Note

A Smallest-Fibre-Size to Poset-Size Ratio Approaching $\frac{8}{15}$

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DEFINITIONS. A fibre of a poset is a subset which meets every maximal antichain. A splitting element of a poset is one which is comparable to every other element in the poset.

The question we address here is: If $P$ is a finite poset with no splitting element, and $F$ is a fibre of $P$ of minimal cardinality, what is the greatest possible value of $|F|/|P|$? Duffus, Kierstead, and Trotter [1] have shown that this value cannot exceed $\frac{2}{3}$. Duffus, Sands, Sauer, and Winkler [2] have found a poset in which this value is $\frac{9}{17}$. We will now explain how to stack copies of this poset so that a value of $(8n+1)/(15n+2)$ can be achieved for any $n$.

Let $P$ be the 17-element poset described in [2] in which each fibre has at least nine elements. Let $n$ be some positive integer. For each $i$ such that $1 < i < n$, let $Q_i$ be a poset isomorphic to $P$. Denote the element $j$ of $Q_i$ by the ordered pair $(i, j)$. It will be convenient to also define a poset $Q_0 = \{(0, 14), (0, 15)\}$, a two-element antichain. Stack the posets $Q_i$ by identifying maximals and minimals of consecutive posets as in Fig. 1. That is, if $1 < i < n$, then $(i, 4) = (i - 1, 14)$ and $(i, 3) = (i - 1, 15)$.

Let $Q$ denote the union of $Q_0, Q_1, \ldots, Q_n$ with the above partial order. Note that any maximal antichain in any $Q_i$ is also a maximal antichain in $Q$, since any element in $Q_i$ is either greater than some maximal of $Q_i$ or less than some minimal of $Q_i$. Notice also that within each $Q_i$, $\{1, 2\}$, $\{2, 3\}$, $\{3, 4\}$, $\ldots$, $\{16, 17\}$ are all maximal antichains. (It is convenient to

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refer to points using only their second coordinates, where this is not ambiguous.)

Let $F$ be a fibre of $Q$. We want to show that $|F| \geq 8n + 1$. Let $R_i = Q_i - Q_{i-1}$ for $1 \leq i \leq n$, so that $|R_i| = 15$. If $|R_i \cap F| \geq 8$ whenever $1 \leq i \leq n$ then $|F| \geq 8n + 1$ and we are done. But suppose there is a $k$ such that $|R_k \cap F| < 8$, where $1 \leq k \leq n$. Since any maximal antichain in $Q_k$ is a maximal antichain in $Q$, one of $\{1, 2\}$ must be in $F$. There must also be at least six of $\{5, 6, 7, \ldots, 17\}$ in $F$. In fact, there must be exactly six of $\{5, 6, 7, \ldots, 17\}$ in order to obtain $|R_k \cap F| < 8$ and the only way to do that is to have $\{6, 8, 10, 12, 14, 16\} \subseteq F$. Since we need one of $\{1, 9, 17\}$ in $F$,
the only way to obtain $|R_k \cap F| < 8$ is to have $R_k \cap F = \{1, 6, 8, 10, 12, 14, 16\}$ so that $|R_k \cap F| = 7$. But then we need both 3 and 4 in $F$ because of the maximal antichains $\{2, 3\}$ and $\{4, 5\}$. That is, in $Q_{k-1}$, we need both 14 and 15 in $F$.

So in the case where $k = 1$, we have that $R_{k-1} \cap F = \{(0, 14), (0, 15)\}$. Otherwise, we make the following observations: $(k - 1, 16)$ must also be in $F$, since $\{(k - 1, 16), (k, 5)\}$ is a maximal antichain and $(k, 5)$ is not in $F$. We also need $(k - 1, 13)$ in $F$, since $\{(k - 1, 13), (k, 2)\}$ is a maximal antichain and $(k, 2)$ is not in $F$. So the elements of $R_{k-1}$ in $F$ include 13, 14, 15, and 16. $F$ must also include at least four of 5, 6, ..., 12 and at least one of 1 and 2. Therefore, we can achieve $|R_k \cap F| < 8$ only if $|R_k \cap F| = 7$ and $|R_{k-1} \cap F| \geq 9$.

These observations establish that $|Q \cap F|$ must be at least $8n + 1$ while $|Q| = 15n + 2$. Also note that $\{(i, j) \in Q: j$ odd$\}$ is a fibre of $Q$ of size exactly $8n + 1$. Thus, as $n$ becomes large, the smallest-fibre-size to poset-size ratio of $Q$ asymptotically approaches $\frac{8}{15}$.

REFERENCES