Computation of the Maximal Degree of the Inverse of a Cubic Automorphism of the Affine Plane with Jacobian 1 via Gröbner Bases

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In this paper we propose to compute the maximal degree of the inverse of a cubic automorphism of the affine plane with Jacobian 1 via Gröbner Bases. This degree is equal to 9 and we give coefficients of the inverse.

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1. Introduction

If \( k \) is any commutative ring, \( k[X,Y] \) will denote the algebra of polynomials with coefficients in \( k \) in the indeterminates \( X, Y \) and \( k^2 \) the affine plane over \( k \). A \( k \)-endomorphism \( f \) of \( k^2 \) will be identified with its coordinate functions \( f = (f_1, f_2) \), where \( f_i \) (\( i = 1, 2 \)) belongs to \( k[X,Y] \). We define the Jacobian of \( f \) by \( \text{Jac}(f) = \frac{\partial f_1}{\partial X} \frac{\partial f_2}{\partial Y} - \frac{\partial f_1}{\partial Y} \frac{\partial f_2}{\partial X} \) and the degree of \( f \) by \( \deg(f) = \max_{1 \leq i \leq 2} \deg(f_i) \).

Let \( d \) be a non-negative integer and \( f \) an endomorphism of \( k^2 \) whose degree is less than or equal to \( d \). The Jacobian Conjecture in degree \( d \) (\( CJ(d) \)) states that \( f \) is invertible if and only if its Jacobian \( \text{Jac}(f) \) is a non-zero constant.

Let \( C_d \) be the smallest integer \( C \) such that if \( k \) is a \( Q \)-algebra and \( f \) a \( k \)-endomorphism of \( k^2 \) satisfying \( \text{Jac}(f) = 1 \) and \( \deg(f) \leq d \), then we have \( \deg(f^{-1}) \leq C \).

Bass has proven the following result in Bass (1983):

**Theorem 1.1.** The three following assertions are equivalent:

(i) \( CJ(d) \) is true,

(ii) if \( k \) is any \( Q \)-algebra and \( f \) any \( k \)-endomorphism of \( k^2 \) whose degree is less than or equal to \( d \), then \( f \) is invertible if and only if \( \text{Jac}(f) \) is an invertible element of \( k[X,Y] \),

(iii) \( C_d < \infty \).

If \( k \) is a reduced \( Q \)-algebra and \( f \) a \( k \)-automorphism of \( k^2 \) satisfying \( \text{Jac}(f) = 1 \) and \( \deg(f) \leq d \), it follows from a formula of Gabber (see Bass et al. (1982) and Cheng et al. (1994)) that \( \deg(f^{-1}) = \deg(f) \). What happens if \( k \) is not reduced? Is it true that \( C_d = d \) (see Question 2.19 of the paper by van den Essen (1991))?
A negative answer to this question is given in Furter (to appear) where it is proven that $C_d \geq d + 1$ as soon as $d \geq 3$. Also, Moh has proven that $CJ(d)$ is true when $d \leq 100$ (see Moh (1983)). It then follows from Theorem 1.1 that $C_d$ is finite for $d \leq 100$.

We could easily check that $C_1 = 1$. Theorem 2 of Furter (to appear) shows us that $C_2 = 2$. The purpose of this paper is to establish the following result:

**Theorem 1.2.** $C_3 = 9$.

As far as we know, there is no explicit upper bound for $C_d$ when $d \geq 4$ and there is not even a conjectured upper bound. An investigation of $C_4$ seems rather important to us in order to acquire some insight into the behaviour of $C_d$ in general.

2. Computation of $C_3$

Let $k$ be the algebra of polynomials with coefficients in $\mathbb{Q}$ in the indeterminates $a_1, a_2, a_3, b_1, b_2, b_3, b_4, c_1, c_2, c_3, d_1, d_2, d_3, d_4$ and let $f = (f_1, f_2)$ be the $k$-endomorphism of $\mathbb{A}^2_k$ whose coordinate functions are

$$
\begin{cases}
  f_1 = X + a_3 X^2 + a_2 XY + a_1 Y^2 + b_4 X^3 + b_3 X^2 Y + b_2 XY^2 + b_1 Y^3, \\
  f_2 = Y + c_3 X^2 + c_2 XY + c_1 Y^2 + d_4 X^3 + d_3 X^2 Y + d_2 XY^2 + d_1 Y^3.
\end{cases}
$$

Let $g = (g_1, g_2)$ be the formal inverse of $f$. The formal series $g_1$ and $g_2$ have expressions of the form

$$
\begin{cases}
  g_1 = X + \sum_{(i,j) \in \mathbb{N}^2, i+j \geq 2} x_{i,j} X^i Y^j, \\
  g_2 = Y + \sum_{(i,j) \in \mathbb{N}^2, i+j \geq 2} y_{i,j} X^i Y^j,
\end{cases}
$$

where $x_{i,j}, y_{i,j}$ belong to $k$.

The Jacobian of $f$ is a polynomial with coefficients in $k$ in the indeterminates $X, Y$. Its constant term is equal to 1 and we could check that its other non-trivial coefficients are equal to

$$
\begin{align*}
  &-3b_3 d_4 + 3b_4 d_3, \\
  &-6b_2 d_4 + 6b_3 d_2, \\
  &-9b_1 d_4 - 3b_2 d_3 + 3b_3 d_2 + 9b_4 d_1, \\
  &-6b_1 d_3 + 6b_2 d_1, \\
  &-3b_1 d_2 + 3b_2 d_1, \\
  &-3a_2 d_4 + 2a_3 d_3 - 2b_3 c_3 + 3b_4 c_2, \\
  &-6a_1 d_4 - a_2 d_3 + 4a_3 d_2 - 4b_2 c_3 + b_3 c_2 + 6b_4 c_1, \\
  &-4a_1 d_3 + a_2 d_2 + 6a_3 d_1 - 6b_1 c_3 - b_2 c_2 + 4b_3 c_1, \\
  &-2a_1 d_2 + 3a_2 d_1 - 3b_1 c_2 + 2b_2 c_1, \\
  &d_3 - 2a_2 c_3 + 2a_3 c_2 + 3b_4, \\
  &2d_2 - 4a_1 c_3 + 4a_3 c_1 + 2b_3, \\
  &3d_1 - 2a_1 c_2 + 2a_2 c_1 + 2b_2, \\
  &c_2 + 2a_3, \\
  &2c_1 + a_2.
\end{align*}
$$

Let $I$ be the ideal of $k$ generated by the 14 polynomials given above.

Let us set $k = k/I$. By reducing all the coefficients of $f$ modulo $I$, we obtain a
On the Jacobian Conjecture

$k$-endomorphism of $\mathbb{A}^2_k$ which we will denote by $\bar{f}$. Clearly, $\bar{f}$ is the generic cubic endomorphism with Jacobian 1 of the affine plane with the following meaning. Let $A$ be any $\mathbb{Q}$-algebra and $\alpha$ be any cubic $A$-endomorphism of $A^2$ with Jacobian 1. Up to an affine change of coordinates, we can always suppose that $\alpha(0) = 0$ and $\alpha'(0) = \text{Id}$. Therefore, there exists a canonical algebra-homomorphism $\phi : k \to A$ such that the $A$-endomorphism of $A^2$ obtained by replacing the coefficients of $\bar{f}$ by their image under $\phi$ will be equal to $\alpha$. As CJ(3) is true, the endomorphism $\bar{f}$ by their image under $\phi$ will be equal to $\alpha$. As CJ(3) is true, the endomorphism $\bar{f}$ is an automorphism and we clearly have $C_3 = \deg(\bar{f})^{-1}$. Hence, the integer $C_3$ is the smallest integer $C$ such that $x_{i,j}, y_{i,j}$ belongs to $I$ as soon as $i + j > C$.

Using a computer, we found that the smallest integer $N$ such that $x_{i,j}, y_{i,j}$ belongs to $I$ as soon as $i + j = N$, is equal to 10. This encouraged us to believe that $C_3 = 9$ (and this already proved that $C_3 \geq 9$). Let $h$ denote the $k$-endomorphism obtained from $\bar{f}$ by truncating its terms of degree bigger than or equal to 10. Then, to show that $C_3 = 9$, we only had to check that all coefficients of the endomorphism $\bar{f} \circ h - \text{Id}$ of $\mathbb{A}^2_k$ (whose degree is $9^3 = 729$) belong to $I$. Indeed, denoting by $\bar{h} = (\bar{h}_1, \bar{h}_2)$ the $k$-endomorphism of $\mathbb{A}^2_k$ obtained by reducing the coefficients of $h$ modulo $I$, the latter fact is equivalent to saying that the endomorphism $\bar{f} \circ \bar{h} - \text{Id}$ of $\mathbb{A}^2_k$ is identically zero, which is well known to ensure that $\bar{h} = (\bar{f})^{-1}$.

All computations were done using the computer algebra system AXIOM (see Jenks and Sutor (1983)).

3. Inversion Formula

Let us endow $k = \mathbb{Q}[a_1, \ldots, d_4]$ with the total degree-inverse lexicographical order (see Davenport et al. (1993)) for the following order of the indeterminates:

$$a_1 < a_2 < a_3 < c_1 < c_2 < c_3 < b_1 < b_2 < b_3 < b_4 < d_1 < d_2 < d_3 < d_4.$$

Considering the automorphism $(Y, X) \circ \bar{f} \circ (Y, X)$, one could easily show that the coefficient of $X^i Y^j$ in $\bar{h}_2$ is obtained from the coefficient of $X^i Y^j$ in $\bar{h}_1$ by replacing $a_1, a_2, a_3, c_1, c_2, c_3, b_1, b_2, b_3, b_4$ by $c_3, c_2, c_1, a_3, a_2, a_1, d_4, d_3, d_2, d_1, b_4, b_3, b_2, b_4$.

<table>
<thead>
<tr>
<th>Coefficients of degree 2</th>
<th></th>
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</thead>
<tbody>
<tr>
<td>coefficient of $X^2$</td>
<td>$\frac{1}{2}c_2$</td>
</tr>
<tr>
<td>coefficient of $XY$</td>
<td>$2c_1$</td>
</tr>
<tr>
<td>coefficient of $Y^2$</td>
<td>$-a_1$</td>
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<table>
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<tr>
<th>Coefficients of degree 3</th>
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<tbody>
<tr>
<td>coefficient of $X^3$</td>
<td>$\frac{1}{2}b_4 + \frac{1}{2}d_3$</td>
</tr>
<tr>
<td>coefficient of $X^2 Y$</td>
<td>$d_2$</td>
</tr>
<tr>
<td>coefficient of $XY^2$</td>
<td>$-\frac{1}{2}b_2 + \frac{3}{2}d_1$</td>
</tr>
<tr>
<td>coefficient of $Y^3$</td>
<td>$-b_1$</td>
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### Coefficients of degree 4

<table>
<thead>
<tr>
<th>Term</th>
<th>Coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X^4$</td>
<td>$\frac{1}{2}c_2b_4 - \frac{1}{2}c_3d_2 + \frac{3}{8}c_2d_3 - \frac{1}{8}c_1d_4$</td>
</tr>
<tr>
<td>$X^3Y$</td>
<td>$c_1b_4 - 2c_3d_1 + c_1d_3$</td>
</tr>
<tr>
<td>$X^2Y^2$</td>
<td>$-\frac{7}{8}(a_1b_4 + c_2d_1 + a_1d_3)$</td>
</tr>
<tr>
<td>XY$^3$</td>
<td>$\frac{1}{2}c_1b_2 - 3c_3d_1 - \frac{7}{8}a_1d_2$</td>
</tr>
<tr>
<td>$Y^4$</td>
<td>$c_1b_1 + \frac{3}{8}a_1b_2 - \frac{1}{8}a_1d_1$</td>
</tr>
</tbody>
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### Coefficients of degree 5

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<tbody>
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<td>$X^5$</td>
<td>$\frac{3}{4}b_1^2 + \frac{1}{4}d_1^2 - \frac{1}{4}b_2d_2 - \frac{1}{4}d_2d_4$</td>
</tr>
<tr>
<td>$X^4Y$</td>
<td>$\frac{1}{2}b_2b_4 + \frac{1}{2}b_3d_3 + \frac{3}{2}d_2d_3 + 2b_2d_4 - \frac{1}{8}d_1d_4$</td>
</tr>
<tr>
<td>$X^3Y^2$</td>
<td>$\frac{1}{2}b_2b_4 - \frac{3}{8}b_3d_2 + \frac{1}{2}d_2d_3 + 2b_2d_4 + \frac{3}{8}d_1d_3 + 6b_1d_4$</td>
</tr>
<tr>
<td>$X^2Y^3$</td>
<td>$-\frac{1}{2}b_1b_4 + 2d_2d_1 + \frac{3}{2}b_1d_3$</td>
</tr>
<tr>
<td>XY$^4$</td>
<td>$-\frac{1}{2}b_1b_4 + \frac{1}{2}d_1^2 - \frac{1}{2}b_1d_2$</td>
</tr>
<tr>
<td>$Y^5$</td>
<td>$-\frac{1}{2}b_1b_2 - \frac{3}{2}b_1d_1$</td>
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### Coefficients of degree 6

<table>
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<tr>
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<th>Coefficient</th>
</tr>
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<tbody>
<tr>
<td>$X^6$</td>
<td>$\frac{1}{4}c_2b_1^2 - \frac{1}{4}c_3d_2d_4 + \frac{1}{2}c_2d_3^2 + \frac{1}{2}c_1b_4d_4 + \frac{7}{8}c_2d_3d_4 - \frac{5}{8}c_2d_2d_4$</td>
</tr>
<tr>
<td>$X^5Y$</td>
<td>$\frac{3}{8}c_1b_4^2 - 5c_3d_1d_3 + \frac{1}{8}c_2d_2d_3 - 2c_1d_2^2 + 12c_1b_3d_4 + 33a_1b_4d_4$</td>
</tr>
<tr>
<td>$X^4Y^2$</td>
<td>$-\frac{19}{8}a_1b_3^2 - \frac{5}{2}c_3d_3d_3 - \frac{5}{8}c_2d_3d_3 - 2c_1d_3^2 - 16c_1b_3d_4 + 33a_1b_4d_4$</td>
</tr>
<tr>
<td>$X^3Y^3$</td>
<td>$-\frac{16}{27}c_1b_3d_4 + \frac{26}{27}c_1b_4d_4 + \frac{26}{27}c_1d_1d_4 - \frac{15}{2}c_1d_2d_4 + 33a_1b_4d_4$</td>
</tr>
<tr>
<td>$X^2Y^4$</td>
<td>$-\frac{9}{2}a_1b_3b_4 - \frac{5}{2}c_1b_2d_2 - \frac{5}{2}a_1b_3d_3 + \frac{13}{2}c_1d_1d_3$</td>
</tr>
<tr>
<td>$XY^5$</td>
<td>$-\frac{16}{27}c_1b_3d_4 - \frac{26}{27}c_1b_4d_4 - \frac{26}{27}c_1d_1d_4 + \frac{15}{2}c_1d_2d_4 + 33a_1b_4d_4$</td>
</tr>
<tr>
<td>$Y^6$</td>
<td>$\frac{1}{4}c_1b_1b_2 + \frac{1}{4}a_1b_1d_3 + c_1b_1d_1 - \frac{1}{4}a_1d_1^2 + \frac{3}{8}a_1b_1d_2$</td>
</tr>
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### Coefficients of degree 7

<table>
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<tr>
<th>Term</th>
<th>Coefficient</th>
</tr>
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<tbody>
<tr>
<td>$X^7$</td>
<td>$\frac{5}{12}d_2^3 + \frac{3}{2}b_3b_4d_4 - \frac{5}{12}b_2d_3d_4 - \frac{1}{2}d_2d_3d_4 + 2b_2d_2^2 + \frac{7}{8}d_1d_4^2$</td>
</tr>
<tr>
<td>$X^6Y$</td>
<td>$\frac{7}{24}d_2^2d_3^2 + \frac{11}{3}b_2d_4d_4 - 4d_2d_4^2 - \frac{1}{2}b_2d_3d_4 + 12d_1d_3d_4 + 18b_1d_4^2$</td>
</tr>
<tr>
<td>$X^5Y^2$</td>
<td>$\frac{15}{12}d_1d_3^2 + \frac{11}{24}b_2b_3d_4 - \frac{7}{8}b_4d_4^3 + \frac{53}{6}b_2d_2d_4$</td>
</tr>
<tr>
<td>$X^4Y^3$</td>
<td>$\frac{7}{24}d_1d_3d_4 - \frac{11}{3}b_1d_3d_4$</td>
</tr>
<tr>
<td>$X^3Y^4$</td>
<td>$\frac{5}{36}d_1d_3d_4 - \frac{35}{16}b_1b_4^2 + \frac{35}{8}b_1b_3d_4 + \frac{395}{36}b_1b_2d_4 + \frac{1445}{12}b_1d_2d_4$</td>
</tr>
</tbody>
</table>

(cont.)
Coefﬁcients of degree 7 (Continued)

\begin{align*}
\text{coefficient of } X^3 Y^4 & \quad \frac{4}{9} b_1 b_2 d_3 + \frac{105}{8} c_2 d_1^2 d_3 + 5 b_1 d_2 d_3 + \frac{25}{8} b_2 b_3 d_1 + \frac{135}{8} b_1 d_1 d_3 \\
\text{coefficient of } X^2 Y^5 & \quad -\frac{1}{2} b_2^2 d_2 + \frac{53}{8} d_2^2 d_1 - \frac{4}{9} b_1 d_2^2 + 2 b_1 b_2 d_3 + \frac{23}{2} b_1 d_1 d_3 + \frac{22}{7} b_2^2 d_3 \\
\text{coefficient of } X Y^6 & \quad -\frac{5}{3} b_1 b_2 b_3 + \frac{3}{8} b_1^2 b_4 + \frac{21}{8} d_1^3 - \frac{3}{8} b_2 b_2 d_2 - \frac{3}{2} b_1^2 d_3 \\
\text{coefficient of } Y^7 & \quad -\frac{1}{2} b_1 b_2^2 - \frac{9}{8} b_1 d_1^2 - \frac{2}{7} b_2^2 d_2 \\
\end{align*}

\begin{align*}
\text{Coefficients of degree 8}
\end{align*}

\begin{align*}
\text{coefficient of } X^8 & \quad -\frac{1}{144} c_2 d_3^3 - \frac{23}{192} c_2 d_2 d_3 d_4 + \frac{29}{48} c_1 d_2^2 d_4 - \frac{11}{16} c_1 b_3 d_2^2 - \frac{33}{16} a_1 b_1 d_4 \\
& \quad + \frac{207}{72} c_2 d_1 d_1^2 - \frac{31}{8} c_2 d_3 d_1^2 - \frac{11}{8} a_1 d_2 d_4 \\
\text{coefficient of } X^7 Y & \quad -\frac{5}{9} c_1 d_1^3 - \frac{11}{6} c_1 b_3 d_3 d_4 - \frac{23}{12} c_1 d_2 d_3 d_4 - 3 a_1 d_3^2 d_4 + \frac{11}{12} c_1 b_1 d_4 \\
& \quad + \frac{91}{3} c_1 d_1^2 d_2^2 + 9 a_1 d_2 d_4 \\
\text{coefficient of } X^6 Y^2 & \quad \frac{7}{18} a_1 c_1 d_3^3 + \frac{21}{4} c_1 b_3 d_3 d_4 + \frac{201}{72} a_1 b_3 d_3 d_4 + \frac{33}{24} a_1 d_2 d_3 d_4 \\
& \quad - \frac{139}{9} c_1 b_1 d_4^2 - \frac{203}{9} a_1 b_2 d_1^2 - \frac{189}{9} a_1 d_1 d_2^2 \\
\text{coefficient of } X^5 Y^3 & \quad \frac{7}{18} a_1 d_1 d_2 d_4^2 + 21 c_1 b_3 d_2 d_4 + \frac{21}{3} a_1 d_2^2 d_4 - 63 c_1 b_1 d_4 \\
& \quad + \frac{203}{12} a_1 b_2 d_3 d_4 - 35 a_1 d_1 d_3 d_4 - \frac{609}{12} a_1 b_1 d_4 \\
\text{coefficient of } X^4 Y^4 & \quad \frac{35}{18} a_1 d_1 d_2 d_3^2 - \frac{145}{4} c_1 b_1 d_4 d_4 - \frac{2975}{144} a_1 b_3 d_4 d_4 - \frac{805}{16} a_1 b_1 d_4 \\
& \quad - \frac{445}{4} c_1 d_1^2 d_4 - \frac{315}{2} c_1 b_1 d_3 d_4 - \frac{2905}{14} a_1 b_1 d_3 d_4 \\
& \quad - \frac{245}{3} a_1 b_1 d_2 d_4 - \frac{477}{24} a_1 b_1 d_3 d_4 \\
\text{coefficient of } X^3 Y^5 & \quad \frac{21}{2} c_1 b_1 d_3 d_3 + \frac{217}{36} a_1 d_1 d_2 d_3 + \frac{581}{12} a_1 b_1 d_3^2 - \frac{7}{8} a_1 b_2^2 d_4 + \frac{7}{4} a_1 b_1 b_3 d_4 \\
& \quad - \frac{189}{2} c_1 b_1 d_1 d_4 - \frac{217}{8} a_1 b_2 d_4 - \frac{581}{6} a_1 b_1 d_4 \\
\text{coefficient of } X^2 Y^6 & \quad -\frac{7}{9} c_1 b_1 d_2 d_3 - \frac{7}{3} a_1 b_1 b_2 d_2 - \frac{7}{4} a_1 b_1 d_2^2 + 9 c_1 b_1^2 d_3 \\
& \quad - \frac{33}{12} a_1 b_1 b_3 d_3 + \frac{11}{4} a_1 b_1 d_3 d_4 + \frac{81}{4} a_1 b_1 d_4^2 \\
\text{coefficient of } X Y^7 & \quad -3 c_3 b_1 b_2 d_2 - \frac{7}{9} a_1 b_2 d_2 - \frac{7}{4} a_1 b_1 d_2^2 + \frac{7}{2} a_1 b_1 b_2 d_3 + 9 c_1 b_1^2 d_3 \\
& \quad - \frac{33}{12} a_1 b_1 b_3 d_3 + \frac{11}{4} a_1 b_1 d_3 d_4 + \frac{81}{4} a_1 b_1 d_4^2 \\
\text{coefficient of } Y^8 & \quad \frac{9}{8} c_1 b_1 b_3 d_3 + \frac{5}{32} a_1 b_1 b_2 b_3 + \frac{51}{32} a_1 b_1 d_1^2 + \frac{27}{8} c_1 b_1 d_3^2 - \frac{7}{4} a_1 d_4^2 \\
& \quad + \frac{9}{8} c_1 b_1^2 d_2 + \frac{7}{16} a_1 b_1 b_2 d_2 + \frac{7}{8} a_1 b_1 d_1 d_2 + \frac{11}{16} a_1 b_1 d_3 d_3 \\
\end{align*}

\begin{align*}
\text{Coefficients of degree 9}
\end{align*}

\begin{align*}
\text{coefficient of } X^9 & \quad -\frac{131}{144} d_2^3 d_4^2 - \frac{181}{288} b_2 d_3 d_2 d_4^2 + \frac{341}{144} d_1 d_3 d_4^2 + \frac{131}{32} b_1 d_4^3 \\
\text{coefficient of } X^8 Y & \quad \frac{131}{32} b_3 d_3 d_2^3 - \frac{1177}{288} b_1 b_4 d_4^3 + \frac{1151}{12} b_2 d_2 d_4^3 - \frac{303}{16} b_1 d_3 d_4^3 \\
\text{coefficient of } X^7 Y^2 & \quad - \frac{131}{32} b_3 d_3 d_2^3 - \frac{1177}{288} b_1 b_4 d_4^3 + \frac{1151}{12} b_2 d_2 d_4^3 - \frac{303}{16} b_1 d_3 d_4^3 \\
\text{coefficient of } X^6 Y^3 & \quad \frac{917}{12} b_3 d_3 d_4^3 d_3 + \frac{917}{24} b_1 b_2 d_3 d_4^3 - \frac{917}{24} b_1 b_2 d_3^2 d_4 - \frac{2771}{24} b_1 d_3 d_4^3 \\
\text{coefficient of } X^5 Y^4 & \quad \frac{917}{8} b_4 d_2 d_4^4 + \frac{917}{16} b_1 b_2 d_3 d_4^3 - \frac{2771}{8} b_1 d_4 d_3 d_4^2 - \frac{885}{16} b_1^2 d_4^2 \\
& \quad (cont).
Coefficients of degree 9 (Continued)

<table>
<thead>
<tr>
<th>Coefficient of $X^4Y^5$</th>
<th>$\frac{8253}{32}b_4^2d_4 + \frac{8253}{32}d_1^3d_4 + \frac{8253}{32}b_1d_1d_2d_4 + \frac{8253}{32}b_1^2d_3d_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coefficient of $X^3Y^6$</td>
<td>$\frac{917}{32}b_4^2d_3 + \frac{917}{32}b_1b_2^2d_4 - \frac{917}{32}b_1^2b_3d_4 - \frac{917}{32}b_1^2d_2d_4$</td>
</tr>
<tr>
<td>Coefficient of $X^2Y^7$</td>
<td>$-\frac{131}{16}b_4^2d_3 - \frac{131}{8}b_1^2d_2d_3 + \frac{393}{16}b_1^2d_2d_4 + \frac{1179}{8}b_1^2d_3d_4$</td>
</tr>
<tr>
<td>Coefficient of $XY^8$</td>
<td>$-\frac{131}{10}b_4^2d_2 - \frac{131}{32}b_1^2b_2d_3 + \frac{293}{10}b_1^2d_3d_4 + \frac{1179}{8}b_1^2d_4$</td>
</tr>
<tr>
<td>Coefficient of $Y^9$</td>
<td>$-\frac{131}{64} (b_4b_3 + b_1d_1^2 + b_2^2d_1d_2 + b_1^2d_3)$</td>
</tr>
</tbody>
</table>

$b_1$, respectively. Now we give the coefficients of $h_1$, or, to be more precise, the coefficients of $h_1$ reduced modulo the Gröbner basis of $I$ (see Davenport et al. (1993)).

References


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