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# Complete reducibility of integrable modules for the affine Lie (super)algebras

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**Abstract**

We prove complete reducibility for an integrable module for an affine Lie algebra where the canonical central element acts non-trivially. We further prove that integrable modules does not exists for most of the superaffine Lie algebras where the center acts non-trivially.


*Keywords:* Affine Lie algebras; Superaffine Lie algebras; Integrable modules

## Introduction

Let $\mathcal{G}$ be simple finite-dimensional Lie algebra. Let $\widehat{\mathcal{G}}$ be the corresponding affine Lie algebra and let $K$ be the canonical central element. A module $V$ of $\widehat{\mathcal{G}}$ is called integrable if the Chevalley generators act locally nilpotently on $V$. In [1] the irreducible integrable modules for $\widehat{\mathcal{G}}$ with finite-dimensional weight spaces has been classified. In particular any irreducible integrable module with finite-dimensional weight spaces where $K$ acts by positive integer is isomorphic to an highest weight module. In this work we prove that any integrable module with finite-dimensional weight spaces where $K$ acts by non-zero scalars is completely reducible (Theorem 1.10).

The integrable modules where $K$ acts trivially, need not be completely reducible. For example, consider the $\widehat{\mathcal{G}}$ (without the derivation) module $\mathcal{G} \otimes \mathbb{C}[t, t^{-1}]/(t-1)^2$ where $K$ acts by zero which is not completely reducible. (See [2] for the graded version.)

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In Section 2 we consider affine Lie superalgebras and prove that most often integrable modules with finite-dimensional weight spaces do not exist. We use stronger definition of the integrability than that of [8]. Let $\mathcal{G}$ be simple finite-dimensional Lie superalgebra. Let $\widehat{\mathcal{G}}$ be the corresponding affine Lie superalgebra. Assume that it has non-degenerate symmetric invariant bilinear form. Assume that the semisimple part of the even part of $\mathcal{G}$ is at least two components. Then integrable modules for $\widehat{\mathcal{G}}$ with finite-dimensional weight spaces where center acts by non-zero scalar does not exist (Theorem 2.6). Certainly integrable modules with $K$ acting zero exists. For example loop modules. Our techniques work only with the notion of stronger integrability. We do not know whether such a result hold with the weaker integrability of [8].

In Theorem 2.9, we prove that an integrable irreducible module for $\widehat{\mathcal{G}}$ with finite-dimensional weight spaces where center $K$ acts by positive integer is necessarily a highest weight module, assuming the semisimple part of the finite even part is only one component. In this case we note that (Remark 2.11) the module is completely reducible for the even part. That class includes the affine Lie superalgebras associated with basic Lie superalgebras of types $A(0,n)$, $B(0,n)$ and $C(n)$.

**1.1.** We will fix some notations. All our algebras are over complex numbers $\mathbb{C}$. Let $\mathring{\mathcal{G}}$ be simple finite-dimensional Lie algebra. Let $\mathring{h}$ be a Cartan subalgebra. Let $\mathring{Q}$ and $\mathring{\Lambda}$ be root and weight lattice of $\mathring{\mathcal{G}}$. Let $\mathring{\Lambda}^+$ be dominant integral weights of $\mathring{\mathcal{G}}$. Let $\alpha_1, \dots, \alpha_n$ be simple roots and let $\beta$ be highest root of $\mathring{\mathcal{G}}$; $\alpha_1^\vee, \dots, \alpha_n^\vee$ be the corresponding simple roots. We choose non-degenerate bilinear form on $\mathring{h}^*$ such that $(\beta, \beta) = 2$.

Let $\widehat{\mathcal{G}} = \mathcal{G} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d$ be the corresponding untwisted affine Lie algebra. Let $\widehat{h} = \mathring{h} \oplus \mathbb{C}K \oplus \mathbb{C}d$ be the Cartan subalgebra of $\widehat{\mathcal{G}}$. Let $Q$ and $\Lambda$ be the root and weight lattice of $\widehat{\mathcal{G}}$. Let $\delta$ be the null root. Let $\Lambda_0$ be an element of $\widehat{h}^*$ such that $\Lambda_0(\mathring{h}) = 0$, $\Lambda_0(K) = 1$ and $\Lambda_0(d) = 0$. An element $\lambda$ in $\mathring{h}^*$ can be treated as an element of $\widehat{h}^*$ by extending as $\lambda(K) = 0$ and $\lambda(d) = 0$. Let $\bar{\lambda}$ be the restriction to $\mathring{h}$. Given $\lambda \in \widehat{h}^*$, $\lambda$ can be uniquely written as

$$\lambda = \bar{\lambda} + \lambda(d)\delta + \lambda(K)\Lambda_0. \tag{1.2}$$

**1.3. Definition.** An element $\lambda$ in $\mathring{\Lambda}^+$ is called minimal if for every $\mu \in \mathring{\Lambda}^+$ such that $\mu \underset{0}{\leq} \lambda$ implies $\mu = \lambda$. Here $\mu \underset{0}{\leq} \lambda$ means $\lambda - \mu = \sum_{i=1}^n n_i \alpha_i,\ n_i \in \mathbb{N}$.

**1.4. Lemma** [5]. *Let $\lambda$ be minimal in $\mathring{\Lambda}^+$. Then $\lambda(\beta^\vee) = 0$ or 1.*

**Proof.** See Exercise 13 of Chapter III of [5]. □

**1.5.** Let $\lambda \in \Lambda^+$ and let $V(\lambda)$ be the irreducible integrable highest weight module for $\widehat{\mathcal{G}}$. Let $P(\lambda)$ be the set of weights of $V(\lambda)$. Define $\mu \leqslant \lambda$ if $\lambda - \mu = \sum_{i=0}^n n_i \alpha_i,\ n_i \in \mathbb{N}$ ($\alpha_0$ is the additional simple root of $\widehat{\mathcal{G}}$). Let $\overline{P}(\lambda) = \{\bar{\mu} \mid \mu \in P(\lambda)\}$. Clearly $\overline{P}(\lambda)$ determines a unique coset in $\mathring{\Lambda}/\mathring{Q}$. Let $\bar{\mu}_0$ be the minimal element in $\mathring{\Lambda}^+$ in the above coset. Let $s$ be a complex number such that $\lambda(d) - s$ is a non-negative integer.

**1.6. Lemma.** *Let $\mu_0 = \bar{\mu}_0 + s\delta + \lambda(K)\Lambda_0$. Then $\mu_0 \in P(\lambda)$.*

**Proof.** First note that by minimality of $\overline{\mu}_0$ we have $\overline{\mu}_0 \underset{0}{\leqslant} \bar{\lambda}$. Then clearly $\mu_0 \leqslant \lambda$.

**Claim.** $\mu_0 \in \Lambda^+$.

Consider $\alpha_i^\vee$ for $1 \leqslant i \leqslant n$. Then $\mu_0(\alpha_i^\vee) = \overline{\mu}_0(\alpha_i) \in \mathbb{N}$ as $\overline{\mu}_0 \in \mathring{\Lambda}^+$. Now $\alpha_0^\vee = K - \beta^\vee$ and

$$\mu_0\big(\alpha_0^\vee\big) = \overline{\mu}_0\big(K - \beta^\vee\big) = \lambda(K) - \overline{\mu}_0\big(\beta^\vee\big).$$

Now $\lambda(K)$ is positive integer and hence $\lambda(K) \geqslant 1$. We know from Lemma 1.4 that $\overline{\mu}_0(\beta^\vee) = 0$ or 1. That means

$$\mu_0\big(\alpha_{n+1}^\vee\big) \in \mathbb{N}.$$

This prove the claim. Thus $\mu_0$ is dominant integral and $\leqslant \lambda$. By Proposition 12.5(a) of [7] it follows that $\mu_0 \in P(\lambda)$. □

We need the following from [3].

**1.7. Lemma** (Lemma 2.6 of [3]). *Let $V$ be integrable module for $\widehat{\mathcal{G}}$ with finite-dimensional weight spaces. Let $P(V)$ be the set of weights of $V$. Let $\lambda \in P(V)$. Then*

(1) *There exists $\eta_0 \underset{0}{\geqslant} 0$, $\eta_0 \in \mathring{Q}$ such that $\lambda + \eta_0 + \eta \notin P(V)$ for all $0 \neq \eta \underset{0}{\geqslant} 0$, $\eta \in \mathring{\Lambda}$.*
(2) *There exists $\eta_0^1 \underset{0}{\leqslant} 0$, $\eta_0^1 \in \mathring{\Lambda}$ such that $\lambda + \eta_0^1 + \eta \notin P(V)$ for all $0 \neq \eta \underset{0}{\leqslant} 0$, $\eta \in \mathring{\Lambda}$.*

**Proof.** (1) follows from the proof of Lemma 2.6 of [3]. The proof of (2) is similar. □

**1.8. Proposition.** *Let $V$ be integrable $\widehat{\mathcal{G}}$-module with finite-dimensional weight spaces. Assume the canonical central element $K$ acts by positive integers. Let $\lambda \in P(V)$. Then there exists $\eta \geqslant 0$ such that $\lambda + \eta \in \Lambda^+$ and the irreducible integrable highest weight module $V(\lambda + \eta) \subseteq V$.*

**Proof.** By previous lemma there exists $\eta_0 \underset{0}{\geqslant} 0$ such that $\lambda + \eta_0 + \eta \notin P(V)$ for $0 \neq \eta \underset{0}{\geqslant} 0$. Now by arguments similar to the proof of Theorem 2.4(i) of [1] will produce an highest weight module with highest weight $\lambda + \eta_0 + \eta_1$ for some $\eta_1 \geqslant 0$. Note that $(\lambda + \eta_0 + \eta_1)(d) \geqslant \lambda(d)$. □

In the above proof we need our Lemma 1.7 as the proof of Lemma 2.6(ii) of [1] is incomplete. We now recall the following variation of a standard result from [7].

**1.9. Proposition.** *Let $V$ be integrable module for $\widehat{\mathcal{G}}$ with finite-dimensional weight spaces. Let $K$ act by positive integer. Suppose for every $v$ in $V$, there exist $N > 0$ such that $U(\mathcal{G})_{\alpha + n\delta} = 0$ for all $n > N$ and for $\alpha \in \mathring{\Delta} U \{0\}$. Then $V$ is completely reducible.*

We need to recall some standard notations from [7] and prove two lemmas.

The Cartan subalgebra $h$ carries a non-degenerate bilinear form $( \mid )$. Let $\nu : h \to h^*$ be an isomorphism such that $\nu(h)(h_1) = (h \mid h_1)$. Let $\langle , \rangle$ be the induced bilinear form on $h^*$. Recall the Casimir operator from Section 2.1, $\rho$ in $h^*$ from (2.5) from [7]. Also recall the notion of primitive weights from (9.3) of [7]. Note that in an integrable module the primitive weights are dominant integral.

**Lemma A.** *Let $V$ be as above. Suppose $\lambda, \mu$ are primitive weights such that $\lambda - \mu = \beta \in Q^+ - \{0\}$. Then $2\langle \lambda + \rho, \nu^{-1}(\beta)\rangle \neq (\beta, \beta)$.*

**Proof.** Follows from the proof of Theorem 10.7 of [7]. See 10.7.3 and the next equation in [7]. □

**Lemma B.** *Let $V$ be as above. Let $v$ be a weight vector of weight $\lambda$ such that $(\Omega_0 - aI_V)^k v = 0$ for some $k \in \mathbb{Z}_+$ and $a \in \mathbb{C}$. Presumably $v^1 \in U_{-\beta}(\widehat{\mathcal{G}})v,\ \beta \in Q$. Then*

$$\big(\Omega_0 - \big(a + 2\big\langle \lambda + \rho, \nu^{-1}(\beta)\big\rangle - (\beta, \beta)\big) I_V\big) v^1 = 0.$$

**Proof.** Follows from (2.6.1) and (3.4.1) of [7]. Also see (9.10.2) of [7]. Note that $V$ is restricted in the sense of [7]. □

**Proof of the Proposition.** Let $\widehat{\mathcal{G}} = n^- \oplus h \oplus n^+$ be the standard triangular decomposition. Let $V^0 = \{v \in V \mid n^+ v = 0\}$. Clearly $V^0$ is $h$-invariant and hence decomposes under $h$. Let $V^1 = U(\widehat{\mathcal{G}})V^0$. It is standard fact that in an integrable module, each highest weight generate an irreducible integrable module. Thus $V^1$ is completely reducible. We will now prove that $V = V^1$.

Clearly the Casimir operator $\Omega_0$ acts on $V$ and leaves each finite dimensional weight space invariant. Thus $\Omega_0$ is locally finite on $V$. Suppose $V \neq V^1$. Then there exists $v$ in $V_1 - V^1$ such that $n^+ v \subseteq V^1$ and $(\Omega_0 - aI_V)^k v = 0$ for some $k \in \mathbb{Z}_+$ and $a \in \mathbb{C}$. Since, clearly $\Omega_0 v \in V^1$, we have $a = 0$ and hence $\Omega_0^k v = 0$.

From the hypothesis it follows that $U(n^+)v$ is finite-dimensional. So it contains vector $u_\beta v$ such that $u_\beta \in U(\widehat{\mathcal{G}})_\beta$ and $n^+ u_\beta v = 0$, $\beta \in Q^+ - \{0\}$. Let $\mu = \lambda + \beta$ and note that $\lambda, \mu$ are primitive roots. Thus by Lemma A

$$2\big\langle \mu + \rho, \nu^{-}(\beta)\big\rangle \neq (\beta, \beta). \tag{$*$}$$

Now by Lemma B it follows that $2\langle \lambda + \rho, -\nu^{-}(\beta)\rangle = (\beta, \beta)$ as $\Omega_0(u_\beta v) = 0$. This is a contradiction to $(*)$. Thus $V = V^1$ and $V$ is completely reducible. □

**1.10. Theorem.** *Let $V$ be integrable module with finite-dimensional weight spaces for $\widehat{\mathcal{G}}$. Suppose all eigenvalues of $K$ are non-zero. Then $V$ is completely reducible as $\widehat{\mathcal{G}}$-module.*

**Proof.** First decompose $V$ with $K$ action. As $K$ commutes with $\widehat{\mathcal{G}}$, each eigenspace is $\widehat{\mathcal{G}}$-module. Thus we can assume that $K$ acts by single scalar. It is well known that the central element $K$ acts by integer (see, for example, [3]). Without loss of generality we can assume that $K$ acts by positive integer. We now decompose the module

$$V = \bigoplus_{\lambda \in \Lambda/Q} W_\lambda,$$

where $\mu_1, \mu_2$ weight occurs in $W_\lambda$ then $\mu_1 - \mu_2 \in Q$. Clearly each $W_\lambda$ is a $\widehat{\mathcal{G}}$-module. Thus we can assume that the weights $P(V)$ of $V$ lie in single coset of $\Lambda$.

**Claim.** *Let $\lambda \in P(V)$. Then there exists $\eta \geqslant 0$ such that $\lambda + \alpha \notin P(V)$ for all positive roots $\alpha$ such that $\alpha > \eta$.*

**Proof of the Claim.** Suppose there exists infinitely many positive roots $\alpha$ such that $\lambda + \alpha \in P(V)$. First by Proposition 1.8 there exists $\eta \geqslant 0$ such that $\lambda + \eta \in \Lambda^+$, $\lambda + \eta \in P(V)$, $(\lambda + \eta)(d) \geqslant \lambda(d)$ and the irreducible integrable highest module $V(\lambda + \eta) \subseteq V$.

Let $\overline{P}(V) = \{\bar{\lambda} \mid \lambda \in P(V)\}$. Clearly $\overline{P}(V)$ defines a unique coset in $\mathring{\Lambda}$. Let $\bar{\mu}_0$ be the minimal weight for this coset. Let $\mu_0 = \bar{\mu}_0 + \lambda(d)\delta + \lambda(K)w \leqslant \lambda + \eta$. By Lemma 1.6 we have $\mu_0 \in P(\lambda + \eta) \subseteq P(V)$.

First note that the number positive roots $\alpha_1$ such that $\alpha_1 \not> \eta$ is finite.

Now choose positive root $\alpha_1 > \eta$ such that $\lambda + \alpha_1 \in P(V)$. (This is due to our supposition.) Now by above arguments there exists $\eta_1 \geqslant 0$ such that $\lambda + \alpha_1 + \eta_1 \in P(V), \lambda + \alpha_1 + \eta_1 \in \Lambda^+$ and $V(\lambda + \alpha_1 + \eta_1) \subseteq V$. Further $\mu_0 \leqslant \lambda + \alpha_1 + \eta_1$ and $\mu_0 \in P(\lambda + \alpha_1 + \eta_1) \subseteq P(V)$. Note that $\lambda + \alpha_1 + \eta_1 > \lambda + \eta$ (note the strict inequality). Thus $V(\lambda + \alpha_1 + \eta_1) \neq V(\lambda + \eta)$. Both modules have common weight $\mu_0$. Thus we have proved that $\dim V_{\mu_0} \geqslant 2$. By repeating $n$ times the above argument we get $\dim V_{\mu_0} \geqslant n$. But $\dim V_{\mu_0}$ is finite and thus this process has to stop. This proves our claim. □

It follows from the claim and that the module $V$ satisfies the conditions of Proposition 1.9 and hence it is completely reducible. □

**1.11. Remark.** Theorem 1.10 imply that an integrable module with finite-dimensional weight spaces in which $K$ acts by positive integer belongs to the category $\mathcal{O}$.

**2.0.** Let $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ be simple finite-dimensional Lie superalgebra $\mathcal{G}_0$ (respectively, $\mathcal{G}_1$) being its even (respectively, odd) part. We assume that $\mathcal{G}_0$ is reductive. We further assume that $\mathcal{G}$ carries a non-degenerate invariant "symmetric" bilinear form. Such Lie superalgebras are called basic. We give the list of basis Lie superalgebras from Proposition 1.1 of [6] (see Table 1).

In this section we study the integrable representations of the untwisted affine Lie superalgebras of basic Lie superalgebras.

Let $\mathcal{G}$ be a basic Lie superalgebra. Then the restriction to the even part need not be positive definite. In fact we choose the form in such a way that the restriction to the first component of the even part is positive definite and the restriction to the second component is negative definite (see Section 6 of [8]). We normalize the form in such a way that $(\alpha, \alpha) = 2$ where $\alpha$ is the highest root of the first component of the even part of $\mathcal{G}_0$ and

Table 1

| $\mathcal{G}$ | $\mathcal{G}_0$ |
|---|---|
| $A(m,n)$ | $A_m + A_n + \mathbb{C}$, |
| $C(n)$ | $C_n + \mathbb{C}$ |
| $B(m,n)$ | $B_m + C_n$ |
| $D(m,n)$ | $D_m + C_n$, |
| $D(2,1:a)$ | $D_2 + A_1$ |
| $F(4)$ | $B_3 + A_1$ |
| $G(3)$ | $G_2 + A_1$ |

$(\beta,\beta) = -2$ where $\beta$ is the highest root of the second component. Let $h$ be the Cartan subalgebra of $\mathcal{G}$ which is contained in the even part.

**2.1.** Define affine superalgebra $\widehat{\mathcal{G}}$

$$\widehat{\mathcal{G}} = \mathcal{G} \otimes \mathbb{C}\big[t, t^{-1}\big] \oplus \mathbb{C}K \oplus \mathbb{C}d.$$

The Lie bracket is given by the following. Write $x(n) = x \otimes t^n$.

$$\big[x(n), y(m)\big] = [x, y](m+n) + n(x, y)\delta_{m+n,0}K,$$
$$\big[d, x(n)\big] = nx(n) \quad x, y \in \mathcal{G}, \quad m, n \in \mathbb{Z}, \quad K \text{ is central.}$$

Let

$$\widehat{h} = h \oplus \mathbb{C}K \oplus \mathbb{C}d.$$

**2.2. Definition.** A module $V$ of $\widehat{\mathcal{G}}$ is called integrable if

(1) $V = \bigoplus_{\lambda \in \widehat{h}^*} V_\lambda$, $V_\lambda = \{v \in V \mid hv = \lambda(h)v, \ \forall h \in \widehat{h}\}$.
(2) $V$ is integrable as a $\widehat{\mathcal{G}}_0$ module.
(3) For any $v \in V$, $U(\mathcal{G})v$ is finite-dimensional.

Here $U(\mathcal{G})$ is the universal enveloping algebra of $\mathcal{G}$.

**2.3. Remark.** In [8] integrable modules are studied with weaker condition. In [8] integrability means the module is integrable only with the affinization of one simple part of $\mathcal{G}_0$. Then they have classified irreducible highest weight module which are integrable in the above sense. See Theorems 6.1 and 6.2 of [8].

The purpose of this section is to classify irreducible integrable modules for $\widehat{\mathcal{G}}$ where center $K$ acts non-trivially.

Let $\mathcal{G}_{01}$ and $\mathcal{G}_{02}$ be the first and second simple component of $\mathcal{G}_0$ as above. (In case of $D(2,n)$ and $D(2,1;\alpha)$ the first component is not simple. Then we take one of the simple component.) Let $h_1$ and $h_2$ be the respective Cartan subalgebras. Let $\Delta_1$ and $\Delta_2$ be the corresponding root system. The following is very standard. Does not matter whether the form is positive definite or negative definite.

**2.4.** For any root $\alpha \in \mathring{\Delta}_i$, let $\alpha^\vee$ be the co-root. Let $x_\alpha$ be the corresponding root vector. Choose $x_{-\alpha}$ in the negative root space such that $(x_\alpha, x_{-\alpha}) = \frac{2}{(\alpha,\alpha)}$. Then $x_\alpha, x_{-\alpha}, \alpha^\vee$ is an $s\ell_2$ triple. Let $\gamma = \alpha + n\delta,\ \alpha \in \mathring{\Delta}_i$. Let $\gamma^\vee = \alpha^\vee + \frac{2n}{(\alpha,\alpha)}K$ be the co-root. Then it is easy to check that $x_\alpha(n), x_{-\alpha}(-n), \gamma^\vee$ is an $s\ell_2$ triple.

**2.5. Lemma.** *Let $V$ be an integrable $\widehat{\mathcal{G}}$-module. Let $\lambda$ be a weight of $V$. Let $\gamma = \alpha + n\delta, \alpha \in \Delta_i$, such that $\lambda(\gamma^\vee) > 0$. Then $\lambda - \gamma$ is a weight of $V$.*

**Proof.** Follows from standard $s\ell_2$ theory. □

**2.6. Theorem.** *Notation as above. Assume the semi simple part of $\mathcal{G}_0$ has at least two components. Let $V$ be integrable module with finite-dimensional weight spaces. Let the central element $K$ act by non-zero scalar. Then $V$ is necessarily trivial module.*

Without loss of generality we can assume that $K$ acts by positive integer. We can establish the following by the arguments similar to the proof of Theorem 1.10.

**2.7.** For any $\lambda \in P(V)$ there exists $N > 0$ such that

$$\lambda + \alpha + n\delta \notin P(V) \quad \text{for all } n \geqslant N \text{ and for all } \alpha \in \mathring{\Delta}_1 \cup \{0\}.$$

**2.8.** There is one problem. The module $V$ need not have finite-dimensional weight spaces for $\widehat{\mathcal{G}}_{01}$ as $h_1 \oplus \mathbb{C}K \oplus \mathbb{C}d$ could be much smaller than the Cartan $h = h_1 \oplus h_2 \oplus \mathbb{C}K \oplus \mathbb{C}d$. To overcome this problem, first observe that $\widehat{\mathcal{G}}_{01}$ commutes with $h_2$. Now decompose the module $V$ with respect to $h_2$ and $h_2$ weight space is a $\widehat{\mathcal{G}}_{01}$-module with finite-dimensional weight spaces. Now apply arguments similar to the proof of Theorem 1.10 to conclude 2.7.

**Claim.** *There exists a weight vector $v$ of weight $\lambda$ such that $x_\alpha(n)v = 0$ for $n < 0$ and for all $\alpha \in \Delta_2 \cup (0)$.*

First we complete the proof assuming the claim. From the claim we have $h(n)v = 0$ for $n < 0$ and $h \in h_2$. From the standard Heisenberg highest weight module theory it follows that $h(n)v \neq 0$ for all $n > 0$ and for all $h$ in $h_2$. Thus it follows that $\lambda + m\delta$ is a weight for all $m > 0$ contradicting 2.7. Thus the module $V$ has to be trivial.

**Proof of the Claim.** From Lemma 1.7 (2) it follows that there exists $\lambda \in P(V)$ such that $\lambda - \alpha \notin P(V)$ for all $\alpha \in \Delta_2^+$. Let $\Delta_2^{-ar}$ be the negative real roots of $\widehat{\mathcal{G}}_{02}$. Define $\Delta(\lambda) = \{\gamma \in \Delta_2^{-ar} \mid \lambda(\gamma^\vee) \leqslant 0\}$. Then $\Delta(\lambda)$ is finite set. Indeed, let $\gamma = \alpha - n\delta,\ \alpha \in \mathring{\Delta}_2,\ n > 0$, be an element of $\Delta_2^{-ar}$. Then $\lambda(\gamma^\vee) = \lambda(\alpha^\vee) - n\lambda(K)/(\alpha,\alpha) > 0$ for $n$ sufficiently large (recall $(\alpha,\alpha) < 0$ for all $\alpha \in \mathring{\Delta}_2$). Fix a positive integer $r$ such that $\alpha - s\delta \in \Delta_2^{-a} - \Delta(\lambda)$ for $s \geqslant r$.

**Subclaim 1.** *$\lambda - s\delta \notin P(V)$ for $s \geqslant r$.*

Suppose $\lambda - s\delta \in P(V)$ for some $s \geqslant r$ we have $\lambda((\alpha - s\delta)^\vee) > 0$ then by Lemma 2.5, $\lambda - s\delta - (\alpha - s\delta) = \lambda - \alpha \in P(V)$ which is a contradiction to the choice of $\lambda$.

Fix a positive integer $p$ such that $\lambda - s\delta \notin P(V)$ for $s > p$ and $\lambda - p\delta \in P(V)$.

**Subclaim 2.** *$\lambda - \alpha - (m + p)\delta \notin P(V)$ for $m > 0$ and $\alpha \in \mathring{\Delta}_2^+$.*

Suppose the claim is false. Consider $(\lambda - \alpha - (m + p)\delta)(\alpha^\vee) < 0$ since $\lambda(\alpha^\vee) < 0$ and $\alpha(\alpha^\vee) = 2$. Then by Lemma 2.5 we have $\lambda - \alpha - (m + p)\delta + \alpha = \lambda - (m + p)\delta \in P(V)$ contradiction the choice of $p$.

**Subclaim 3.** *$\lambda + \alpha - (m + p + 1)\delta \notin P(V)$ for $m > r$ and $\alpha \in \mathring{\Delta}_2^+$.*

Suppose the claim is false. Consider $(\lambda + \alpha - (m + p + 1)\delta)(\alpha - m\delta)^\vee > 0$ as $\alpha - m\delta \notin \Delta(\lambda)$. Thus by Lemma 2.5 we have

$$\lambda + \alpha - (m + 1 + p)\delta - \alpha + m\delta = \lambda - (1 + p)\delta \in P(V)$$

contradicts the choice of $p$.

Thus we have proved

$$\widehat{\mathcal{G}}_{02,-r\delta} V_{\lambda - p\delta} = 0, \quad r > 0, \quad \text{and} \quad \widehat{\mathcal{G}}_{02,\alpha - s\delta} V_{\lambda - p\delta} = 0$$

for all but finitely many negative roots. Since $V$ is integrable $\overline{W} = U(\widehat{\mathcal{G}}_{02}^-)V_{\lambda - p\delta}$ is finite-dimensional. Let $\mu$ be the lowest weight of $\overline{W}$. This weight satisfies all the requirements of the claim. □

**2.9. Theorem.** *Let $\widehat{\mathcal{G}}$ be the affine super algebra defined earlier. Assume that the semisimple part of the finite even part has only one component. Further assume that the non-degenerate form restricted to this simple Lie algebra $\mathcal{G}_0$ is positive definite. Let $V$ be irreducible integrable module with finite-dimensional weight spaces. Assume the central element $K$ acts as positive integer. Then $V$ is an highest weight module.*

**Proof.** From the proof of Theorem 2.6 we have 2.7. Let $\beta_1, \ldots, \beta_k$ be odd roots of $\mathcal{G}$. Let $v$ be a weight vector of $V$ of weight $\lambda$.

**Claim.** *The following vectors span is a finite-dimensional space $W$*

$$\left\{x_{\beta_{i_1}}(m_1) \cdots x_{\beta_{i_k}}(m_k)v,\ i_j \leqslant i_k,\ m \geqslant 0\right\},$$

*where $x_{\beta_i}$ is a root vector for the odd root space $\mathcal{G}_{\pm\beta_i}$. In the above we take negative roots first and positive roots next. The indices are decreasing order.*

It is sufficient to prove that the vector space $T$ spanned by the following vector is finite-dimensional,

$$\{x_\beta(m_1)\cdots x_\beta(m_k)w,\ m_i \geqslant 0\},$$

where $w$ is any weight vector of $V$. This is because there are only finitely many odd roots in $\mathcal{G}$.

First note that if $k\beta$ is a root for $k > 0$ then $k = 1$ or 2. Consider $x_\beta(m)w = [h(m), x_\beta]w = h(m)x_\beta(w) \pm x_\beta h(m)w$. By 2.7 both vectors are zero for large $m$. Let $m_0$ be such that $x_{k\beta}(m)w = 0$ for $m > m_0$ and $k = 1, 2$. Then it is easy to see that $T$ is spanned by

$$\{x_\beta(m_1)\cdots x_\beta(m_k);\ 0 \leqslant m_i \leqslant m_0,\ m_i \neq m_j,\ i \neq j\}$$

which is clearly finite-dimensional.

Let $H$ be the center of the reductive Lie algebra $\mathcal{G}_0$. Consider

$$S = U(\widehat{\mathcal{G}}_0^-)U(h)U(\widehat{\mathcal{G}}_0^+)U\left(\bigoplus_{n>0} H \otimes t^n\right)W. \tag{2.10}$$

Then

$$V = U\left(\bigoplus_{n<0} H \otimes t^n\right)U(\widehat{\mathcal{G}}_1^-)S$$

by PBW basis theorem.

By 2.7 we conclude that

$$U\left(\bigoplus_{n>0} H \otimes t^n\right)W = W_1$$

is finite-dimensional. Clearly $S$ is $\widehat{\mathcal{G}}_0$-module and by Theorem 1.10 $S$ is completely reducible. In fact it is direct sum of highest weight modules. Since $W_1$ is finite dimensional it intersects only finitely many of them. Say $V(\lambda_1)\cdots V(\lambda_k)$. Thus $S = \bigoplus V(\lambda_i)$ a finite sum. Thus $S$ has a maximal weight. (Here the ordering is the following $\mu_1 \leqslant \mu_2$ means $\mu_2 - \mu_1 = \sum n_i\alpha_i,\ n_i \in \mathbb{N},\ \alpha_i$'s are small roots of $\widehat{\mathcal{G}}$. $\widehat{\mathcal{G}}$ is a generalized Kac–Moody Lie superalgebra and it does admit simple roots. See [8].) The maximal weight is in fact maximal for $V$ as the rest of the space brings the weights down.

The maximal weight is in fact highest weight. As $V$ is irreducible, it is irreducible highest weight module. □

**2.11. Remark.** In the process we also established that an irreducible integrable highest weight module for $\widehat{\mathcal{G}}$ is completely reducible for $\widehat{\mathcal{G}_0 \oplus H}$.

**Proof.** Let $V$ be irreducible highest weight module for $\widehat{\mathcal{G}}$. Let

$$\Omega(V) = \left\{v \in V \mid h(k)v = 0 \text{ for all } h \in H,\ k > 0\right\}.$$

Let $M(k)$ be the irreducible highest weight module for $H \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$ where $K$ acts by $k$. Then by Theorem 1.7.3 of [4] we have $V = \Omega(V) \otimes M(k)$. Now $\Omega(V)$ is an integrable module and hence by Theorem 1.10 decomposes into irreducible modules for $\widehat{\mathcal{G}}_0$. Thus the Remark follows. □

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