i-2}, w_{i-1} + 1], \\ &= \frac{p''_{k+1}}{q''_{k+1}}, \quad \text{say}. \end{aligned} \tag{5.2}$$

Accordingly,

$$\begin{aligned} r &= p'_{k+2}, \\ &= \beta p'_{k+1} + p'_k, \\ &= \beta \left(m_0 p'_k + p'_{k-1} \right) + p'_k, \\ &= (\beta m_0 a_k + a_k + \beta)\, p_{k-1} + (\beta m_0 + 1)\, p_{k-2}. \end{aligned}$$

Similarly,

$$\begin{aligned} s &= q'_{k+2} = (\beta m_0 a_k + a_k + \beta)\, q_{k-1} + (\beta m_0 + 1)\, q_{k-2}, \\ u &= p''_{k+1} = (\Psi a_k + \Psi w_0 + 1)\, p_{k-1} + \Psi p_{k-2}, \quad \text{and} \\ v &= q''_{k+1} = (\Psi a_k + \Psi w_0 + 1)\, q_{k-1} + \Psi q_{k-2}. \end{aligned}$$

We have

$$\begin{aligned} \mathbb{G}_{Q,\frac{e}{g}} &= rv - us, \\ &= (p_{k-2}q_{k-1} - p_{k-1}q_{k-2})\,(\beta m_0 - \Psi\beta + \Psi w_0 + \Psi\beta m_0 w_0 + 1)\,, \\ &= (-1)^k\,(\beta m_0 - \Psi\beta + \Psi w_0 + \Psi\beta m_0 w_0 + 1)\,. \end{aligned}$$

That is, $\mathbb{G}_{Q,\frac{e}{g}}$ is not dependent on any of the a_i s in (5.1). The result follows. The case for k even proceeds similarly.

(ii) We now show that $\mathbb{G}_{Q',\frac{e}{g}} = (-1)^{k-1}\,(\beta m_0 - \Psi\beta + \Psi w_0 + \Psi\beta m_0 w_0 + 1)$ establishing (ii).

Assume that $\frac{r}{s}$ and $\frac{u}{v}$ are defined as in Definition 16, but with Q' (as in Definition 17) instead of Q. Then

$$\begin{aligned} \frac{r}{s} &= R^{a_0}\cdots L^{a_k+m_0}R^{m_1}L^{m_2}\cdots(L \text{ or } R)^{m_{n-1}}\,, \\ &= [a_0, a_1, \ldots, a_{k-1}, a_k + m_0, \beta] \quad \text{where} \\ \beta &= [m_1, m_2, \ldots, m_{n-2}, m_{n-1} + 1]\,, \quad \text{and} \\ \frac{u}{v} &= [a_0, a_1, \ldots, a_k, w_0, w_1, \ldots, w_{n-2}, w_{n-1} + 1]\,, \\ &= [a_0, a_1, \ldots, a_k, w_0, \Psi] \quad \text{where } \Psi = [w_1, w_2, \ldots, w_{i-2}, w_{i-1} + 1]. \end{aligned}$$

It follows that

$$\begin{aligned} r &= (\beta a_k + \beta m_0 + 1)\, p_{k-1} + \beta p_{k-2}, \\ s &= (\beta a_k + \beta m_0 + 1)\, q_{k-1} + \beta q_{k-2}, \\ u &= (\Psi w_0 a_k + a_k + \Psi)\, p_{k-1} + (\Psi w_0 + 1)\, p_{k-2}, \\ v &= (\beta w_0 a_k + a_k + \Psi)\, q_{k-1} + (\Psi w_0 + 1)\, q_{k-2}, \end{aligned}$$

and

$$\begin{aligned} \mathbb{G}_{Q',\frac{e}{g}} &= rv - us, \\ &= (p_{k-1}q_{k-2} - p_{k-2}q_{k-1})\,(\beta m_0 - \Psi\beta + \Psi w_0 + \Psi\beta m_0 w_0 + 1)\,, \\ &= (-1)^{k-1}\,(\beta m_0 - \Psi\beta + \Psi w_0 + \Psi\beta m_0 w_0 + 1)\,. \quad \square \end{aligned}$$

Example 8. Let $Q = \begin{cases} R^2L^2R^1L^1 \\ L^3 \end{cases}$. Then $Q' = \begin{cases} L^2R^2L^1R^1 \\ R^3 \end{cases}$ and

$$\mathbb{G}_{Q,\frac{1}{2}} = 19.5 - 1.27 = 68$$

$$\mathbb{G}_{Q',\frac{1}{2}} = 8.5 - 27.4 = -68$$

$$\mathbb{G}_{Q,\frac{2}{3}} = 27.9 - 5.35 = 68$$

$$\mathbb{G}_{Q',\frac{2}{3}} = 27.6 - 5.46 = -68.$$

Corollary 4. *Let $\frac{r}{s}$ and $\frac{u}{v}$ be two terms in the same level of the Stern–Brocot tree that are found on different but complementary paths based on $\frac{e}{g}$. Then $\frac{e}{g} = \frac{r}{s}\tilde{\bigoplus}\frac{u}{v}$.*

Proof. Let $\frac{e}{g}$, $\frac{r}{s}$ and $\frac{u}{v}$ be defined as in the proof of Theorem 16. If $\frac{r}{s}$ and $\frac{u}{v}$ are in the same level, then $\Psi = \beta = [m_1, m_2, \ldots, m_{n-2}, m_{n-1} + 1]$ in (5.2). Accordingly,

$$\frac{r}{s} \oplus \frac{u}{v} = \frac{(\beta m_0 + 1 + \beta)\left((a_k + 1)\, p_{k-1} + p_{k-2}\right)}{(\beta m_0 + 1 + \beta)\left((a_k + 1)\, q_{k-1} + q_{k-2}\right)} = \frac{(\beta m_0 + 1 + \beta)\, e}{(\beta m_0 + 1 + \beta)\, g}.$$

That is, $\frac{e}{g} = \frac{r}{s} \tilde{\oplus} \frac{u}{v}$. $\square$

Example 9. In level 7, $\frac{3}{13}$ and $\frac{7}{17}$ are found on complementary paths based on $\frac{1}{3}$. This is confirmed by the fact that $\frac{3}{13} \oplus \frac{7}{17} = \frac{10}{30} = \frac{1}{3}$.

Definition 18 (*Intra-Path Cross-Differences*)**.** Let $M_{\mathcal{B}_{L(\mu),t},\Omega}$ be the set of end terms formed when each term in $\mathcal{B}_{L(\mu),t}$ is the base for the path Ω.

Let $\frac{u}{v}$ and $\frac{r}{s}$ be the ith and $(i+1)$th elements respectively in $M_{\mathcal{B}_{L(\mu),t},\Omega}$.

The ith intra-path cross-difference of $M_{\mathcal{B}_{L(\mu),t},\Omega}$, called $\mathbb{V}_{\mathcal{B}_{L(\mu),t},\Omega,i}$, is given by

$$\mathbb{V}_{\mathcal{B}_{L(\mu),t},\Omega,i} = rv - us.$$

Similarly, let $M_{\mathcal{B}_{R(\mu),t},\Omega}$ be the set of end terms formed when each term in $\mathcal{B}_{R(\mu),t}$ is the base for the path Ω.

Let $\frac{u}{v}$ and $\frac{r}{s}$ be the ith and $(i+1)$th elements respectively in $M_{\mathcal{B}_{R(\mu),t},\Omega}$.

The ith intra-path cross-difference of $M_{\mathcal{B}_{R(\mu),t},\Omega}$, called $\mathbb{V}_{\mathcal{B}_{R(\mu),t},\Omega,i}$, is given by

$$\mathbb{V}_{\mathcal{B}_{R(\mu),t},\Omega,i} = rv - us.$$

Theorem 17. *Let Ω and Ω' represent complementary paths, that is,*

$$\Omega = R^{m_1} L^{m_2} R^{m_3} \cdots (R \text{ or } L)^{m_n} \quad \text{and}$$
$$\Omega' = L^{m_1} R^{m_2} L^{m_3} \cdots (L \text{ or } R)^{m_n}\,.$$

Then for $t > 0$,

(i) $\mathbb{V}_{\mathcal{B}_{L(\mu),t},\Omega,i} = -\mathbb{V}_{\mathcal{B}_{R(\mu),t},\Omega',i}$

(ii) $\mathbb{V}_{\mathcal{B}_{L(\mu),t},\Omega',i} = -\mathbb{V}_{\mathcal{B}_{R(\mu),t},\Omega,i}$

(iii) $\mathbb{V}_{\mathcal{B}_{L(\mu),t},Q,i}$ *is independent of* μ.
That is, $\mathbb{V}_{\mathcal{B}_{L(\mu),t},Q,i} = \mathbb{V}_{\mathcal{B}_{L(\varpi),t},Q,i}$ *for any path* Q

(iv) $\mathbb{V}_{\mathcal{B}_{R(\mu),t},Q,i}$ *is independent of* μ.
That is, $\mathbb{V}_{\mathcal{B}_{R(\mu),t},Q,i} = \mathbb{V}_{\mathcal{B}_{R(\varpi),t},Q,i}$ *for any path* Q.

Proof. We have

(i) The node of the ith and $(i+1)$th terms in $M_{\mathcal{B}_{L(\mu),t},\Omega}$ is the ith term in the left branch of μ. Similarly, the node of the ith and $(i+1)$th terms in $M_{\mathcal{B}_{R(\mu),t},\Omega'}$ is the ith term in the right branch of μ. For $t > 0$, the two paths that connect the ith and $(i+1)$th terms in $M_{\mathcal{B}_{L(\mu),t},\Omega}$ with their node, and the two paths that connect the ith and $(i+1)$th terms in $M_{\mathcal{B}_{R(\mu),t},\Omega'}$ with their node form a symmetry couplet. The result follows from Theorem 16.

(ii) The argument follows for (i) with Ω and Ω' interchanged.

(iii) and (iv) follow from Theorem 16(i). $\square$

Theorem 18. *For $k = 2^i\,(2m-1) + 1$ and $r = 1, 2, \ldots, 2^i$,*

$$h_{n,k} = h_{n,k-r} \tilde{\oplus} h_{n,k+r}\,.$$

Proof. By Theorem 6(v) and Definition 2,

$$\begin{aligned} h_{n,k} &= h_{n-i-1,m} \oplus h_{n-i-1,m+1}, \\ &= h_{n,2^i(2m-1)+1-2^i} \oplus h_{n,2^i(2m-1)+1+2^i}, \\ &= h_{n,k-2^i} \oplus h_{n,k+2^i}. \end{aligned}$$

This is the $r = 2^i$ case (the only case when $i = 0$).

For $r < 2^i$ there are two cases:

(1) $i = 1$ for which $k = 4m - 1$ and $r = 1$.
By Theorem 6(vi) and Definition 2,

$$h_{n-1,2m} = h_{n,4m-1} = h_{n,4m-2} \tilde{\oplus} h_{n,4m}.$$

That is,

$$h_{n,k} = h_{n,k-r} \tilde{\oplus} h_{n,k+r}.$$

(2) $i > 1$. Then $h_{n,k-r}$ and $h_{n,k+r}$ are found on complementary paths based on $h_{n,k}$. The result follows by Corollary 4. □

6. Summation

We now explore summation of terms within any one level of the tree to demonstrate that progressive child's addition of consecutive terms in a level of the tree (whether forward addition or backward addition) yields a value that is dependent only on the last two terms of the addition. That is, the addition is not dependent on the number or size of earlier summands.

Definition 19. $\tilde{\oplus} \sum$ is summation over a sequence of summands performed using $\tilde{\oplus}$, and where summation proceeds sequentially from left to right.

Example 10.

$$\left(\frac{1}{4} \tilde{\oplus} \frac{2}{5}\right) \tilde{\oplus} \frac{3}{5} = \frac{1}{3} \tilde{\oplus} \frac{3}{5} = \frac{1}{2} = \tilde{\oplus} \sum_{l=1}^{3} h_{4,2l}.$$

Definition 20. ($T_{n,j,k}$ and $T'_{n,j,k}$).

- $T_{n,j,k}$ is defined as the sum by child's addition of the j terms beginning with the $(k - j + 1)$th term and concluding with the k th term in level n of the Stern–Brocot tree and where the summation proceeds successively *left to right*, and at each summation all fractions are reduced. That is,

$$\begin{aligned} T_{n,j,k} &= \left(\left(\cdots \left(\left(h_{n,2(k-j+1)} \tilde{\oplus} h_{n,2(k-j+2)}\right) \tilde{\oplus} h_{n,2(k-j+3)}\right) \tilde{\oplus} \cdots\right) \tilde{\oplus} h_{n,2k-2}\right) \tilde{\oplus} h_{n,2k}, \\ &= \tilde{\oplus} \sum_{l=k-j+1}^{k} h_{n,2l}. \end{aligned}$$

- $T'_{n,j,k}$ is defined as the sum by child's addition of the j terms beginning with the $(k + j - 1)$th term and concluding with the kth term in level n of the Stern–Brocot tree and where the summation proceeds successively *right to left*, and at each summation all fractions are reduced. That is,

$$\begin{aligned} T'_{n,j,k} &= h_{n,2k} \tilde{\oplus} \left(h_{n,2k+2} \tilde{\oplus} \left(\cdots \tilde{\oplus} \left(h_{n,2(k+j-3)} \tilde{\oplus} \left(h_{n,2(k+j-2)} \tilde{\oplus} h_{n,2(k+j-1)}\right)\right) \cdots\right)\right), \\ &= \tilde{\oplus} \sum_{l=0}^{j-1} h_{n,2(k+j-1-l)}. \end{aligned}$$

Example 11.

$$T_{6,3,7} = \left(\frac{4}{11}\tilde{\oplus}\frac{5}{13}\right)\tilde{\oplus}\frac{5}{12} = \frac{3}{8}\tilde{\oplus}\frac{5}{12} = \frac{2}{5}.$$

$$T'_{6,3,7} = \frac{5}{12}\tilde{\oplus}\left(\frac{4}{9}\tilde{\oplus}\frac{5}{9}\right) = \frac{5}{12}\tilde{\oplus}\frac{1}{2} = \frac{3}{7}.$$

We are now able to state our main result for child's addition across a level in the tree.

Theorem 19 (*Main Result for Child's Addition Across a Level*)**.** (i) *For* $2 \le j \le k$*,*

$$T_{n,j,k} = h_{n-1,k} = h_{n,2k-1}$$

(ii) *For* $2 \le j \le 2^{n-1} - k + 1$, $k \ge 1$,

$$T'_{n,j,k} = h_{n-1,k+1} = h_{n,2k+1}.$$

Proof. We examine each of the two restrictions on *j*.

(i) We prove this by induction on *j*.

$T_{n,2,k} = h_{n-1,k}$ is true by Theorem 18. Suppose our result is true for $T_{n,v,k}$ where $v < k$. By Definition 20,

$$\begin{aligned}
T_{n,v+1,k} &= \tilde{\oplus}\sum_{l=k-v}^{k} h_{n,2l},\\
&= \left(\cdots\left(\left(h_{n,2(k-v)}\tilde{\oplus}h_{n,2(k-v+1)}\right)\tilde{\oplus}h_{n,2(k-v+2)}\right)\tilde{\oplus}\cdots\right)\tilde{\oplus}h_{n,2k},\\
&= \left(\cdots\left(h_{n,2(k-v)+1}\tilde{\oplus}h_{n,2(k-v+2)}\right)\tilde{\oplus}\cdots\right)\tilde{\oplus}h_{n,2k} \text{by Theorem 18},\\
&= \left(\ldots(h_{n,2(k-v)+3}\tilde{\oplus}h_{n,2(k-v+3)})\tilde{\oplus}\cdots\right)\tilde{\oplus}h_{n,2k} \quad \text{by Theorem 6(viii)},\\
&= \left(\cdots\left((h_{n,2(k-v+1)}\tilde{\oplus}h_{n,2(k-v+2)})\tilde{\oplus}h_{n,2(k-v+3)}\right)\tilde{\oplus}\cdots\right)\tilde{\oplus}h_{n,2k} \quad \text{by Theorem 18},\\
&= \tilde{\oplus}\sum_{l=k-v+1}^{k} h_{n,2l},\\
&= T_{n,v,k}.
\end{aligned}$$

Hence $T_{n,2,k} = T_{n,3,k} = \cdots = T_{n,k-1,k} = T_{n,k,k} = h_{n-1,k} = h_{n,2k-1}$.

(ii) A similar induction proof holds for $T'_{n,j,k}$ except that Theorem 6(ix) is used instead of Theorem 6(viii). □

Example 12.

$$T_{6,2,7} = \frac{5}{13}\tilde{\oplus}\frac{5}{12} = \frac{2}{5}.$$

$$T_{6,3,7} = \left(\frac{4}{11}\tilde{\oplus}\frac{5}{13}\right)\tilde{\oplus}\frac{5}{12} = \frac{3}{8}\tilde{\oplus}\frac{5}{12} = \frac{2}{5}.$$

And so, $T_{6,4,7} = T_{6,5,7} = T_{6,6,7} = T_{6,7,7} = \frac{2}{5}$.

$$T'_{6,2,7} = \frac{5}{12}\tilde{\oplus}\frac{4}{9} = \frac{3}{7}$$

$$T'_{6,3,7} = \left(\frac{5}{9}\tilde{\oplus}\frac{4}{9}\right)\tilde{\oplus}\frac{5}{12} = \frac{1}{2}\tilde{\oplus}\frac{5}{12} = \frac{3}{7}.$$

And so $T'_{6,4,7} = T'_{6,5,7} = T'_{6,6,7} = T'_{6,7,7} = \frac{3}{7}$.

Corollary 5. *For* $2 \le j \le k$, $2 \le m \le 2^{n-1} - k + 1$,

$$T_{n,j,k} \oplus T'_{n,m,k} = h_{n,2k}.$$

Proof. Combining Theorem 19 with Theorem 18 gives the result.

7. Child's addition of continued fractions

One of the shortcomings of continued fractions, and probably the reason for their obscurity, is their inability to be readily subject to basic arithmetic operations (addition, subtraction, multiplication and division). However using our earlier results, we show that some general results are possible under child's addition.

Theorem 20 (*Main Result for Child's Addition of Continued Fractions*)**.**

(I) *For* $a_0 \ge 0$, $b_1 \ge 1$, $a_i \ge 1$, $i = 1, 2, \ldots$, *and* β *any continued fraction greater than or equal to 1,*

$$[a_0, a_1, \ldots, a_{i-1}, a_i, b_1, \beta] \,\tilde{\oplus}\, [a_0, a_1, \ldots, a_{i-1}, a_i + b_1, \beta]$$
$$= [a_0, a_1, \ldots, a_{i-1}, a_i + 1].$$

(II) *For* $k, m \ge 0$,

$$[a_0, a_1, \ldots, a_i, k] \,\tilde{\oplus}\, [a_0, a_1, \ldots, a_i - 1, 1, m] = [a_0, a_1, \ldots, a_i].$$

(III) *For* $m \ge 0$,

$$[a_0, a_1, \ldots, a_i] \oplus [a_0, a_1, \ldots, a_i, m] = [a_0, a_1, \ldots, a_i, m + 1].$$

(IV) *For* $m \ge 0$,

$$[a_0, a_1, \ldots, a_i, m] \,\tilde{\oplus}\, [a_0, a_1, \ldots, a_{i-1}, a_i - 1] = [a_0, a_1, \ldots, a_i].$$

(V) *For* $m \ge 0$,

$$[a_0, a_1, \ldots, a_i, m] \,\tilde{\oplus}\, [a_0, a_1, \ldots, a_i, m + 2] = [a_0, a_1, \ldots, a_i, m + 1].$$

(VI) *For* $r, s \ge 0$, $m \ge 1$,

$$[a_0, a_1, \ldots, a_{i-1}, a_i - 1, 1, r, m] \,\tilde{\oplus}\, [a_0, a_1, \ldots, a_i, s, m] = [a_0, a_1, \ldots, a_i].$$

(VII) *For* $r \ge 0$, $m \ge 1$,

$$[a_0, a_1, \ldots, a_{i-1}, a_i - 1, 1, r, m] \,\tilde{\oplus}\, [a_0, a_1, \ldots, a_{i-1}, a_i - 1, 1, r + 2, m]$$
$$= [a_0, a_1, \ldots, a_{i-1}, a_i - 1, 1, r + 1, m].$$

(VIII) *For* $r \ge 0$, $m \ge 1$,

$$[a_0, a_1, \ldots, a_i, r, m] \,\tilde{\oplus}\, [a_0, a_1, \ldots, a_i, r + 2, m] = [a_0, a_1, \ldots, a_i, r + 1, m].$$

Proof. We have

(I) There are two cases:

(1) i odd. By Theorem 4,

$$\begin{aligned}&[a_0, a_1, \ldots, a_{i-1}, a_i, b_1, b_2, \ldots, b_m] \\ &\quad = R^{a_0+1} L^{a_1} \cdots R^{a_{i-1}} L^{a_i} R^{b_1} L^{b_2} \cdots (L \text{ or } R)^{b_m - 1}\end{aligned} \tag{7.1}$$

and

$$\begin{aligned}&[a_0, a_1, \ldots, a_{i-1}, a_i + b_1, b_2, \ldots, b_m] \\ &\quad = R^{a_0+1} L^{a_1} \cdots R^{a_{i-1}} L^{a_i + b_1} R^{b_2} \cdots (R \text{ or } L)^{b_m - 1}, \\ &\quad = R^{a_0+1} L^{a_1} \cdots R^{a_{i-1}} L^{a_i} L^{b_1} R^{b_2} \cdots (L \text{ or } R)^{b_m - 1}.\end{aligned} \tag{7.2}$$

Comparing (7.1) and (7.2),

$[a_0, a_1, \ldots, a_{i-1}, a_i, b_1, b_2, \ldots, b_m]$ and $[a_0, a_1, \ldots, a_{i-1}, a_i + b_1, b_2, \ldots, b_m]$ are on the same level of the tree and are found on complementary paths based on

$$R^{a_0+1}L^{a_1} \cdots R^{a_{i-1}}L^{a_i} = [a_0, a_1, \ldots, a_{i-1}, a_i + 1] .$$

Letting $m \to \infty$ and setting $\beta = [b_2, b_3, \ldots]$ where $b_i \geq 1, i = 2, 3, \ldots$ the result follows by Corollary 4.

(2) i even. By Theorem 4,

$$[a_0, a_1, \ldots, a_{i-1}, a_i, b_1, b_2, \ldots, b_m] = R^{a_0+1}L^{a_1} \cdots L^{a_{i-1}}R^{a_i}L^{b_1}R^{b_2} \cdots (L \text{ or } R)^{b_m-1} \tag{7.3}$$

and

$$\begin{aligned}[a_0, a_1, \ldots, a_{i-1}, a_i + b_1, b_2, \ldots, b_m] &= R^{a_0+1}L^{a_1} \cdots L^{a_{i-1}}R^{a_i+b_1}L^{b_2} \cdots (R \text{ or } L)^{b_m-1} , \\ &= R^{a_0+1}L^{a_1} \cdots L^{a_{i-1}}R^{a_i}R^{b_1}L^{b_2} \cdots (L \text{ or } R)^{b_m-1} .\end{aligned} \tag{7.4}$$

Comparing (7.3) and (7.4), $[a_0, a_1, \ldots, a_{i-1}, a_i, b_1, b_2, \ldots, b_m]$ and $[a_0, a_1, \ldots, a_{i-1}, a_i + b_1, b_2, \ldots, b_m]$ are on the same level of the tree and are found on complementary paths based on

$$R^{a_0+1}L^{a_1} \cdots L^{a_{i-1}}R^{a_i} = [a_0, a_1, \ldots, a_{i-1}, a_i + 1] .$$

Letting $m \to \infty$ and setting $\beta = [b_2, b_3, \ldots]$ where $b_i \geq 1, i = 2, 3, \ldots$ the result follows by Corollary 4.

(II) By Theorem 9, $[a_0, a_1, \ldots, a_i, k]$ and $[a_0, a_1, \ldots, a_i - 1, 1, m]$ are the kth and mth terms respectively of the two branches of $[a_0, a_1, \ldots, a_i]$. Their common parent is therefore $[a_0, a_1, \ldots, a_i]$. By Theorem 8 the result follows.

(III) There are two cases:

(1) i even. There are two subcases:

(a) $a_i = 1$. Then $[a_0, a_1, \ldots, a_i] = [a_0, a_1, \ldots, a_{i-1} + 1]$ by which Theorem 9 has right branch $\{[a_0, a_1, \ldots, a_i, m] \mid m \geq 1\}$.

(b) $a_i > 1$. Then by Theorem 9 $[a_0, a_1, \ldots, a_i]$ has right branch $\{[a_0, a_1, \ldots, a_i, m] \mid m \geq 1\}$.

Combining (a) and (b): for i even, $[a_0, a_1, \ldots, a_i]$ and $[a_0, a_1, \ldots, a_i, m]$ are the left and right parents respectively of $[a_0, a_1, \ldots, a_i, m + 1]$. The result follows.

(2) i odd. There are two subcases:

(a) $a_i = 1$. Then, as in (1)(a), $[a_0, a_1, \ldots, a_i] = [a_0, a_1, \ldots, a_{i-1} + 1]$ by which Theorem 9 has left branch $\{[a_0, a_1, \ldots, a_i, m] \mid m \geq 1\}$.

(b) $a_i > 1$. Then by Theorem 9 $[a_0, a_1, \ldots, a_i]$ has left branch $\{[a_0, a_1, \ldots, a_i, m] \mid m \geq 1\}$.

Combining (a) and (b): for i odd, $[a_0, a_1, \ldots, a_i]$ and $[a_0, a_1, \ldots, a_i, m]$ are the right and left parents respectively of $[a_0, a_1, \ldots, a_i, m + 1]$. The result follows.

Combining (1) and (2), the result follows.

(IV) There are two cases:

(1) i even. By Theorem 9, for both subcases $a_i = 1$ and $a_i > 1$, $[a_0, a_1, \ldots, a_i]$ has right branch $\{[a_0, a_1, \ldots, a_i, m] \mid m \geq 1\}$. By Theorem 9 again, $[a_0, a_1, \ldots, a_i]$ is the a_ith term on the left branch of $[a_0, a_1, \ldots, a_{i-1}]$. Thus the left parent of $[a_0, a_1, \ldots, a_i]$ is $[a_0, a_1, \ldots, a_{i-1}, a_i - 1]$. (Note that $[a_0, a_1, \ldots, a_{i-1}, a_i - 1] = [a_0, a_1, \ldots, a_{i-2}]$ for $a_i = 1$). The result follows from Theorem 8.

(2) i odd. The argument is identical to that for (1) except that instances of *right* become *left*, and instances of *left* become *right*.

(V) The result follows from Theorems 9 and 14.

(VI) The result follows from Theorem 11(iii) and 15.

(VII) and (VIII) The results follow from Theorems 14 and 15. □

8. Further developments

The relationships formed through child's addition in the Stern–Brocot have not been exhausted in this paper. For example, there are interesting patterns formed when a term in a branch is child's added to distant terms in the same branch or to terms in distant branches. Similar comments hold for offset branches. These patterns have yet to be formulated algebraically. Also, we have yet to explore the patterns that may result from the child's addition of right diagonals with left diagonals. These relationships may, in turn, give rise to further continued fraction results under child's addition. Moreover, child's addition in the Calkin–Wilf tree, a related tree to the Stern–Brocot tree (see [3]), has yet to be explored.

Acknowledgement

We thank the referee for offering helpful suggestions to improve the flow and prioritisation of ideas in this paper.

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