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THE GEOMETRY OF CUBE COMPLEXES AND THE COMPLEXITY OF THEIR FUNDAMENTAL GROUPS

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We investigate the geometry of geodesics in CAT(0) cube complexes. A group which acts cocompactly and properly discontinuously on such a complex is shown to have a biautomatic structure. There is a family of natural subgroups each of which is shown to be rational. © 1997 Elsevier Science Ltd. All rights reserved

1. INTRODUCTION

The idea of formulating non-positive curvature for general path-metric spaces goes back to the work of Busemann, Aleksandrov, and Topogonov. In [8] Gromov introduced the term CAT(0) for a criterion applicable to path metric spaces, based on Topogonov's comparison theorem in Riemannian geometry. The CAT(0) condition has proved to be very fruitful in geometric terms, particularly in the context of piecewise Euclidean cell complexes, and CAT(0) cube complexes (which were also introduced by Gromov in [8]) form, a class of spaces which seem particularly amenable to investigation.

Various classes of groups are known to act properly discontinuously and cocompactly on CAT(0) cubed complexes, among them the right angled Coxeter groups and the right angled Artin groups. In [10] we show how to construct a CAT(0) cube complex for any finitely generated word hyperbolic Coxeter group, (in fact the construction works for any finitely generated Coxeter group, but the action is not always cocompact). The Cayley graph of a free group is a simplicial tree, which may be viewed as a 1-dimensional CAT(0) cube complex, and the free abelian group of rank n acts on the integer lattice cubing of $\mathbb{R}^n$. Cube complexes of dimension 2 have been called $A_1 \times A_1$ complexes by Gersten and Short [7], and the groups which act freely on such complexes are the T(4)-C(4)-P small cancellation groups. Aitchison and Rubinstein [1] showed that the universal cover of many 3-manifolds can be given a CAT(0) cubing on which, of course, their fundamental groups act; such groups were studied by Skinner [15], and Reeves [12].

All of these classes of groups are known to be automatic or biautomatic, that is they admit normal forms for their group elements which form a regular language satisfying the k-fellow-traveller property for some constant k (see Section 2 for definitions). In the case of the fundamental groups of CAT(0) cubed 3-manifolds Reeves established results analogous to the ones in this paper, but in general the techniques used have varied from class to class. Our main theorem may be regarded as a generalisation of these results, giving a uniform

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way to construct a synchronous biautomatic structure for any group acting properly discontinuously and cocompactly on a CAT(0) cube complex.

THEOREM 5.3. *Let X be a simply connected, non-positively curved cube complex, and G be a group acting effectively, cellularly, properly discontinuously and cocompactly on X. Then there is a synchronously biautomatic structure on G.*

The paper is organised as follows: in Section 2 we will establish the notation and definitions used in the rest of the paper, discuss the idea of the proof of our main theorem and prove some technical results on the geometry of hyperplanes in a cubing which are used to prove it. In Section 3 we define the class of "normal cube-paths" in any CAT(0) cube complex, which is used in Sections 4 and 5 to construct the biautomatic structure on the group. Finally in Section 6 we discuss some examples.

2. PRELIMINARIES

2.1. Biautomatic structures

We first introduce some of the terminology from the theory of automatic groups. For a complete introduction see, for example, [6] or [3].

Let G be a group and $\mathscr{A}$ a set of semi-group generators for G such that $\mathscr{A}$ is closed under the operation of taking inverses, and let $\mathscr{A}^*$ denote the free monoid on $\mathscr{A}$. There is a natural map from $\mathscr{A}^*$ to G, denoted $w \mapsto \bar{w}$. A normal form for G is a language $\mathscr{L} \subseteq \mathscr{A}^*$ which surjects onto G. The language is called *regular* if there is a finite-state automaton for which $\mathscr{L}$ is the accepted language.

Given the generating set $\mathscr{A}$ we may construct the Cayley graph $\Gamma_{\mathscr{A}}(G)$ of G; since $\mathscr{A}$ is closed under inverses, the word metric d on G with respect to $\mathscr{A}$ coincides with the simplicial metric on $\Gamma_{\mathscr{A}}(G)$. Given a word $w \in \mathscr{L}$ there is a sequence of vertices $\bar{w}_n$ in $\Gamma_{\mathscr{A}}(G)$, each at distance 1 from its predecessor, where w_n denotes the initial segment of w of length n. This gives rise to a continuous path $\hat{w}: [0, \infty) \to \Gamma_{\mathscr{A}}(G)$ where $\hat{w}(t) = \bar{w}_t$ for each positive integer t is less than the length, $l = |w|$, of w, and $\hat{w}(t) = \bar{w}_l$ for t greater than or equal to l. The group G acts (on the left) on the Cayley graph, so given any element $g \in G$ and any path $\hat{w}$ we obtain a map $g\hat{w}: [0, \infty) \to \Gamma_{\mathscr{A}}(G)$ which represents a path based at g. Two paths p_1 and p_2 are (synchronous) δ-fellow-travellers if for all $t > 0$, $d(p_1(t), p_2(t)) \leqslant \delta$.

Definition 2.1. A *biautomatic structure* on G is a normal form such that the language $\mathscr{L}$ is regular and there exists $\delta \in \mathbb{R}$ such that for any pair of words u and v in $\mathscr{L}$, and any generators $g, h \in \mathscr{A}$ such that $\bar{u} = \overline{gvh}$, the paths $\hat{u}$ and $g\hat{v}$ are δ-fellow-travellers.

2.2. Cube complexes

The groups we will study all act by isometries on CAT(0) cube complexes, so we spend some time introducing these complexes, in particular developing the theory of the codimension-1 hyperplanes that they contain. These hyperplanes will play a fundamental role in the construction of the biautomatic structure.

A *cube complex* is a metric polyhedral complex in which each cell is isometric to a Euclidean cube and the glueing maps are isometries. If there is a bound on the dimension of the cubes then such a complex carries a complete geodesic metric [4].

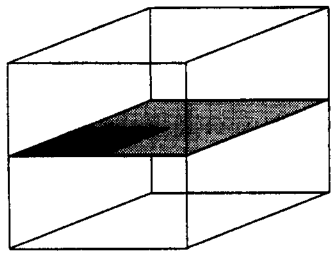

Fig. 1. A quadrant in a midplane in a cube.

Definition 2.2. A cube complex is *non-positively curved* if for any cube C the following conditions on the link of C, $lk(C)$, are satisfied:

(i) (*no bigons*) For each pair of vertices in $lk(C)$ there is at most one edge containing them.
(ii) (*no triangles*) Every edge cycle of length three in $lk(C)$ is contained in a 2-simplex of $lk(C)$.

The following theorem of Gromov relates the combinatorics and the geometry of the complex.

LEMMA 2.3 (Gromov [8]). *X is locally $CAT(0)$ if and only if it is non-positively curved, and it is $CAT(0)$ if and only if it is simply connected and non-positively curved.*

2.3. Hyperplanes

A *midplane* of a cube $[-1/2, 1/2]^n$ is its intersection with a codimension-1 coordinate hyperplane, so every n-cube contains n midplanes which pairwise intersect. For a fixed midplane these intersections decompose it into 2^{n-1} components and a *quadrant* Q of a midplane M is the closure of one of these components (Fig. 1).

Definition 2.4. Each midplane M defines a unique cube in X, the cube of least dimension in X which contains M, and we will denote this by $C(M)$. The star of a point $x \in X$, denoted by $St_X(x)$ (or just $St(x)$), is the union of the cubes containing x. More generally the star of a cube C, denoted $St(C)$, is the union of the cubes containing C and the star of a hyperplane is the union of the cubes which intersect it. The *carrier* of a point $p \in X$ is the unique cube containing p in its interior.

Two midplanes M and N are *hyperplane equivalent* if there is a sequence of midplanes $M = M_1, M_2, \dots, M_n = N$ such that $M_i \cap M_{i+1}$ is a midplane for each $i = 1, 2, \dots, (n-1)$.

Definition 2.5. A *hyperplane* in X is an equivalence class of midplanes.

In fact, it is convenient to regard the hyperplane as a "codimension-1" subspace consisting of the union of the midplanes in its equivalence class. Suppose that S is an equivalence class of midplanes; each midplane is a cube and we define H to be the cube complex whose cubes are the elements of S, and whose glueing maps are the restrictions of the glueing maps in X.

It is easy to show that this cubing satisfies the combinatorial non-positive curvature condition, so that the hyperplane is also a cube complex of non-positive curvature. Notice that, as for X, H is a geodesic metric space. Let $\phi : H \to X$ be the inclusion which sends each cube in H to the corresponding midplane in X; we will show that ϕ is an isometry, which will enable us to treat hyperplanes as totally geodesic, codimension-1 subspaces.

LEMMA 2.6. *The map ϕ is an isometry onto its image.*

Proof. It is easy to show that each cube in X is isometrically embedded, so $\phi|_M$ is an isometry for each cube $M \in H$. We wish to show that if $x \in H$, then there is a neighbourhood U of x such that $\phi|_U$ is an isometry. Then using the results of Gromov [8, Section 4] one observes that a local isometry into a CAT(0) space is a global isometry.

Suppose that M is a midplane in X, define $\rho_M : C(M) \to C(M)$ to be the reflection in M. If M_1 and M_2 are cubes of H which each contain x, then the reflections $\rho_{\phi(M_1)}$ and $\rho_{\phi(M_2)}$ agree on $C(\phi(M_1 \cap M_2))$, so we may define a "reflection" $\mathrm{p}: St(\phi(x)) \to St(\phi(x))$ where for each cube M in $St_H(x)$, $\rho|_{C(\phi(M))} = \rho_{\phi(M)}$. To see that this defines ρ we need to establish that for any cube C in $St(\phi(x))$ there is a cube M in $St_H(x)$ with $C = C(\phi(M))$. This follows from the observation that the minimal cube in H containing $\phi(x)$ is precisely $C(\phi(M))$ where M is the minimal cube in H containing x. There is an $\varepsilon > 0$ such that B, the ball of radius ε in X about $\phi(x)$, is convex (see [4]) and contained in $St(\phi(x))$. It follows that ρ preserves B, and therefore restricts to an isometry with fixed point set $\phi(St_H(x)) \cap B$. The fixed point set of an isometry is totally geodesic, hence a geodesic in B with endpoints in $\phi(St_H(x)) \cap B$ lies entirely in $\phi(St_H(x)) \cap B$. Putting $U = \phi^{-1}(\phi(St_H(x)) \cap B)$, we have that $\phi|_U$ is an isometry.

As a consequence of the above, we shall regard hyperplanes as totally geodesic subspaces in X. Notice that it follows from the proof that the intersection of any hyperplane with a cube is a single midplane, since otherwise the hyperplane intersects itself transversely. It therefore divides each cube it meets into two components, a standard cohomological argument, using the fact that the cubing is simply connected, then shows:

LEMMA 2.7. *Every hyperplane in X separates X into exactly two components.*

2.4. Sketch of the main theorem

We can now outline the idea in this paper. Let G be a group acting cellularly, properly discontinuously, freely and cocompactly on a CAT(0) cube complex X. To simplify the arguments for the moment assume that the action has a single orbit G_v of vertices, where v is a vertex of X so that we can identify each element $g \in G$ with the corresponding translate gv of v. We can also identify G with the fundamental group $\pi_1(G \backslash X, Gv)$, and it is easy to show that each homotopy class of a loop in $G \backslash X$ can be represented by a piecewise linear path composed of loops each of which is the oriented diagonal of a cube in the quotient and which lifts to a path in X joining v to gv. Since G acts cocompactly on X this gives us a finite alphabet $\mathscr{A}$ (the oriented diagonals in $G \backslash X$) which generates G, that is, the obvious map from $\mathscr{A}^*$ to G is surjective. Of course, each element of G can be represented in many different ways as a product in this alphabet; we will show how to pick out a unique representative for each element as a "normal cube-path" giving a language $\mathscr{L}$ in $\mathscr{A}^*$ which maps bijectively to G. The normal cube-path is actually constructed in X.

Any loop representing g lifts to a path in X joining v to gv, and the sequence of diagonals used in the normal cube-path representing g is chosen to ensure that the sequence of

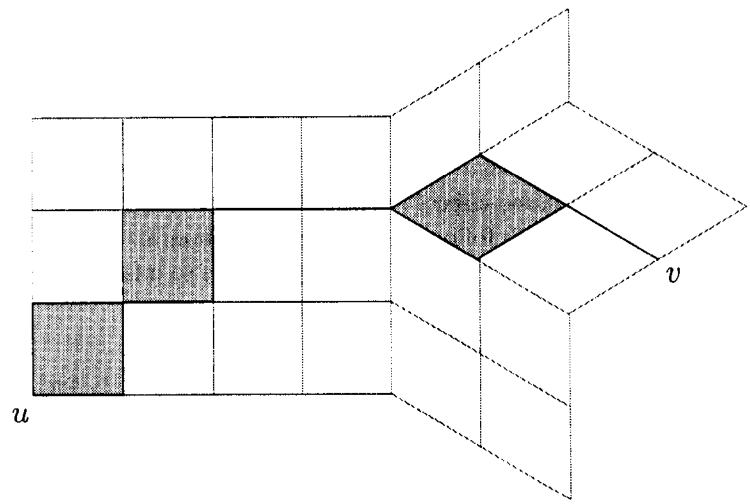

Fig. 2. A normal cube-path from u to v.

diagonals used to connect v to gv travels "as diagonally as possible". To interpret this we note that the two vertices are separated by a finite family of hyperplanes, and that any path joining them must cross these hyperplanes. We will insist that the first diagonal traversed from v to gv crosses all of the hyperplanes separating v and gv which cut edges adjacent to v. That we can do this depends on the fact that all of these hyperplanes have a common intersection in a cube containing v as a vertex, and this fact is the key technical result in our paper (Lemma 2.14 and Proposition 2.15). This will get us from v to some vertex v_1 in X and the second diagonal in the normal cube-path is then required to cross all of the hyperplanes separating v_1 from gv which cut an edge adjacent to v_1 and so on (see Fig. 2).

Our main theorem is then proved in three parts. First we need to show that any two vertices in X can be joined by a normal cube-path (Proposition 3.3), so that the corresponding language surjects on G. Second we need to show that the corresponding language is regular (Proposition 5.1), which we do by showing that the condition defining normality is a purely local condition. Finally, we need to show that normal cube-paths satisfy the k-fellow-traveller property for some k. In fact we show that we can put $k = 1$ (Proposition 5.2).

In general we do not need to assume that the action is free, merely that there is a free orbit of vertices. Provided the action has no kernel and is properly discontinuous we can arrange this by subdividing the cubing. Of course G no longer acts transitively on the vertices in this case, but we do not need this either. We introduce the fundamental groupoid of $G\backslash X$, and use the same argument to show that it is synchronously biautomatic. It follows from [ECHLPT] that any vertex group of a biautomatic groupoid is itself biautomatic, and therefore that the fundamental group G is biautomatic as required.

As remarked above, the key point is that hyperplanes which meet the star of a given vertex and separate it from another vertex gv intersect in that star (Lemma 2.14). The remainder of this section is dedicated to proving this fact.

2.5. Intersections of hyperplanes

Observe that given an edge, $e \in X$, there is a unique hyperplane, $H(e)$, which meets e and moreover $e \cap H(e)$ is the midpoint of the edge. We shall use the following results about neighbourhoods of midpoints:

LEMMA 2.8. *Suppose that x is the midpoint of an edge e, and denote by B the open ball of radius $\frac{1}{2}$ about x. Then for all $p \in B$, the carrier of p contains e. In particular, B contains no vertices of X.*

Proof. Consider the geodesic segment γ from x to p. For small ε, the initial segment of γ of length ε is contained in some cube C, which we assume to have minimal dimension amongst all such cubes. The result follows from the observation that, for a Euclidean cube with side length 1, the distance from the midpoint of an edge to a face not containing that edge is at least $\frac{1}{2}$. □

LEMMA 2.9. *Suppose that x_1 and x_2 are midpoints of distinct edges e_1 and e_2 respectively, and denote by B_i the open $\frac{1}{2}$-ball about x_i. If $B_1 \cap B_2 \neq \emptyset$, then there is a 2-cube which contains e_1 and e_2 as adjacent edges.*

Proof. Choose a point $p \in B_1 \cap B_2$. Applying the preceding lemma, the carrier, C, of p has both e_1 and e_2 as edges. In a Euclidean cube the distance between non-adjacent edges is at least 1, so e_1 and e_2 must be adjacent in C and are therefore contained in some two-dimensional face of C. □

Since it has been shown that hyperplanes intersect each cube in a single midplane, Lemma 2.9 implies the following (in the notation of the lemma):

COROLLARY. *If $H(e_1) = H(e_2)$, then $B_1 \cap B_2 = \emptyset$.*

PROPOSITION 2.10. *Let e_1 and e_2 be distinct edges of X which have a common vertex v. Then $H(e_1) \neq H(e_2)$.*

Proof. The edge intervals joining x_i to v connect to form a path γ in X from x_1 to x_2 of length 1. Assume that $H(e_1) = H(e_2)$. Then from the corollary it follows that γ is a geodesic, which contradicts the fact that hyperplanes are totally geodesic.

LEMMA 2.11. *Let H be a hyperplane which intersects the star of a given vertex v. Then $H \cap St(v)$ is connected, and there is a unique edge which meets both v and $H \cap St(v)$.*

Proof. By definition, $H \cap St(v)$ is composed of midplanes each of which intersects a unique edge incident to v. Applying Proposition 2.10 we see that each midplane intersects the same edge and connectedness is also clear.

Let F denote the union of all quadrants which meet the edge, e, given by the above lemma. We shall call F a *facet* and e the edge *dual to* F. Both F and e are determined by H and v. The point at which F meets the edge dual to it will be called *the midpoint of the facet* F. The above lemmas can be extended to the case of open $\frac{1}{2}$-neighbourhoods of facets.

LEMMA 2.12. *Denote by B the open $\frac{1}{2}$-neighbourhood of F. Then for all $p \in B$, p is contained in the interior of a cube which contains e, the edge dual to F.*

Proof. Let γ be a shortest path from p to F, so γ ends in a quadrant Q of F and is orthogonal to it. We may assume that p is contained in the $\frac{1}{2}$-neighbourhood of some quadrant, Q, of a midplane intersecting e. The proof then proceeds as in Lemma 2.8, with the observation that for a Euclidean cube the distance from a quadrant to a face not meeting the quadrant is at least $\frac{1}{2}$.

LEMMA 2.13. *Suppose that F_1 and F_2 are distinct facets inside the star of some vertex v, and denote by B_i the open $\frac{1}{2}$-neighbourhood of F_i. If $B_1 \cap B_2 \neq \emptyset$, then F_1 and F_2 intersect.*

Proof. Let e_i denote the edge dual to F_i, and choose a point $p \in B_1 \cap B_2$. By the preceding lemma, p is in the interior of a cube which contains both e_1 and e_2. The centre of this cube lies in both facets.

Now, let H_1 and H_2 be distinct hyperplanes which both intersect the star of a given vertex v; for each i let F_i denote the facet determined by H_i and v, and let x_i denote the midpoint of F_i. Then there is an arc from x_1 and x_2 consisting of the two half edges from x_1 to v and from v to x_2, and this arc has length 1, so the distance from x_1 to x_2 is at most 1. On the other hand if it is less than 1 then the open balls of radius $\frac{1}{2}$ around x_1 and x_2 intersect non-trivially, so by the above lemma the facets intersect. Notice that in this case the distance between x_1 and x_2 is exactly $1/\sqrt{2}$.

LEMMA 2.14. *If H_1, H_2 are distinct hyperplanes which both intersect the link of a given vertex v, then either they intersect in that link or they are disjoint.*

Proof. Let F_i denote the facet determined by H_i and v, and x_i its midpoint. If $d(x_1, x_2) < 1$ then H_1 meets H_2 in the link of v by the previous comment. We can therefore assume that the distance is 1. As in the comment above we then have a geodesic arc from x_1 to x_2 which runs along the edges dual to F_1 and F_2. We will denote this arc by α. We will further assume that the two hyperplanes intersect in some cube C and will show that this is a contradiction. Let x denote the centre of the cube C (so x must lie in the intersection $H_1 \cap H_2$) and choose geodesics γ_i from x to x_i. We will assume, without loss of generality, that γ_1 has length less than or equal to γ_2, and it is clear that $\text{length}(\gamma_2) \leqslant \text{length}(\gamma_1) + 1$. In fact, the difference in length between γ_1 and γ_2, which we will denote Δ, is strictly less than 1, since if γ_2 is longer than γ_1 by 1, then the arc γ_1 followed by α is a geodesic with both endpoints on H_2 and must therefore lie entirely in H_2 since hyperplanes are totally geodesic; this contradicts the fact that α does not lie in a hyperplane.

Now extend γ_1 backward into C by $\Delta/2$, and shorten γ_2 by the same amount, to obtain new geodesics (which by abuse of notation we will also denote γ_1 and γ_2) of equal length l from points y_1 and y_2 in C with $d(y_1, y_2) \leqslant \Delta < 1$. Parameterising the new geodesics by length from their initial points y_1 and y_2 we obtain a function $f: [0, l] \to \mathbb{R}$ given by $f(t) = d(\gamma_1(t), \gamma_2(t))$. From Lemma 2.13 above, we know that the $\frac{1}{2}$-neighbourhoods about the facets F_i are disjoint, but for all $t \in [l - 1/2, l]$ we have $\gamma_i(t) \in F_i$, so $f(t) \geqslant 1$. Since $f(l) = 1$, f achieves a local minimum at l. Because X is CAT(0) it is a Busemann space (see [4]) and so this function is convex and any local minimum for it is also a global minimum. This contradicts the fact that $f(0) < 1$.

We also note the following.

PROPOSITION 2.15. *Suppose that $F_1, \ldots, F_n$ are facets in the link of some vertex, v, which intersect pairwise. Then there is an n-cube $C \in St(v)$ which intersects all of the F_i. Moreover, any cube in $St(v)$ which intersects some subset of the F_i is a face of C.*

Proof. Both statements follow from Lemma 2.14 above and the no-triangles curvature condition on links of vertices in X.

Definition 2.16. The cube C in the above proposition shall be referred to as *the cube spanned by the hyperplanes* $H_1, \ldots, H_n$. If a pair of cubes C and D are contained in some cube, it will be useful to talk of the cube spanned by them, by which we mean the unique cube of minimal dimension which contains them both.

3. CUBE-PATHS AND NORMAL CUBE-PATHS

Recall that for a cube $C \in X$, $St(C)$ is the union of all cubes which contain C as a subface (including C itself). Consider a sequence of cubes $\{C_i\}_0^n$, each of dimension at least 1, such that each cube meets its successor in a single vertex, $v_i = C_{i-1} \cap C_i$; we call this sequence a *cube-path* if C_i is the (unique) cube of minimal dimension containing v_i and v_{i+1}. Equivalently, v_i and v_{i+1} are diagonally opposite vertices of C_i. We define v_0 to be the vertex of C_0 which is diagonally opposite v_1, and v_{n+1} the vertex of C_n diagonally opposite v_n. We call the v_i, vertices of the cube-path, with v_0 the initial vertex and v_{n+1} the terminal vertex. Notice that for every vertex v_i, we have $C_{i-1}, C_i \in St(v_i)$. The length of a cube-path is defined to be the number of cubes in the sequence. A cube-path defines a family of edge paths from v_0 to v_{n+1} which travel from v_i to v_{i+1} via a geodesic in the 1-skeleton of C_i.

Definition 3.1. A cube-path is called a *normal cube-path* if $C_i \cap St(C_{i-1}) = v_i$.

Given a hyperplane in X, one or more of the cubes in a normal cube-path may intersect the hyperplane, however, as we will now see, the normal cube-path crosses the hyperplane at most once.

LEMMA 3.2. *The intersection of a normal cube-path and a hyperplane is connected.*

Proof. Assume that this is not the case, and let H be the first hyperplane to be recrossed by the path, and C_i the second cube in the path which meets H. Then every hyperplane which meets C_{i-1} separates C_i from all preceding cubes, since these hyperplanes do not cross any of the preceding cubes (here we are using the fact that hyperplanes separate X into two components). In particular each plane which meets C_{i-1} separates v_{i+1} from the initial vertex of the path. By hypothesis, H does not separate v_{i+1} and the initial vertex, so H must intersect each of the planes which meet C_{i-1}. However, it then follows from Proposition 2.15 that all of these hyperplanes intersect in $St(v_i)$. A set of facets in $St(v_i)$ which pairwise intersect must all intersect in some cube C in $St(v_i)$, by Proposition 2.15. Moreover, C contains C_{i-1} and an edge of C_i. This however contradicts the definition of a normal cube-path.

PROPOSITION 3.3. *Given two vertices* $\iota, \tau \in V(X)$, *there is a unique normal cube-path from* ι *to* τ. (The order is important here, since in general normal cube-paths are not reversible.)

Proof. We will inductively construct a normal cube-path from ι to τ and then show that it is unique. Let $v_0 = \iota$ and suppose that we have reached a vertex v_i. The next cube is uniquely determined by the midplanes in $St(v_i)$ which separate the vertex from τ. Let S be the set of midplanes in $St(v_i)$ which separate v_i from τ. If S is empty then clearly $v_i = \tau$. If $S \neq \emptyset$ then by Proposition 2.15 there is some cube in $St(v_i)$ which contains all the midplanes in S. The unique minimal such cube is chosen as the next cube in the path: C_i. We claim that $C_i \cap St(C_{i-1}) = v_i$ where this condition is taken to be vacuous in the case $v_i = \iota$. As in the proof of the above lemma, it follows from this and Proposition 2.15 that no hyperplane which meets C_i has been crossed before, and so after a finite number of steps we will reach the vertex τ. It also follows from the above claim that the cube-path is normal. To establish the claim, assume that there is some edge $e \in C_i \cap St(C_{i-1})$, and let H be the hyperplane dual to this edge. By construction, H separates v_i from τ (since it meets C_i). Then we have a contradiction: H separates v_{i-1} from $\tau \Leftrightarrow H$ meets $C_{i-1} \Leftrightarrow H$ separates v_{i-1} from $v_i \Leftrightarrow H$ does not separate v_{i-1} from τ.

Suppose now that we have constructed such a cube-path, and let $\{C'_i\}$ be another normal cube-path from ι to τ. If they are different, there is some vertex v_i which lies on both but with $C_i \neq C'_i$. Let S be the set of planes which meet C_i, and S' the set which meet C'_i. We need to show that $S = S'$. It is clear that $S' \subseteq S$ since otherwise the cube-path $\{C'_i\}$ crosses a hyperplane which does not separate its endpoints.

Let H be a plane in S. Since H separates ι and τ, it must intersect the cube-path $\{C'_i\}$ in some cube C'_j. Then, as in the proof of the above lemma, consider the set of hyperplanes which meet C'_{j-1}; each of these separates v'_j from ι. The plane H does not separate v'_j from ι, and must therefore intersect each of the planes which meet C'_{j-1}. From Proposition 2.15 it follows that they are all contained in some cube $C \in St(v'_j)$ which contains C'_{j-1} and shares a face of dimension at least 1 with C'_j. This contradicts the fact that $\{C'_i\}$ is a normal cube-path. So $S \subseteq S'$ and the proof is complete. □

Remark. The observation in the above proof can be expressed as: given a vertex v on a normal cube-path, which terminates at τ the cube following v is spanned by the planes which meet $St(v)$ and separate v from τ.

The preceding results suggest that normal cube-paths are in some sense geodesics, and it is possible to show that a normal cube-path achieves the minimum length among all cube-paths joining the endpoints [13].

4. THE FUNDAMENTAL GROUPOID

In order to define the biautomatic structure on the group G it is convenient to consider the fundamental groupoid associated to the action. We will show how to define a biautomatic structure on this groupoid, and it is then possible to obtain a structure for G, which may be identified with a vertex group in the groupoid [6]. We use the obvious notion of rationality for groupoids. The vertex group of a rational subgroupoid is rational in the vertex group of the larger groupoid, [13].

We denote by $G \backslash X$ the quotient of the complex X by the action of G. The fundamental groupoid $\pi(G \backslash X)$ is the groupoid whose objects are the points of $G \backslash X$ and morphisms between points, v, v' are homotopy classes of paths in $G \backslash X$ beginning at v and ending at v'. The multiplication in $\pi(G \backslash X)$ is induced by composition of paths. Given a subset Y of points in $G \backslash X$ we obtain a subgroupoid $\pi(G \backslash X, Y)$ whose objects are the points in Y and whose morphisms are all the morphisms of $\pi(G \backslash X)$ between the objects in Y. In particular if Y consists of a single point v then $\pi(G \backslash X, \{v\})$ is the fundamental group of $G \backslash X$ based at the point v. For each $Gv \in V(G \backslash X)$ we fix a representative v in X. Let p be a path in $G \backslash X$ between Gv, $Gv' \in V(G \backslash X)$; then there is a unique lift of p which begins at v. This ends at a translate gv' of v' where g is determined by the homotopy class of p. Moreover, any path in X from v to gv' will project to a path in $G\backslash X$ which is homotopic to p.

We now define an alphabet $\mathscr{A}$ which will map to a set of generators for the groupoid $\pi(G\backslash X, V(G\backslash X))$. A *directed cube* is a cube with two (ordered) diagonally opposite vertices specified. An element of the alphabet is a directed cube in $G\backslash X$, and projects to the homotopy class (relative to its endpoints) of the diagonal in $G\backslash X$. Since in X the diagonal between two vertices is unique (by the non-positive curvature condition), different elements of $\mathscr{A}$ project to different elements of the groupoid.

The directed cubes in X can be labelled equivariantly by $\mathscr{A}$, so each cube-path in X spells out a word in $\mathscr{A}^*$. The path obtained from a cube-path by replacing each directed cube by the corresponding diagonal will be called the *derived path.* Let Gv and Gv' be vertices in $G\backslash X$ and p be a path from Gv to Gv'. From Proposition 3.3 we know that there is a unique normal cube-path in X whose derived path runs from v to gv' and so the corresponding word over $\mathscr{A}$ projects to the class of p. Define the language $\mathscr{L}$ to be the subset of $\mathscr{A}^*$ which is given by all words which label normal cube-paths. It follows from the above that we have a bijective map from $\mathscr{L}$ to morphisms in $\pi(G\backslash X, V(G\backslash X))$.

Recall that we have fixed an object $Gv \in \pi(G\backslash X, V(G\backslash X))$ and a representative $v \in V(X)$ of the corresponding orbit in X. We shall let $\Gamma(\mathscr{G})$ denote the Cayley graph of the groupoid $\mathscr{G}$ with respect to the set $\mathscr{A}$, and we shall think of this as being immersed in X via the map φ which sends a vertex of the Cayley graph given by the homotopy class $[p]$ to the endpoint of the unique lift of p which begins at v. This map defines a bijection between the vertex set of $\Gamma(\mathscr{G})$ and the vertex set of X. An edge in $\Gamma(\mathscr{G})$ is sent to the diagonal in X between the images of its endpoints. The images of two edges may intersect in the centre of a cube.

For two vertices $u, v \in X$ let $d'(u, v)$ be the length of the normal cube-path from u to v. It can be easily checked that d' is a metric on the set of vertices of X.

LEMMA 4.1. *φ is an isometry between $V(\Gamma(\mathscr{G}))$ with the word metric and $V(X)$ with the metric d'.*

Proof. Let $\varphi(u_1), \varphi(u_2) \in V(X)$ be such that $d'(\varphi(u_1), \varphi(u_2)) = n$. By definition there is a cube-path of length n joining them, and hence a path in $\Gamma(\mathscr{G})$ of length n between u_1 and u_2. Conversely if there is a path of length n in $\Gamma(\mathscr{G})$ joining u_1 and u_2 then we lift to a sequence of cubes $\{C_i'\}_0^n$ in X. This need not be a cube-path since the cube $C_{i+1} \cap C_i$ may have dimension greater than zero, however it can be replaced by a cube-path from $\varphi(u_1)$ to $\varphi(u_2)$ of length less than or equal n as follows: Let i be the smallest possible so that $\dim(C_{i+1} \cap C_i) > 0$. If $C_i = C_{i+1} \cap C_i$ then we omit the cube C_i and renumber. Otherwise, notice that $C_i \cap C_{i-1} \not\subseteq C_{i+1} \cap C_i$ and replace C_i by the face C_i which is orthogonal to $C_{i+1} \cap C_i$ and contains $C_i \cap C_{i-1}$. After at most n steps we will have a cube-path which joins $\varphi(u_1)$ and $\varphi(u_2)$ and has length at most n.

5. PROVING THAT THE GROUPOID IS BIAUTOMATIC

We claim that the language $\mathscr{L}$ defined above determines a biautomatic structure for $\mathscr{G}$. It has already been noted that this language maps onto the groupoid, it remains to establish that $\mathscr{L}$ is regular and has the fellow-traveller property.

PROPOSITION 5.1. *The language $\mathscr{L}$ over $\mathscr{A}$ determined by the set of all normal cube-paths is regular.*

Proof. The argument is basically that used by Cannon [5]. We shall construct a non-deterministic finite-state automaton, M, over $\mathscr{A}$ which has $\mathscr{L}$ as the set of accepted words. The set of states of M is $\mathscr{A}$; all states are initial states and all states are accept states. For $a \in \mathscr{A}$ choose a diagonal $\tilde{a} \in X$ labelled by a. There is a transition labelled b from a to b if and only if there is a diagonal $b \in X$ with label b which starts at the tail of $\tilde{a}$ and the cubes C and D in X, of which $\tilde{a}$ and $\tilde{b}$ are the diagonals satisfy $St(C) \cap D = \{\tau\}$. We remark that for each diagonal $\tilde{a}$ there are only finitely many possibilities for b, since X is locally finite.

Since the condition defining the transitions is the same as the condition defining normal cube-paths (Definition 3.1) the language $\mathscr{L}$ is precisely the language accepted by the finite-state automaton.

In order to establish that $\mathscr{L}$ provides a biautomatic structure, we will now show that it has the 1-fellow-traveller property.

PROPOSITION 5.2. *Suppose that* $\gamma = \{C_i\}_0^m$, $\delta = \{D_i\}_0^n$ *are normal cube-paths such that* $d'(\iota(\gamma), \iota(\delta)) \leqslant 1$ *and* $d'(\tau(\gamma), \tau(\delta)) \leqslant 1$, *then* $d'(\gamma(t), \delta(t)) \leqslant 1$ *for all* t.

Before we begin the proof notice that vertices u and v are distance one apart if and only if they are diagonally opposite one another in some cube of X, so it suffices to show that if $u_0, \dots, u_n$ $v_0, \dots, v_m$ are the vertices of γ and δ, respectively, then for each i, u_i and v_i lie on a common cube; furthermore it is obviously sufficient to show this for u_1 and v_1 since any subpath of a normal cube-path is itself normal and the general result will then follow by induction on the length of the normal cube-path.

Proof. Now suppose that $d(u_1, v_1) \geqslant 2$ so there are non-intersecting hyperplanes H and K both separating u_1 and v_1. Denote the components of the complement of the hyperplanes by $H^{\pm}$ and $K^{\pm}$ where $u_1 \in H^+$ and $v_1 \in K^+$ and $H^+ \cap K^+ = \emptyset$ so that H and K separate the vertices of X into three disjoint sets: those in H^+, those in K^+ and those in $H^- \cap K^-$.

Since $d(u_0, v_0) = 1$ at most one of the hyperplanes H, K separates u_0 from v_0. Suppose that neither separates them, so they both lie in one of H^+, K^+ or $H^- \cap K^-$. Now if $u_0 \in K^+$ then H and K both separate u_0 from u_1, which contradicts the fact that $d(u_0, u_1) = 1$, so $u_0 \in K^-$ and similarly $v_0 \in H^-$ and so they both lie in $H^- \cap K^-$. But then the normal path from u_0 to u_m crosses H, so H must separate u_0 from u_m, i.e., $u_m \in H^+$ and likewise $v_n \in K^+$. Hence $d(u_m, v_n) \geqslant 2$, which is a contradiction.

Now assume without loss of generality that H separates u_0 from v_0. If H separates u_0 and u_m then, by the remark following Proposition 3.3, H separates u_0 and u_1, so $u_0 \in H^-$. It follows that $v_0 \in H^+$ and since $H^+ \cap K^+ = \emptyset$, $v_0 \in K^-$. But then H and K both separate v_0 from v_1 contradicting the fact that $d(v_0, v_1) = 1$.

We are now able to prove our main theorem:

THEOREM 5.3. *Let X be a simply connected, non-positively curved cube complex, and G be a group acting effectively, cellularly, properly discontinuously and cocompactly on X. Then* $(\mathscr{A}, \mathscr{L})$ *induces a synchronously biautomatic structure on G.*

Proof. $\mathscr{L}$ surjects onto $\mathscr{G}$ by Proposition 3.3, is regular by Proposition 5.1 and has the 1-fellow-traveller property by Proposition 5.2, hence the groupoid $\mathscr{G}$ is biautomatic. It follows that its vertex groups are also biautomatic so G is biautomatic as required.

Without much further effort we can also deduce:

THEOREM 5.4. *The subgroups of G which stabilise the hyperplanes are rational.*

Proof. The quotient of a hyperplane H by the action of its stabiliser is compact. Denote by $St(H)$ the union of all (closed) cubes in X which meet H. Since $St(H)$ is a regular neighbourhood of H and the action is properly discontinuous, $Stab_G(H)$ has finite index in $Stab_G(St(H))$.

Now we will show that $St(H)$ is convex with respect to the set of normal cube-paths. Let C be a cube which has at least one vertex lying in $St(H)$, and let S be the set of hyperplanes which intersect C. If all the planes in S intersect H then by Proposition 2.12 there is a cube in $St(H)$ which has C as a subcube, and hence C is in $St(H)$. Now assume, for a contradiction, that we have a normal cube-path between two vertices of $St(H)$, which does not lie in $St(H)$. From the above comment there must be a cube in the path which crosses a plane which does not intersect H. Since the plane does not intersect H, the path must cross it twice which contradicts Lemma 3.2.

We have seen that $St(H)$ is convex with respect to the language of normal cube-paths. Define $\mathscr{K} \in \mathscr{G}$ to be the full subgroupoid spanned by $Stab_G(St(H)) \subseteq G\backslash X$. It follows that $\mathscr{K} \in \mathscr{G}$ is a $\mathscr{L}$-rational subgroupoid of $\mathscr{G}$. If $v_0 \in Obj(\mathscr{K})$ then $\mathscr{K}_{v_0}$ is an $\mathscr{L}$-rational subgroup of $\mathscr{G}_{v_0}$, that is $\mathscr{K}_{v_0}$ is an $\mathscr{L}$-rational subgroup of G. The result then follows from the fact that finite index subgroups of rational subgroups are rational [6].

6. EXAMPLES

If G is a finitely generated free group then its Cayley Graph with respect to a free generating set is a tree, so it is a simply connected one-dimensional non-positively curved cube complex. The regular language given by Theorem 5.3 in this case is the set of reduced words in the generators of G.

The free abelian group of rank n acts by translations on the integer lattice in $\mathbb{E}^n$. A fundamental domain for the action is given by $C = [0, 1]^n$, and the quotient is an n-torus. Neumann and Shapiro [11] have classified the automatic structures of a free abelian group of rank n, associating to each a rational triangulation of the $n - 1$ sphere. They show that rational subgroups correspond to great subspheres of the triangulation. The automatic structure $(\mathscr{A}, \mathscr{L})$ given by Theorem 5.3 corresponds to the barycentric subdivision of the $n - 1$ sphere ∂C. The hyperplane subgroups correspond to the great subspheres parallel to the faces of the Euclidean n-cube, C.

We illustrate the case of $\mathbb{Z}^3$ in Fig. 3. The finite state automaton accepting the language $\mathscr{L}$ can be obtained by adding a loop at each vertex of the triangulation, and in this case the alphabet $\mathscr{A}$ has 26 letters, and the structure is shortlex.

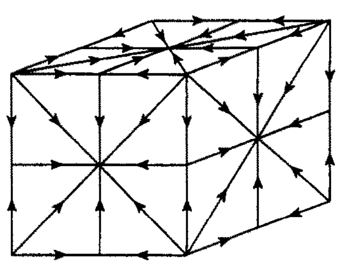

Fig. 3. Neumann-Shapiro triangulation of S^2 corresponding to the structure on $\mathbb{Z}^3$ given by Theorem 5.3.

Some explicit examples of non positively curved cubed 3-manifolds are described in [12]. In particular, a family of examples is provided by certain canonical surgeries defined by link diagrams (see [2]).

Cube complexes of dimension 2 have been called $A_1 \times A_1$ complexes by Gersten and Short [7]. Groups which act on such complexes are the T(4)-C(4)-P small cancellation groups and an automatic language is given which is precisely that resulting from our construction.

Right angled Coxeter groups act freely and cocompactly on CAT(0) cube complexes as do right angled Artin groups, so these too are biautomatic.

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